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Research Article

Volterra-Type Operators on Zygmund Spaces

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The boundedness and the compactness of the two integral operators $J_g f(z) = \int_0^z f(\xi)g'(\xi)d\xi$; $I_g f(z) = \int_0^z f'(\xi)g(\xi)d\xi$, where g is an analytic function on the open unit disk in the complex plane, on the Zygmund space are studied.

1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ its boundary. Denote by $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$.

Let $\mathcal{Z}$ denote the space of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\|_{\mathcal{Z}} = \sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty, \tag{1.1}$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$. By a Zygmund theorem (see [1, Theorem 5.3]) and the closed graph theorem, we have that $f \in \mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{D}} (1 - |z|^2) |f''(z)| < \infty, \tag{1.2}$$

moreover the following asymptotic relation holds:

$$\|f\|_{\mathcal{Z}} \asymp \sup_{z\in\mathbb{D}} (1 - |z|^2) |f''(z)|. \tag{1.3}$$

Therefore, $\mathcal{Z}$ is called Zygmund class. Since the quantities in (1.3) are semi norms (they do not distinguish between functions differing by a linear polynomial), it is natural to add them to the quantity $|f(0)| + |f'(0)|$ to obtain two equivalent norms on the Zygmund class of functions. Zygmund class with such defined norm will be called Zygmud space. This norm will be again denoted by $\|\cdot\|_{\mathcal{Z}}$.

By (1.3), we have

$$|f'(z)-f'(0)| \le C\|f\|_{\mathcal{Z}} \ln\frac{1}{1-|z|}. \tag{1.4}$$

Also, we have

$$\begin{aligned}
&|f(z)-f(0)-zf'(0)| \\
&\quad = \left|\int_0^z\int_0^1 f''(t\zeta)\zeta\,dt\,d\zeta\right| \le \|f\|_{\mathcal{Z}}\left|\int_0^z\int_0^1 \frac{|\zeta|dt}{1-t|\zeta|}|d\zeta|\right| \\
&\quad \le \|f\|_{\mathcal{Z}}\left|\int_0^{|z|}\ln\frac{1}{1-s}ds\right| = \|f\|_{\mathcal{Z}}\left(|z|+(|z|-1)\ln\frac{1}{1-|z|}\right),
\end{aligned} \tag{1.5}$$

for every $z\in\mathbb{D}$. From this and since the quantity

$$\sup_{x\in[0,1)}\left(x+(x-1)\ln\frac{1}{1-x}\right) \tag{1.6}$$

is bounded, it follows that

$$\|f\|_\infty \le C\|f\|_{\mathcal{Z}}, \tag{1.7}$$

for every $f\in\mathcal{Z}$, and for some positive constant C independent of f.

We introduce the little Zygmund space $\mathcal{Z}_0$ in the following natural way:

$$f\in\mathcal{Z}_0 \Longleftrightarrow \lim_{|z|\to 1}(1-|z|)|f''(z)| = 0. \tag{1.8}$$

It is easy to see that $\mathcal{Z}_0$ is a closed subspace of $\mathcal{Z}$.

Suppose that $g:\mathbb{D}\to\mathbb{C}$ is a holomorphic map, $f\in H(\mathbb{D})$. The integral operator, called Volterra-type operator,

$$J_g f(z) = \int_0^z f\,dg = \int_0^1 f(tz)zg'(tz)dt = \int_0^z f(\xi)g'(\xi)d\xi, \quad z\in\mathbb{D}, \tag{1.9}$$

was introduced by Pommerenke in [2].

Another natural integral operator is defined as follows:

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)d\xi. \tag{1.10}$$

The importance of the operators J_g and I_g comes from the fact that

$$J_g f + I_g f = M_g f - f(0)g(0), \tag{1.11}$$

where the multiplication operator M_g is defined by

$$(M_g f)(z) = g(z)f(z), \quad f\in H(\mathbb{D}),\ z\in\mathbb{D}. \tag{1.12}$$

In [2] Pommerenke showed that J_g is a bounded operator on the Hardy space H^2 if and only if $g \in \mathrm{BMOA}$. The boundedness and compactness of J_g and I_g between some spaces of analytic functions, as well as their n-dimensional extensions, were investigated in [3–16] (see also the related references therein).

The purpose of this paper is to study the boundedness and compactness of integral operators J_g and I_g on the Zygmund space and the little Zygmund space.

