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Research Article

Perturbed Iterative Algorithms for Generalized Nonlinear Set-Valued Quasivariational Inclusions Involving Generalized m-Accretive Mappings

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A new class of generalized nonlinear set-valued quasivariational inclusions involving generalized m-accretive mappings in Banach spaces are studied, which included many variational inclusions studied by others in recent years. By using the properties of the resolvent operator associated with generalized m-accretive mappings, we established the equivalence between the generalized nonlinear set-valued quasi-variational inclusions and the fixed point problems, and some new perturbed iterative algorithms, proved that its proximate solution converges strongly to its exact solution in real Banach spaces.

1. Introduction

In 1994, Hassouni and Moudafi [1] introduced and studied a class of variational inclusions and developed a perturbed algorithm for finding approximate solutions of the variational inclusions. Since then, Adly [2], Ding [3], Ding and Luo [4], Huang [5, 6], Huang et al. [7], Ahmad and Ansari [8] have obtained some important extensions of the results in various different assumptions. For more details, we refer to [1–29] and the references therein.

In 2001, Huang and Fang [16] were the first to introduce the generalized m-accretive mapping and give the definition of the resolvent operator for the generalized m-accretive mappings in Banach spaces. They also showed some properties of the resolvent operator for the generalized m-accretive mappings in Banach spaces. For further works, we refer to Huang [15], Huang et al. [19] and Huang et al. [20].

Recently, Huang and Fang [17] introduced a new class of maximal η-monotone mapping in Hilbert spaces, which is a generalization of the classical maximal monotone mapping, and studied the properties of the resolvent operator associated with the maximal

η-monotone mapping. They also introduced and studied a new class of nonlinear variational inclusions involving maximal η-monotone mapping in Hilbert spaces.

Motivated and inspired by the research work going on in this field, we introduce and study a new class of generalized nonlinear set-valued quasivariational inclusions involving generalized m-accretive mappings in Banach spaces, which include many variational inclusions studied by others in recent years. By using the properties of the resolvent operator associated with generalized m-accretive mappings, we establish the equivalence between the generalized nonlinear set-valued quasivariational inclusions and the fixed point problems, and some new perturbed iterative algorithms, prove that its proximate solution converges to its exact solution in real Banach spaces. The results presented in this paper extend and improve the corresponding results in the literature.

2. Preliminaries

Throughout this paper, we assume that X is a real Banach space equipped with norm $\|\cdot\|$, X^* is the topological dual space of X, $CB(X)$ is the family of all nonempty closed and bounded subset of X, 2^X is a power set of X, $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$D(A,B) = \max\left\{\sup_{u\in A} d(u,B), \sup_{v\in B} d(A,v)\right\} \quad \forall A,B \in CB(X), \tag{2.1}$$

where $d(u,B) = \inf_{v\in B} d(u,v)$ and $d(A,v) = \inf_{u\in A} d(u,v)$.

Suppose that $\langle\cdot,\cdot\rangle$ is the dual pair between X and X^*, $J : X \to 2^{X^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in X^* : \langle x,f\rangle = \|x\|^2, \|x\| = \|f\|\}, \quad x \in X, \tag{2.2}$$

and j is a selection of normalized duality mapping J.

Definition 2.1. A single-valued mapping $g : X \to X$ is said to be k-strongly accretive if there exists $k > 0$ such that for any $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle g(x) - g(y), j(x-y)\rangle \geq k\|x-y\|^2. \tag{2.3}$$

Definition 2.2. A single-valued mapping $N : X \times X \to X$ is said to be γ-Lipschitz continuous with respect to the first argument if there exists a constant $\gamma > 0$ such that

$$\|N(x,\cdot) - N(y,\cdot)\| \leq \gamma\|x-y\| \quad \forall x,y \in X. \tag{2.4}$$

In a similar way, we can define Lipschitz continuity of $N(\cdot,\cdot)$ with respect to the second argument.

