Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 41820, 13 pages
doi:10.1155/2007/41820

*Research Article*

# Functional Inequalities Associated with Jordan-von Neumann-Type Additive Functional Equations

Choonkil Park, Young Sun Cho, and Mi-Hyen Han

Received 27 September 2006; Accepted 1 November 2006

Recommended by Sever S. Dragomir

We prove the generalized Hyers-Ulam stability of the following functional inequalities: $\|f(x)+f(y)+f(z)\| \leq \|2f((x+y+z)/2)\|$, $\|f(x)+f(y)+f(z)\| \leq \|f(x+y+z)\|$, $\|f(x)+f(y)+2f(z)\| \leq \|2f((x+y)/2+z)\|$ in the spirit of the Rassias stability approach for approximately homomorphisms.



## 1. Introduction and preliminaries

Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

*We are given a group $G$ and a metric group $G'$ with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?*

Hyers [2] considered the case of approximately additive mappings $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \tag{1.1}$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon. \tag{1.3}$$

Rassias [3] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1 (Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.4}$$

*for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$. Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.5}$$

*exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.6}$$

*for all $x \in E$. If $p < 0$, then inequality (1.4) holds for $x, y \neq 0$ and (1.6) for $x \neq 0$.*

Rassias [4] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. Gajda [5], following the same approach as in Rassias [3], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [5] as well as by Rassias and Šemrl [6] that one cannot prove a Rassias-type theorem when $p = 1$. The inequality (1.4) that was introduced for the first time by Rassias [3] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept of stability is known as *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations (cf. the books of Czerwik [7], Hyers et al. [8]).

Rassias [9] followed the innovative approach of Rassias' theorem [3] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

Găvruța [10] provided a further generalization of Rassias' theorem. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [11–14]).

Throughout this paper, let $G$ be a 2-divisible abelian group. Assume that $X$ is a normed space with norm $\|\cdot\|_X$ and that $Y$ is a Banach space with norm $\|\cdot\|_Y$.

In [15], Gilányi showed that if $f$ satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\| \tag{1.7}$$

then $f$ satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f(xy^{-1}), \tag{1.8}$$

see also [16]. Gilányi [17] and Fechner [18] proved the generalized Hyers-Ulam stability of the functional inequality (1.7).

In Section 2, we prove that if $f$ satisfies one of the inequalities $\|f(x)+f(y)+f(z)\| \le \|2f((x+y+z)/2)\|$, $\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\|$, and $\|f(x)+f(y)+2f(z)\| \le \|2f((x+y)/2+z)\|$ then $f$ is Cauchy additive.

In Section 3, we prove the generalized Hyers-Ulam stability of the functional inequality $\|f(x)+f(y)+f(z)\| \le \|2f(x+y+z/2)\|$.

In Section 4, we prove the generalized Hyers-Ulam stability of the functional inequality $\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\|$.

In Section 5, we prove the generalized Hyers-Ulam stability of the functional inequality $\|f(x)+f(y)+2f(z)\| \le \|2f(x+y/2+z)\|$.

## 2. Functional inequalities associated with Jordan-von Neumann-type additive functional equations

Proposition 2.1. *Let $f: G \to Y$ be a mapping such that*

$$||f(x)+f(y)+f(z)||_Y \le \left\|2f\left(\frac{x+y+z}{2}\right)\right\|_Y \tag{2.1}$$

*for all $x,y,z \in G$. Then $f$ is Cauchy additive.*

*Proof.* Letting $x=y=z=0$ in (2.1), we get

$$||3f(0)||_Y \le ||2f(0)||_Y. \tag{2.2}$$

So $f(0)=0$.

Letting $z=0$ and $y=-x$ in (2.1), we get

$$||f(x)+f(-x)||_Y \le ||2f(0)||_Y = 0 \tag{2.3}$$

for all $x \in G$. Hence $f(-x)=-f(x)$ for all $x \in G$.