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to an other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. The boundedness and compactness of $J_g, I_g : \mathscr{Z} \to \mathscr{Z}$

In this section, we consider the boundedness and compactness of the operators J_g and I_g on the Zygmund space. To this end, we need two lemmas. Before formulating these lemmas, we quote the following result from [17].

Theorem 2.1. *Assume that f is a holomorphic function on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Then the modulus of continuity on the closed disk is bounded by a constant times the modulus of continuity on the circle.*

By Theorem 2.1 and standard arguments (see, e.g., [18, Proposition 3.11]), the following lemma follows.

Lemma 2.2. *Assume that g is an analytic function on $\mathbb{D}$. Then J_g (or I_g) $: \mathscr{Z} \to \mathscr{Z}$ is compact if and only if J_g (or I_g) $: \mathscr{Z} \to \mathscr{Z}$ is bounded, and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathscr{Z}$ which converges to zero uniformly on $\overline{\mathbb{D}}$ as $k \to \infty$, $\|J_g f_k\|_{\mathscr{Z}} \to 0$ (or $\|I_g f_k\|_{\mathscr{Z}} \to 0$) as $k \to \infty$.*

Lemma 2.3. *Suppose that $f \in \mathscr{Z}_0$, then*

$$\lim_{|z|\to 1} \frac{|f'(z)|}{\ln\left(1/(1-|z|^2)\right)} = 0. \tag{2.1}$$

Proof. Since $f \in \mathscr{Z}_0$, it follows that for every $\varepsilon > 0$ there is a $\delta \in (1/2, 1)$ such that

$$(1-|z|)\,|f''(z)| < \varepsilon, \tag{2.2}$$

whenever $\delta < |z| < 1$.

From (2.2), when $\delta < |z| < 1$, we have that

$$\begin{aligned} |f'(z) - f'(0)| &= \left| \int_0^1 f''(tz) z\,dt \right| \le \int_0^{\delta/|z|} |f''(tz)|\,|z|\,dt + \int_{\delta/|z|}^1 |f''(tz)|\,|z|\,dt \\ &\le \|f\|_{\mathscr{Z}} \int_0^{\delta/|z|} \frac{|z|\,dt}{1-t|z|} + \varepsilon \int_{\delta/|z|}^1 \frac{|z|\,dt}{1-t|z|} \le \|f\|_{\mathscr{Z}} \ln\frac{1}{1-\delta} + \varepsilon \ln\frac{1}{1-|z|}. \end{aligned} \tag{2.3}$$

Dividing (2.3) by $\ln(1/(1-|z|))$ and letting $|z| \to 1$, we obtain

$$\lim_{|z|\to 1} \frac{|f'(z)|}{\ln(1/(1-|z|))} \le \varepsilon, \tag{2.4}$$

from which the lemma follows. □

Now, we are in a position to formulate and prove the main results of this section.

Theorem 2.4. *Assume that g is an analytic function on $\mathbb{D}$. Then $J_g : \mathfrak{X} \to \mathfrak{X}$ is bounded if and only if $g \in \mathfrak{X}$.*

Proof. Assume that $J_g : \mathfrak{X} \to \mathfrak{X}$ is bounded. Taking the function given by $f(z) = 1$, we see that $g \in \mathfrak{X}$.

Conversely, assume that $g \in \mathfrak{X}$. Employing (1.4) and (1.7), we have

$$\begin{aligned} (1-|z|^2)\,|(J_g f)''(z)| &= (1-|z|^2)\,|f'(z)g'(z) + f(z)g''(z)| \\ &\le C\|f\|_{\mathfrak{X}}(1-|z|^2)\,|g'(z)|\ln\frac{1}{1-|z|^2} + C\|f\|_{\mathfrak{X}}(1-|z|^2)\,|g''(z)| \\ &\le C\|f\|_{\mathfrak{X}}\|g\|_{\mathfrak{X}}\left((1-|z|^2)\left(\ln\frac{1}{1-|z|^2}\right)^2 + 1\right). \end{aligned} \tag{2.5}$$

On the other hand, we have that

$$J_g(f)(0) = 0, \qquad |(J_g(f))'(0)| = |f(0)g'(0)| \le \|f\|_{\mathfrak{X}}\,|g'(0)|. \tag{2.6}$$

From (2.6), by taking the supremum in (2.5) over $\mathbb{D}$ and using the fact that the quantity

$$\sup_{x\in(0,1]} x\left(\ln\frac{1}{x}\right)^2 \tag{2.7}$$

is finite, the boundedness of the operator $J_g : \mathfrak{X} \to \mathfrak{X}$ follows. □