Definition 2.3. A set-valued mapping $S : X \to 2^X$ is said to be ξ-D-Lipschitz continuous if there exists $\xi > 0$ such that

$$D(S(x),S(y)) \leq \xi\|x-y\| \quad \forall x,y \in X. \tag{2.5}$$

Definition 2.4. A mapping $\eta : X \times X \to X^*$ is said to be

(i) accretive if for any $x, y \in X$,

$$\langle x - y, \eta(x, y) \rangle \geq 0; \tag{2.6}$$

(ii) strictly accretive if for any $x, y \in X$,

$$\langle x - y, \eta(x, y) \rangle \geq 0, \tag{2.7}$$

and equality holds if and only if $x = y$;

(iii) α-strongly accretive if there exists a constant $\alpha > 0$ such that

$$\langle x - y, \eta(x, y) \rangle \geq \alpha \|x - y\|^2 \quad \forall x, y \in X; \tag{2.8}$$

(iv) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$||\eta(x, y)|| \leq \beta \|x - y\| \quad \forall x, y \in X. \tag{2.9}$$

Definition 2.5 [16]. Let $\eta : X \times X \to X^*$ be a single-valued mapping. A set-valued mapping $M : X \to 2^X$ is said to be

(i) η-accretive if for any $x, y \in X$,

$$\langle u - v, \eta(x, y) \rangle \geq 0, \quad u \in M(x),\ v \in M(y); \tag{2.10}$$

(ii) strictly η-accretive if for any $x, y \in X$,

$$\langle u - v, \eta(x, y) \rangle \geq 0, \quad u \in M(x),\ v \in M(y), \tag{2.11}$$

and equality holds if and only if $x = y$;

(iii) μ-strongly η-accretive if there exists a constant $\mu > 0$ such that

$$\langle u - v, \eta(x, y) \rangle \geq \mu \|x - y\|^2 \quad \forall x, y \in X,\ u \in M(x),\ v \in M(y); \tag{2.12}$$

(iv) generalized m-accretive if M is η-accretive and $(I + \rho M)(X) = X$ for any $\rho > 0$, where I is the identity mapping.

Remark 2.6. If X is a smooth Banach space, $\eta(x, y) = J(x - y)$ for all x, y in X, then Definition 2.5 reduces to the usual definitions of accretiveness of the set-valued mapping M in smooth Banach spaces.

Lemma 2.7 [30]. *Let X be a real Banach space and let $J : X \to 2^{X^*}$ be the normalized duality mapping. Then for any $x, y \in X$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \quad \forall j(x + y) \in J(x + y). \tag{2.13}$$

Lemma 2.8 [16]. *Let $\eta : X \times X \to X$ be a strictly accretive mapping and let $M : X \to 2^X$ be a generalized m-accretive mapping. Then the following conclusions hold:*

(1) $\langle x-y, \eta(u,v)\rangle \geq 0$ $\forall (y,v) \in graph(M)$ *implies* $(x,u) \in graph(M)$, *where* $graph(M) = \{(x,u) \in X \times X : x \in M(u)\}$;

(2) *the mapping* $(I+\rho M)^{-1}$ *is single-valued for any* $\rho > 0$.

Based on Lemma 2.8, we can define the resolvent operator for a generalized m-accretive mapping M as follows:

$$J_\rho^M(z) = (I+\rho M)^{-1}(z) \quad \forall z \in X, \tag{2.14}$$

where $\rho > 0$ is a constant and $\eta : X \times X \to X^*$ is a strictly accretive mapping.

Lemma 2.9 [16]. *Let $\eta : X \times X \to X^*$ be a δ-strongly accretive and τ-Lipschitz continuous mapping. Let $M : X \to 2^X$ be a generalized m-accretive mapping. Then the resolvent operator J_ρ^M for M is τ/δ-Lipschitz continuous, that is,*

$$\left\|J_\rho^M(u) - J_\rho^M(v)\right\| \leq \frac{\tau}{\delta}\|u-v\| \quad \forall u,v \in X. \tag{2.15}$$

3. Variational inclusions

In this section, by using the resolvent operator for the generalized m-accretive mapping and the results obtained in Section 2, we introduce and study a new class of generalized nonlinear set-valued quasivariational inclusion problem involving generalized m-accretive mappings, and prove that its proximate solution converges strongly to its exact solution in real Banach spaces.

Let $S,T,G : X \to CB(X)$ and $M(\cdot,\cdot) : X \times X \to 2^X$ be set-valued mappings such that for any given $t \in X, M(t,\cdot) : X \to 2^X$ is a generalized m-accretive mapping. Let $g : X \to X$ and $N(\cdot,\cdot) : X \times X \to X$ be nonlinear mappings. For any $f \in X$, we consider the following problem.