Letting $z=-x-y$ in (2.1), we get

$$||f(x)+f(y)-f(x+y)||_Y = ||f(x)+f(y)+f(-x-y)||_Y \le ||2f(0)||_Y = 0 \tag{2.4}$$

for all $x,y \in G$. Thus

$$f(x+y)=f(x)+f(y) \tag{2.5}$$

for all $x,y \in G$, as desired. □

Proposition 2.2. *Let $f : G \to Y$ be a mapping such that*

$$||f(x)+f(y)+f(z)||_Y \leq ||f(x+y+z)||_Y \tag{2.6}$$

*for all $x,y,z \in G$. Then $f$ is Cauchy additive.*

*Proof.* Letting $x = y = z = 0$ in (2.6), we get

$$||3f(0)||_Y \leq ||f(0)||_Y. \tag{2.7}$$

So $f(0) = 0$.

Letting $z = 0$ and $y = -x$ in (2.6), we get

$$||f(x)+f(-x)||_Y \leq ||f(0)||_Y = 0 \tag{2.8}$$

for all $x \in G$. Hence $f(-x) = -f(x)$ for all $x \in G$.

Letting $z = -x-y$ in (2.6), we get

$$||f(x)+f(y)-f(x+y)||_Y = ||f(x)+f(y)+f(-x-y)||_Y \leq ||f(0)||_Y = 0 \tag{2.9}$$

for all $x, y \in G$. Thus

$$f(x+y) = f(x)+f(y) \tag{2.10}$$

for all $x, y \in G$, as desired. □

Proposition 2.3. *Let $f : G \to Y$ be a mapping such that*

$$||f(x)+f(y)+2f(z)||_Y \leq \left\|2f\left(\frac{x+y}{2}+z\right)\right\|_Y \tag{2.11}$$

*for all $x,y,z \in G$. Then $f$ is Cauchy additive.*

*Proof.* Letting $x = y = z = 0$ in (2.11), we get

$$||4f(0)||_Y \leq ||2f(0)||_Y. \tag{2.12}$$

So $f(0) = 0$.

Letting $z = 0$ and $y = -x$ in (2.11), we get

$$||f(x)+f(-x)||_Y \leq ||f(0)||_Y = 0 \tag{2.13}$$

for all $x \in G$. Hence $f(-x) = -f(x)$ for all $x \in G$.

Replacing $x$ by $-2z$ and letting $y = 0$ in (2.11), we get

$$||-f(2z)+2f(z)||_Y = ||f(-2z)+2f(z)||_Y \leq ||f(0)||_Y = 0 \tag{2.14}$$

for all $z \in G$. Thus $f(2z) = 2f(z)$ for all $z \in G$.

Letting $z = -(x+y)/2$ in (2.11), we get

$$||f(x)+f(y)-f(x+y)||_Y = \left\|f(x)+f(y)+2f\left(-\frac{x+y}{2}\right)\right\|_Y \le ||f(0)||_Y = 0 \qquad (2.15)$$

for all $x, y \in G$. Thus

$$f(x+y) = f(x)+f(y) \qquad (2.16)$$

for all $x, y \in G$, as desired. □

## 3. Stability of a functional inequality associated with a 3-variable Jensen additive functional equation

We prove the generalized Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann-type 3-variable Jensen additive functional equation.

Theorem 3.1. *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$||f(x)+f(y)+f(z)||_Y \le \left\|2f\left(\frac{x+y+z}{2}\right)\right\|_Y + \theta(\|x\|_X^r + \|y\|_X^r + \|z\|_X^r) \qquad (3.1)$$

*for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\left\|\frac{f(x)-f(-x)}{2} - h(x)\right\|_Y \le \frac{2^r+2}{2^r-2}\theta\|x\|_X^r \qquad (3.2)$$