Theorem 2.5. *Assume that g is an analytic function on $\mathbb{D}$. Then $I_g : \mathfrak{X} \to \mathfrak{X}$ is bounded if and only if $g \in H^\infty \cap \mathcal{B}_{\log}$, where*

$$\|g\|_{\mathcal{B}_{\log}} = \sup_{z\in\mathbb{D}} (1-|z|^2)\,|g'(z)|\ln\frac{1}{1-|z|^2}. \tag{2.8}$$

Proof. Assume that $g \in H^\infty \cap \mathcal{B}_{\log}$. Then by (1.4), we have

$$\begin{aligned} (1-|z|^2)\,|(I_g f)''(z)| &= (1-|z|^2)\,|f''(z)g(z) + f'(z)g'(z)| \\ &\le C\|f\|_{\mathfrak{X}}\,|g(z)| + C\|f\|_{\mathfrak{X}}(1-|z|^2)\,|g'(z)|\ln\frac{1}{1-|z|^2} \\ &\le C\|f\|_{\mathfrak{X}}\|g\|_\infty + C\|f\|_{\mathfrak{X}}\|g\|_{\mathcal{B}_{\log}}. \end{aligned} \tag{2.9}$$

On the other hand, we have that

$$I_g(f)(0)=0, \qquad \left|\left(I_g(f)\right)'(0)\right| = \left|f'(0)g(0)\right| \le \|f\|_{\mathcal{Z}}\left|g(0)\right|. \tag{2.10}$$

From this, by taking the supremum in (2.9) over $\mathbb{D}$ and using the conditions of the theorem, the boundedness of the operator $I_g:\mathcal{Z}\to\mathcal{Z}$ follows.

Conversely, assume that $I_g:\mathcal{Z}\to\mathcal{Z}$ is bounded. Then there is a positive constant C such that

$$\|I_g f\|_{\mathcal{Z}} \le C\|f\|_{\mathcal{Z}}, \tag{2.11}$$

for every $f\in\mathcal{Z}$. Set

$$h(z)=(z-1)\left[\left(1+\ln\frac{1}{1-z}\right)^2+1\right], \tag{2.12}$$

$$h_a(z)=\frac{h(\overline{a}z)}{\overline{a}}\left(\ln\frac{1}{1-|a|^2}\right)^{-1} \tag{2.13}$$

for $a\in\mathbb{D}$ such that $|a|>\sqrt{1-1/e}$. Then, we have

$$\begin{aligned} h_a'(z)&=\left(\ln\frac{1}{1-\overline{a}z}\right)^2\left(\ln\frac{1}{1-|a|^2}\right)^{-1},\\ h_a''(z)&=\frac{2\overline{a}}{1-\overline{a}z}\left(\ln\frac{1}{1-\overline{a}z}\right)\left(\ln\frac{1}{1-|a|^2}\right)^{-1}. \end{aligned} \tag{2.14}$$

Thus for $\sqrt{1-1/e}<|a|<1$, we have

$$\left|h_a''(z)\right| \le \frac{2}{1-|z|}\left(\ln\frac{1}{1-|a|}+C\right)\left(\ln\frac{1}{1-|a|^2}\right)^{-1} \le \frac{C}{1-|z|}, \tag{2.15}$$

and consequently

$$M_1=\sup_{\sqrt{1-1/e}<|a|<1}\|h_a\|_{\mathcal{Z}}<\infty. \tag{2.16}$$

Therefore, we have that

$$\begin{aligned} \infty &> \|I_g\|\|h_a\|_{\mathcal{Z}} \ge \|I_g h_a\|_{\mathcal{Z}}\\ &\ge \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|h_a''(z)g(z)+h_a'(z)g'(z)\right|\\ &\ge \left(1-|a|^2\right)\left|h_a''(a)g(a)+h_a'(a)g'(a)\right|\\ &\ge \left(1-|a|^2\right)\left|\frac{2\overline{a}}{1-|a|^2}g(a)+g'(a)\ln\frac{1}{1-|a|^2}\right|\\ &\ge -2|a|\left|g(a)\right|+\left(1-|a|^2\right)\left|g'(a)\right|\ln\frac{1}{1-|a|^2}. \end{aligned} \tag{2.17}$$