Find $x \in X$, $w \in S(x)$, $y \in T(x)$, $z \in G(x)$ such that

$$f \in N(w,y) + M(z,g(x)), \tag{3.1}$$

which is called the generalized nonlinear set-valued quasivariational inclusion problem involving generalized m-accretive mappings.

Some special cases of problem (3.1) are as follows.

(I) If $S,T,G : X \to X$ is a single-valued mapping, then problem (3.1) reduced to finding $x \in X$ such that

$$f \in N(S(x),T(x)) + M(G(x),g(x)), \tag{3.2}$$

which is called the nonlinear quasivariational inclusion problem.

(II) If $X = H$ is a Hilbert space and $\eta(u,v) = u-v$, then problem (3.1) becomes the usual nonlinear quasivariational inclusion with a maximal monotone mapping M.

Remark 3.1. For a suitable choice of S, T, G, N, M, g, f, and the space X, a number of known classes of variational inequalities (inclusion) and quasivariational inequalities (inclusion) can be obtained as special cases of generalized nonlinear set-valued quasivariational inclusion (3.1).

LEMMA 3.2. *Problem (3.1) has a solution (x,w,y,z), where $x \in X, w \in S(x)$, $y \in T(x)$, $z \in G(x)$ if and only if (p,x,w,y,z), where $p \in X$, is a solution of implicit resolvent equation*

$$p = g(x) - \rho(N(w,y) - f), \quad g(x) = J_\rho^{M(z,\cdot)}(p), \tag{3.3}$$

where $J_\rho^{M(z,\cdot)} = (I + \rho M(z,\cdot))^{-1}$ and $\rho > 0$ is a constant.

Proof. This directly follows from the definition of $J_\rho^{M(z,\cdot)}$. □

Now Lemma 3.2 and Nadler's theorem [31] allow us to suggest the following iterative algorithm.

ALGORITHM 3.3. *Assume that $S,T,G : X \to CB(X)$, and $M(\cdot,\cdot) : X \times X \to 2^X$ are set-valued mappings such that for any given $t \in X$, $M(t,\cdot) : X \to 2^X$ is a generalized m-accretive mapping and $g : X \to X$ is a strongly accretive and Lipschitz continuous mapping. Let $N(\cdot,\cdot) : X \times X \to X$ be a nonlinear mapping. For any $f \in X$ and for given $p_0 \in X$, $x_0 \in X$ and $w_0 \in S(x_0)$, $y_0 \in T(x_0)$, $z_0 \in G(x_0)$, compute the sequences $\{p_n\}$, $\{x_n\}$, $\{w_n\}$, $\{y_n\}$, and $\{z_n\}$ defined by the iterative schemes*

$$\begin{aligned}
&g(x_n) = J_\rho^{M(z_n,\cdot)} p_n,\\
&w_n \in S(x_n), \quad \|w_n - w_{n+1}\| \le (1 + (1+n)^{-1}) D(S(x_n), S(x_{n+1})),\\
&y_n \in T(x_n), \quad \|y_n - y_{n+1}\| \le (1 + (1+n)^{-1}) D(T(x_n), T(x_{n+1})), \quad n = 0,1,2,\ldots,\\
&z_n \in G(x_n), \quad \|z_n - z_{n+1}\| \le (1 + (1+n)^{-1}) D(G(x_n), G(x_{n+1})),\\
&p_{n+1} = (1-\lambda) p_n + \lambda (g(x_n) - \rho N(w_n, y_n) + \rho f) + \lambda e_n,
\end{aligned} \tag{3.4}$$

where $0 < \lambda \le 1$ is a constant and $e_n \in X$ is the errors while considering the approximation in computation.

If $S,T,G : X \to X$ are single-valued mappings, then Algorithm 3.3 can be degenerated to the following algorithm for problem (3.2).

ALGORITHM 3.4. *For any $f \in X$ and for given $p_0 \in X$, $x_0 \in X$, we can obtain sequences $\{p_n\}$, $\{x_n\}$ satisfying*

$$\begin{aligned}
&g(x_n) = J_\rho^{M(G(x_n),\cdot)} p_n,\\
&p_{n+1} = (1-\lambda) p_n + \lambda (g(x_n) - \rho N(S(x_n), T(x_n)) + \rho f) + \lambda e_n,
\end{aligned} \quad n = 0,1,2,\ldots, \tag{3.5}$$

where $0 < \lambda \le 1$ is a constant and $e_n \in X$ is the errors while considering the approximation in computation.