*for all $x \in X$.*

*Proof.* Letting $y = x$ and $z = -2x$ in (3.1), we get

$$||2f(x)+f(-2x)||_Y \le (2+2^r)\theta\|x\|_X^r \qquad (3.3)$$

for all $x \in X$. Replacing $x$ by $-x$ in (3.3), we get

$$||2f(-x)+f(2x)||_Y \le (2+2^r)\theta\|x\|_X^r \qquad (3.4)$$

for all $x \in X$. Let $g(x) := (f(x)-f(-x))/2$. It follows from (3.3) and (3.4) that

$$||2g(x)-g(2x)||_Y \le (2+2^r)\theta\|x\|_X^r \qquad (3.5)$$

for all $x \in X$. So

$$\left\|g(x)-2g\left(\frac{x}{2}\right)\right\|_Y \le \frac{2+2^r}{2^r}\theta\|x\|_X^r \qquad (3.6)$$

for all $x \in X$. Hence

$$\left\|2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right)\right\|_Y \le \sum_{j=l}^{m-1}\left\|2^j g\left(\frac{x}{2^j}\right) - 2^{j+1} g\left(\frac{x}{2^{j+1}}\right)\right\|_Y \le \frac{2+2^r}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|_X^r \qquad (3.7)$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{2^n g(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{2^n g(x/2^n)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right) \tag{3.8}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.2).

It follows from (3.1) that

$$\begin{aligned}
||h(x)+h(y)+h(z)||_Y &= \lim_{n\to\infty} 2^n \left\|g\left(\frac{x}{2^n}\right)+g\left(\frac{y}{2^n}\right)+g\left(\frac{z}{2^n}\right)\right\|_Y \\
&= \lim_{n\to\infty} \frac{2^n}{2} \left\|f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)+\left(\frac{z}{2^n}\right)-f\left(\frac{-x}{2^n}\right)-f\left(\frac{-y}{2^n}\right)-\left(\frac{-z}{2^n}\right)\right\|_Y \\
&\le \lim_{n\to\infty} \frac{2^n}{2} \left\|2f\left(\frac{x+y+z}{2^{n+1}}\right)-2f\left(\frac{x+y+z}{-2^{n+1}}\right)\right\|_Y \\
&\quad + \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}} \left(\|x\|_X^r+\|y\|_X^r+\|z\|_X^r\right) \\
&= \left\|2h\left(\frac{x+y+z}{2}\right)\right\|_Y
\end{aligned} \tag{3.9}$$

for all $x, y, z \in X$. So

$$||h(x)+h(y)+h(z)||_Y \le \left\|2h\left(\frac{x+y+z}{2}\right)\right\|_Y \tag{3.10}$$

for all $x, y, z \in X$. By Proposition 2.1, the mapping $h : X \to Y$ is Cauchy additive.

Now, let $T : X \to Y$ be another Cauchy additive mapping satisfying (3.2). Then we have

$$\begin{aligned}
||h(x)-T(x)||_Y &= 2^n \left\|h\left(\frac{x}{2^n}\right)-T\left(\frac{x}{2^n}\right)\right\|_Y \\
&\le 2^n \left(\left\|h\left(\frac{x}{2^n}\right)-g\left(\frac{x}{2^n}\right)\right\|_Y+\left\|T\left(\frac{x}{2^n}\right)-g\left(\frac{x}{2^n}\right)\right\|_Y\right) \\
&\le \frac{2(2^r+2)2^n}{(2^r-2)2^{nr}}\theta\|x\|_X^r,
\end{aligned} \tag{3.11}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $h$. Thus the mapping $h : X \to Y$ is a unique Cauchy additive mapping satisfying (3.2). □

Theorem 3.2. *Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (3.1). Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\left\|\frac{f(x)-f(-x)}{2}-h(x)\right\|_Y \le \frac{2+2^r}{2-2^r}\theta\|x\|_X^r \tag{3.12}$$

*for all $x \in X$.*

*Proof.* It follows from (3.5) that

$$\left\|g(x)-\frac{1}{2}g(2x)\right\|_Y \le \frac{2+2^r}{2}\theta\|x\|_X^r \tag{3.13}$$

for all $x \in X$. Hence

$$\left\|\frac{1}{2^l}g(2^l x)-\frac{1}{2^m}g(2^m x)\right\|_Y \le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}g(2^j x)-\frac{1}{2^{j+1}}g(2^{j+1}x)\right\|_Y \le \frac{2+2^r}{2}\sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\theta\|x\|_X^r \tag{3.14}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.14) that the sequence $\{(1/2^n)g(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{(1/2^n)g(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{2^n}g(2^n x) \tag{3.15}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.14), we get (3.12).