Next, let

$$f_a(z) = \frac{h(\bar{a}z)}{\bar{a}}\left(\ln\frac{1}{1-|a|^2}\right)^{-1} - \int_0^z \ln\frac{1}{1-\bar{a}w}dw \tag{2.18}$$

for $a \in \mathbb{D}$ such that $|a| > \sqrt{1-1/e}$. Then, we have

$$\begin{aligned} f_a'(z) &= \left(\ln\frac{1}{1-\bar{a}z}\right)^2\left(\ln\frac{1}{1-|a|^2}\right)^{-1} - \ln\frac{1}{1-\bar{a}z}, \\ f_a''(z) &= \frac{2\bar{a}}{1-\bar{a}z}\left(\ln\frac{1}{1-\bar{a}z}\right)\left(\ln\frac{1}{1-|a|^2}\right)^{-1} - \frac{\bar{a}}{1-\bar{a}z}. \end{aligned} \tag{2.19}$$

Similar to the previous case, we have

$$M_2 = \sup_{\sqrt{1-1/e}<|a|<1} \|f_a\|_{\mathcal{Z}} < \infty. \tag{2.20}$$

From this and by using the facts that

$$f_a'(a) = 0, \qquad f_a''(a) = \frac{\bar{a}}{1-|a|^2}, \tag{2.21}$$

we have that

$$\begin{aligned} \infty &> \|I_g\|\|f_a\|_{\mathcal{Z}} \geq \|I_g f_a\|_{\mathcal{Z}} \\ &\geq \sup_{z\in\mathbb{D}} (1-|z|^2)\,|f_a''(z)g(z) + f_a'(z)g'(z)| \\ &\geq (1-|a|^2)\,|f_a''(a)g(a) + f_a'(a)g'(a)| \\ &= (1-|a|^2)\,|f_a''(a)g(a)| = |a|\,|g(a)|, \end{aligned} \tag{2.22}$$

for $\sqrt{1-1/e} < |a| < 1$. From (2.22), we see that $\sup_{\sqrt{1-1/e}<|z|<1} |g(z)| < \infty$. From this and by the maximum modulus theorem, it follows that $g \in H^\infty$, as desired.

From (2.17) and (2.22), it follows that

$$(1-|a|^2)\,|g'(a)|\ln\frac{1}{1-|a|^2} \leq \|I_g h_a\|_{\mathcal{Z}} + 2\|g\|_\infty \leq M_1\|I_g\|_{\mathcal{Z}\to\mathcal{Z}} + 2\|g\|_\infty < \infty \tag{2.23}$$

for every $\sqrt{1-1/e} < |a| < 1$.

On the other hand, we have that

$$\begin{aligned} \sup_{|a|\leq\sqrt{1-1/e}} (1-|a|^2)\,|g'(a)|\ln\frac{1}{1-|a|^2} &\leq \frac{1}{e}\max_{|a|=\sqrt{1-1/e}} |g'(a)| \\ &\leq \sup_{\sqrt{1-1/e}\leq|a|<1} (1-|a|^2)\,|g'(a)|\ln\frac{1}{1-|a|^2}. \end{aligned} \tag{2.24}$$

From (2.23) and (2.24), we obtain $g \in \mathcal{B}_{\log}$, finishing the proof of the theorem. □

Theorem 2.6. *Assume that g is an analytic function on $\mathbb{D}$. Then, $J_g : \mathfrak{L} \to \mathfrak{L}$ is compact if and only if $g \in \mathfrak{L}$.*

Proof. If $J_g : \mathfrak{L} \to \mathfrak{L}$ is compact, then it is bounded, and by Theorem 2.4 it follows that $g \in \mathfrak{L}$.

Now assume that $g \in \mathfrak{L}$ and that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{L}$ such that $\sup_{n\in\mathbb{N}} \|f_n\|_{\mathfrak{L}} \le L$ and that $f_n \to 0$ uniformly on $\overline{\mathbb{D}}$ as $n \to \infty$. Now note that for every $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$(1-|z|^2)\left(\ln\frac{1}{1-|z|^2}\right)^2 < \varepsilon, \tag{2.25}$$

whenever $\delta < |z| < 1$. Let $K = \{z \in \mathbb{D} : |z| \le \delta\}$. Note that K is a compact subset of $\mathbb{D}$. In view of (1.4), (1.7), and (2.25), we have that

$$\begin{aligned}
\|J_g f_n\|_{\mathfrak{L}} &= \sup_{z\in\mathbb{D}}(1-|z|^2)\,|f_n'(z)g'(z)+f_n(z)g''(z)| + |f_n(0)g'(0)| \\
&\le \sup_{z\in K}(1-|z|^2)\,|f_n'(z)g'(z)| + \sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)\,|f_n'(z)g'(z)| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)\,|f_n(z)g''(z)| + |f_n(0)g'(0)| \\
&\le C\|g\|_{\mathfrak{L}}\sup_{z\in K}|f_n'(z)|\sup_{z\in K}(1-|z|^2)\ln\frac{1}{1-|z|} \\
&\quad + C\|f_n\|_{\mathfrak{L}}\|g\|_{\mathfrak{L}}\sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)\left(\ln\frac{1}{1-|z|}\right)^2 \\
&\quad + \|g\|_{\mathfrak{L}}\sup_{z\in\mathbb{D}}|f_n(z)| + |f_n(0)|\,\|g\|_{\mathfrak{L}} \\
&\le \frac{2C}{e}\|g\|_{\mathfrak{L}}\sup_{z\in K}|f_n'(z)| + C\varepsilon L\|g\|_{\mathfrak{L}} + 2\|g\|_{\mathfrak{L}}\sup_{z\in\mathbb{D}}|f_n(z)|.
\end{aligned} \tag{2.26}$$

Since $f_n \to 0$ uniformly on $\overline{\mathbb{D}}$, by the Cauchy estimate, it follows that $f_n' \to 0$ uniformly on compacts of $\mathbb{D}$, in particular on K. Using this, the fact that the quantity $\sup_{x\in(0,1]} x\ln(1/x)$ is bounded, that ε is an arbitrary positive number, by letting $n \to \infty$ in the last inequality, we obtain that $\lim_{n\to\infty}\|J_g f_n\|_{\mathfrak{L}} = 0$. Therefore, by Lemma 2.2, it follows that $J_g : \mathfrak{L} \to \mathfrak{L}$ is compact. □

Theorem 2.7. *Assume that g is an analytic function on $\mathbb{D}$. Then, $I_g : \mathfrak{L} \to \mathfrak{L}$ is compact if and only if $g = 0$.*

Proof. Assume that $g = 0$. Then, it is clear that $I_g : \mathfrak{L} \to \mathfrak{L}$ is compact.

Conversely, suppose that $I_g : \mathfrak{L} \to \mathfrak{L}$ is compact. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|z_n| \to 1$ as $n \to \infty$, and let $(f_n)_{n\in\mathbb{N}}$ be defined by

$$f_n(z) = \frac{h(\overline{z}_n z)}{\overline{z}_n}\left(\ln\frac{1}{1-|z_n|^2}\right)^{-1} - \int_0^z \ln^3\frac{1}{1-\overline{z}_n w}\,dw\left(\ln\frac{1}{1-|z_n|^2}\right)^{-2}. \tag{2.27}$$

Similar to the proof of Theorem 2.5, we see that $\sup_{n\in\mathbb{N}} \|f_n\|_{\mathcal{Z}} \le C$ and f_n converges to 0 uniformly on $\overline{\mathbb{D}}$ as $n \to \infty$. Since $I_g : \mathcal{Z} \to \mathcal{Z}$ is compact, we have

$$||I_g f_n||_{\mathcal{Z}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.28}$$

Thus

$$\begin{aligned} |z_n|\,|g(z_n)| &\le \sup_{z\in\mathbb{D}} (1-|z|^2)\,|f_n''(z)g(z) + f_n'(z)g'(z)| \\ &= \sup_{z\in\mathbb{D}} (1-|z|^2)\,|(I_g f_n)''(z)| \le ||I_g f_n||_{\mathcal{Z}} \longrightarrow 0 \end{aligned} \tag{2.29}$$

as $n \to \infty$. Hence, we obtain $\lim_{|z|\to 1} |g(z)| = 0$, which by the maximum modulus theorem implies that $g = 0$, as desired. □

3. The boundedness and compactness of $J_g, I_g : \mathcal{Z}_0 \to \mathcal{Z}_0$

In this section, we study the boundedness and compactness of the operator J_g (or I_g) : $\mathcal{Z}_0 \to \mathcal{Z}_0$. Before formulating the main results of this section, we need an auxiliary result which is incorporated in the lemma which follows.

Lemma 3.1. *A closed set K in $\mathcal{Z}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2)\,|f''(z)| = 0. \tag{3.1}$$

The proof is similar to the proof of [19, Lemma 1]. We omit the details.