The rest of the proof is similar to the proof of Theorem 3.1. □

THEOREM 3.3. *Let $r > 1/3$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x)+f(y)+f(z)\|_Y \le \left\|2f\left(\frac{x+y+z}{2}\right)\right\|_Y + \theta\cdot\|x\|_X^r\cdot\|y\|_X^r\cdot\|z\|_X^r \tag{3.16}$$

*for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\left\|\frac{f(x)-f(-x)}{2}-h(x)\right\|_Y \le \frac{2^r\theta}{8^r-2}\|x\|_X^{3r} \tag{3.17}$$

*for all $x \in X$.*

*Proof.* Letting $y = x$ and $z = -2x$ in (3.16), we get

$$\|2f(x)+f(-2x)\|_Y \le 2^r\theta\|x\|_X^{3r} \tag{3.18}$$

for all $x \in X$. Replacing $x$ by $-x$ in (3.18), we get

$$\|2f(-x)+f(2x)\|_Y \le 2^r\theta\|x\|_X^{3r} \tag{3.19}$$

for all $x \in X$. Let $g(x) := (f(x)-f(-x))/2$. It follows from (3.18) and (3.19) that

$$\|2g(x)-g(2x)\|_Y \le 2^r\theta\|x\|_X^{3r} \tag{3.20}$$

for all $x \in X$. So

$$\left\|g(x)-2g\left(\frac{x}{2}\right)\right\|_Y \le \frac{2^r}{8^r}\theta\|x\|_X^{3r} \tag{3.21}$$

for all $x \in X$. Hence

$$\left\|2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right)\right\|_Y \le \sum_{j=l}^{m-1} \left\|2^j g\left(\frac{x}{2^j}\right) - 2^{j+1} g\left(\frac{x}{2^{j+1}}\right)\right\|_Y \le \frac{2^r}{8^r} \sum_{j=l}^{m-1} \frac{2^j}{8^{rj}} \theta \|x\|_X^{3r} \tag{3.22}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$.

It follows from (3.22) that the sequence $\{2^n g(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{2^n g(x/2^n)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right) \tag{3.23}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.22), we get (3.17).

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.4. *Let $r < 1/3$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (3.16). Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\left\|\frac{f(x) - f(-x)}{2} - h(x)\right\|_Y \le \frac{2^r \theta}{2 - 8^r} \|x\|_X^{3r} \tag{3.24}$$

*for all $x \in X$.*

*Proof.* It follows from (3.20) that

$$\left\|g(x) - \frac{1}{2} g(2x)\right\|_Y \le \frac{2^r}{2} \theta \|x\|_X^{3r} \tag{3.25}$$

for all $x \in X$. Hence

$$\left\|\frac{1}{2^l} g(2^l x) - \frac{1}{2^m} g(2^m x)\right\|_Y \le \sum_{j=l}^{m-1} \left\|\frac{1}{2^j} g(2^j x) - \frac{1}{2^{j+1}} g(2^{j+1} x)\right\|_Y \le \frac{2^r}{2} \sum_{j=l}^{m-1} \frac{8^{rj}}{2^j} \theta \|x\|_X^r \tag{3.26}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$.

It follows from (3.26) that the sequence $\{(1/2^n) g(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{(1/2^n) g(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{2^n} g(2^n x) \tag{3.27}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.26), we get (3.24).

The rest of the proof is similar to the proof of Theorem 3.1. □

## 4. Stability of a functional inequality associated with a 3-variable Cauchy additive functional equation

We prove the generalized Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann-type 3-variable Cauchy additive functional equation.