Theorem 3.2. *Assume that g is an analytic function on $\mathbb{D}$. Then*
(a) $J_g : \mathcal{Z}_0 \to \mathcal{Z}_0$ *is bounded;*
(b) $J_g : \mathcal{Z}_0 \to \mathcal{Z}_0$ *is compact;*
(c) $g \in \mathcal{Z}_0$.

Proof. (b)⇒(a) is obvious.

(a)⇒(c). Assume that $J_g : \mathcal{Z}_0 \to \mathcal{Z}_0$ is bounded. Then, by taking $f(z) = 1$, we see that $g \in \mathcal{Z}_0$.

(c)⇒(b). Assume $g \in \mathcal{Z}_0$. Then, for any $f \in \mathcal{Z}_0$, by (1.4) and (1.7), we have

$$\begin{aligned} &(1-|z|^2)\,|(J_g f)''(z)| \\ &\quad= (1-|z|^2)\,|f'(z)g'(z) + f(z)g''(z)| \\ &\quad\le C\|f\|_{\mathcal{Z}} (1-|z|^2)\,|g'(z)| \ln\frac{1}{1-|z|^2} + C\|f\|_{\mathcal{Z}} (1-|z|^2)\,|g''(z)| \\ &\quad\le C\|f\|_{\mathcal{Z}} \frac{|g'(z)|}{\ln(1/(1-|z|^2))} (1-|z|^2)\left(\ln\frac{1}{1-|z|^2}\right)^2 + C\|f\|_{\mathcal{Z}} (1-|z|^2)\,|g''(z)|. \end{aligned} \tag{3.2}$$

Taking the supremum in the last inequality over the set $\{f \in H(\mathbb{D}) \mid \|f\|_{\mathcal{Z}} \le 1\}$, employing Lemmas 2.3 and 3.1, and (2.7), the compactness of the operator $J_g : \mathcal{Z}_0 \to \mathcal{Z}_0$ follows. □

Theorem 3.3. *Assume that g is an analytic function on $\mathbb{D}$. Then, $I_g : \mathcal{Z}_0 \to \mathcal{Z}_0$ is bounded if and only if $g \in H^\infty \cap \mathcal{B}_{\log}$.*

Proof. Assume that $g \in H^\infty \cap \mathcal{B}_{\log}$. Then from Theorem 2.5, $I_g : \mathcal{Z} \to \mathcal{Z}$ is bounded, and hence $I_g : \mathcal{Z}_0 \to \mathcal{Z}$ is bounded. To prove that $I_g : \mathcal{Z}_0 \to \mathcal{Z}_0$ is bounded, it is enough to show that for any $f \in \mathcal{Z}_0$, $I_g f \in \mathcal{Z}_0$. Now, for any $f \in \mathcal{Z}_0$, we have

$$\begin{aligned}
&(1-|z|^2)\,|(I_g f)''(z)| \\
&\quad = (1-|z|^2)\,|f'(z)g'(z) + f''(z)g(z)| \\
&\quad \le \left((1-|z|^2)\,|g'(z)|\ln\frac{1}{1-|z|^2}\right)|f'(z)|/\ln\frac{1}{1-|z|^2} + |g(z)|\,(1-|z|^2)\,|f''(z)| \\
&\quad \le \frac{\|g\|_{\mathcal{B}_{\log}}\,|f'(z)|}{\ln(1/(1-|z|^2))} + \|g\|_\infty (1-|z|^2)\,|f''(z)|.
\end{aligned} \tag{3.3}$$

From (3.3) and by employing Lemma 2.3, we obtain the desired result.

Conversely, assume that $I_g : \mathcal{Z}_0 \to \mathcal{Z}_0$ is bounded. Then it is clear that $I_g : \mathcal{Z}_0 \to \mathcal{Z}$ is bounded. Since the functions defined in (2.13) and (2.18) belong to $\mathcal{Z}_0$, we obtain $g \in H^\infty \cap \mathcal{B}_{\log}$. □

Theorem 3.4. *Assume that g is an analytic function on $\mathbb{D}$. Then, $I_g : \mathcal{Z}_0 \to \mathcal{Z}_0$ is compact if and only if $g = 0$.*

Proof. The sufficiency is obvious. Now we prove the necessity. From the assumption that $I_g : \mathcal{Z}_0 \to \mathcal{Z}_0$ is compact, we see that $I_g : \mathcal{Z}_0 \to \mathcal{Z}$ is compact. Since the functions in (2.27) belong to $\mathcal{Z}_0$, similar to the proof of Theorem 2.7, we obtain the desired result. □

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