THEOREM 4.1. *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x)+f(y)+f(z)\|_Y \le \|f(x+y+z)\|_Y + \theta(\|x\|_X^r + \|y\|_X^r + \|z\|_X^r) \tag{4.1}$$

*for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\left\|\frac{f(x)-f(-x)}{2} - h(x)\right\|_Y \le \frac{2^r+2}{2^r-2}\theta\|x\|_X^r \tag{4.2}$$

*for all $x \in X$.*

*Proof.* Letting $y = x$ and $z = -2x$ in (4.1), we get

$$\|2f(x)+f(-2x)\|_Y \le (2+2^r)\theta\|x\|_X^r \tag{4.3}$$

for all $x \in X$. Replacing $x$ by $-x$ in (4.3), we get

$$\|2f(-x)+f(2x)\|_Y \le (2+2^r)\theta\|x\|_X^r \tag{4.4}$$

for all $x \in X$. Let $g(x) := (f(x)-f(-x))/2$. It follows from (4.3) and (4.4) that

$$\|2g(x)-g(2x)\|_Y \le (2+2^r)\theta\|x\|_X^r \tag{4.5}$$

for all $x \in X$.

The rest of the proof is the same as in the proof of Theorem 3.1. □

THEOREM 4.2. *Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (4.1). Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\left\|\frac{f(x)-f(-x)}{2} - h(x)\right\|_Y \le \frac{2+2^r}{2-2^r}\theta\|x\|_X^r \tag{4.6}$$

*for all $x \in X$.*

*Proof.* It follows from (4.5) that

$$\left\|g(x) - \frac{1}{2}g(2x)\right\|_Y \le \frac{2+2^r}{2}\theta\|x\|_X^r \tag{4.7}$$

for all $x \in X$.

The rest of the proof is the same as in the proofs of Theorems 3.1 and 3.2. □

THEOREM 4.3. *Let $r > 1/3$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x)+f(y)+f(z)\|_Y \le \|f(x+y+z)\|_Y + \theta\cdot\|x\|_X^r\cdot\|y\|_X^r\cdot\|z\|_X^r \tag{4.8}$$

*for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\left\| \frac{f(x) - f(-x)}{2} - h(x) \right\|_Y \leq \frac{2^r \theta}{8^r - 2} \|x\|_X^{3r} \tag{4.9}$$

*for all $x \in X$.*

*Proof.* Letting $y = x$ and $z = -2x$ in (4.8), we get

$$\|2f(x) + f(-2x)\|_Y \leq 2^r \theta \|x\|_X^{3r} \tag{4.10}$$

for all $x \in X$. Replacing $x$ by $-x$ in (4.10), we get

$$\|2f(-x) + f(2x)\|_Y \leq 2^r \theta \|x\|_X^{3r} \tag{4.11}$$

for all $x \in X$. Let $g(x) := (f(x) - f(-x))/2$. It follows from (4.10) and (4.11) that

$$\|2g(x) - g(2x)\|_Y \leq 2^r \theta \|x\|_X^{3r} \tag{4.12}$$

for all $x \in X$.

The rest of the proof is the same as in the proofs of Theorems 3.1 and 3.3. □

Theorem 4.4. *Let $r < 1/3$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (4.8). Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\left\| \frac{f(x) - f(-x)}{2} - h(x) \right\|_Y \leq \frac{2^r \theta}{2 - 8^r} \|x\|_X^{3r} \tag{4.13}$$

*for all $x \in X$.*

*Proof.* It follows from (4.12) that

$$\left\| g(x) - \frac{1}{2} g(2x) \right\|_Y \leq \frac{2^r}{2} \theta \|x\|_X^{3r} \tag{4.14}$$

for all $x \in X$.

The rest of the proof is the same as in the proofs of Theorems 3.1 and 3.4. □

## 5. Stability of a functional inequality associated with the Cauchy-Jensen functional equation

We prove the generalized Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann-type Cauchy-Jensen functional equation.

Theorem 5.1. *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + 2f(z)\|_Y \leq \left\| 2f\left(\frac{x+y}{2} + z\right) \right\|_Y + \theta\left(\|x\|_X^r + \|y\|_X^r + \|z\|_X^r\right) \tag{5.1}$$

*for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\left\|\frac{f(x)-f(-x)}{2}-h(x)\right\|_Y \leq \frac{2^r+1}{2^r-2}\theta\|x\|_X^r \tag{5.2}$$

*for all $x \in X$.*

*Proof.* Replacing $x$ by $2x$ and letting $y = 0$ and $z = -x$ in (5.1), we get

$$\|f(2x)+2f(-x)\|_Y \leq (1+2^r)\theta\|x\|_X^r \tag{5.3}$$

for all $x \in X$. Replacing $x$ by $-x$ in (5.3), we get

$$\|f(-2x)+2f(x)\|_Y \leq (1+2^r)\theta\|x\|_X^r \tag{5.4}$$

for all $x \in X$. Let $g(x) := (f(x) - f(-x))/2$. It follows from (5.3) and (5.4) that

$$\|2g(x)-g(2x)\|_Y \leq (1+2^r)\theta\|x\|_X^r \tag{5.5}$$

for all $x \in X$. So

$$\left\|g(x)-2g\left(\frac{x}{2}\right)\right\|_Y \leq \frac{1+2^r}{2^r}\theta\|x\|_X^r \tag{5.6}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 5.2. *Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (5.1). Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\left\|\frac{f(x)-f(-x)}{2}-h(x)\right\|_Y \leq \frac{1+2^r}{2-2^r}\theta\|x\|_X^r \tag{5.7}$$

*for all $x \in X$.*

*Proof.* It follows from (5.5) that

$$\left\|g(x)-\frac{1}{2}g(2x)\right\|_Y \le \frac{1+2^r}{2}\theta\|x\|_X^r \tag{5.8}$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 3.1 and 3.2. □

## Acknowledgment

This work was supported by the second Brain Korea 21 Project.

## References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[4] Th. M. Rassias, "Problem 16; 2, Report of the 27th International Symp. on Functional Equations," *Aequationes Mathematicae*, vol. 39, pp. 292–293; 309, 1990.

[5] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[6] Th. M. Rassias and P. Šemrl, "On the behavior of mappings which do not satisfy Hyers-Ulam stability," *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.

[7] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.

[8] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser Boston, Boston, Mass, USA, 1998.

[9] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.

[10] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[11] K.-W. Jun and Y.-H. Lee, "A generalization of the Hyers-Ulam-Rassias stability of the pexiderized quadratic equations," *Journal of Mathematical Analysis and Applications*, vol. 297, no. 1, pp. 70–86, 2004.

[12] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[13] C. Park, "Homomorphisms between Poisson $JC^*$-algebras," *Bulletin of the Brazilian Mathematical Society. New Series*, vol. 36, no. 1, pp. 79–97, 2005.

[14] C. Park, "Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras," to appear in *Bulletin des Sciences Mathématiques*.

[15] A. Gilányi, "Eine zur Parallelogrammgleichung äquivalente Ungleichung," *Aequationes Mathematicae*, vol. 62, no. 3, pp. 303–309, 2001.

[16] J. Rätz, "On inequalities associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 66, no. 1-2, pp. 191–200, 2003.

[17] A. Gilányi, "On a problem by K. Nikodem," *Mathematical Inequalities & Applications*, vol. 5, no. 4, pp. 707–710, 2002.

[18] W. Fechner, "Stability of a functional inequality associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 71, no. 1-2, pp. 149–161, 2006.

Choonkil Park: Department of Mathematics, Hanyang University, Seoul 133-791, South Korea
*Email address*: baak@hanyang.ac.kr

Young Sun Cho: Department of Mathematics, Chungnam National University, Daejeon 305-764, South Korea
*Email address*: s9801627@hanmail.net

Mi-Hyen Han: Department of Mathematics, Chungnam National University, Daejeon 305-764, South Korea
*Email address*: hmh1014@hanmail.net