Remark 3.5. If we choose suitable S, T, G, N, M, g, and the space X, then Algorithm 3.3 can be degenerated to a number of algorithm for solving variational inequalities (inclusions).

Theorem 3.6. *Let X be a real Banach space. Let $\eta : X \times X \to X^*$ be δ-strongly accretive and τ-Lipschitz continuous, let $S,T,G : X \to CB(X)$ be α, β and γ-D-Lipschitz continuous, respectively, let $g : X \to X$ be k-strongly accretive and ξ-Lipschitz continuous. Let $N(\cdot,\cdot) : X \times X \to X$ be r,t-Lipschitz continuous with respect to the first and second arguments, respectively. Let $M : X \times X \to 2^X$ be such that for each fixed $t \in X$, $M(t,\cdot)$ is a generalized m-accretive mapping. Suppose that there exist constants $\rho > 0$ and $\mu > 0$ such that for each $x,y,z \in X$,*

$$||J_\rho^{M(x,\cdot)}(z) - J_\rho^{M(y,\cdot)}(z)|| \leq \mu\|x-y\|, \tag{3.6}$$

$$\rho < \frac{\sqrt{k+1.5-\mu^2\gamma^2} - b\xi}{b(r\alpha + t\beta)}, \quad b\xi < \sqrt{k+1.5-\mu^2\gamma^2}, \quad b = \frac{\tau}{\delta}, \tag{3.7}$$

$$\lim_{n\to\infty} ||e_n|| = 0, \qquad \sum_{n=0}^{\infty} ||e_{n+1} - e_n|| < \infty. \tag{3.8}$$

Then there exist $p,x \in X$, $w \in S(x)$, $y \in T(x)$, $z \in G(x)$ satisfy the implicit resolvent equation (3.3) and the iterative sequences $\{p_n\}$, $\{x_n\}$, $\{w_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by Algorithm 3.3 converge strongly to p, x, w, y, and z in X, respectively.

Proof. From condition (3.6), Lemma 2.9, and γ-Lipschitz continuity of G, we have

$$\begin{aligned}
&||J_\rho^{M(z_{n+1},\cdot)}p_{n+1} - J_\rho^{M(z_n,\cdot)}p_n|| \\
&\quad\leq ||J_\rho^{M(z_{n+1},\cdot)}p_{n+1} - J_\rho^{M(z_n,\cdot)}p_{n+1}|| + ||J_\rho^{M(z_n,\cdot)}p_{n+1} - J_\rho^{M(z_n,\cdot)}p_n|| \\
&\quad\leq \mu||z_{n+1} - z_n|| + \frac{\tau}{\delta}||p_{n+1} - p_n|| \\
&\quad\leq \mu\gamma\left(1+\frac{1}{n}\right)||x_{n+1} - x_n|| + \frac{\tau}{\delta}||p_{n+1} - p_n||.
\end{aligned} \tag{3.9}$$

Since g is k-strongly accretive mapping, from Algorithm 3.3, Lemma 2.7, and (3.9), for any $j(x_{n+1}-x_n) \in J(x_{n+1}-x_n)$, we have

$$\begin{aligned}
||x_{n+1} - x_n||^2 &= ||x_{n+1} - x_n + (g(x_{n+1}) - g(x_n)) - (J_\rho^{M(z_{n+1},\cdot)}p_{n+1} - J_\rho^{M(z_n,\cdot)}p_n)||^2 \\
&\leq ||J_\rho^{M(z_{n+1},\cdot)}p_{n+1} - J_\rho^{M(z_n,\cdot)}p_n||^2 - 2\langle g(x_{n+1}) - g(x_n) + x_{n+1} - x_n, j(x_{n+1} - x_n)\rangle \\
&\leq \left(\mu\gamma\left(1+\frac{1}{n}\right)||x_{n+1} - x_n|| + \frac{\tau}{\delta}||p_{n+1} - p_n||\right)^2 \\
&\quad - 2\langle g(x_{n+1}) - g(x_n), j(x_{n+1} - x_n)\rangle - 2\langle x_{n+1} - x_n, j(x_{n+1} - x_n)\rangle \\
&\leq \left(2\mu^2\gamma^2\left(1+\frac{1}{n}\right)^2 - 2k - 2\right)||x_{n+1} - x_n||^2 + 2\frac{\tau^2}{\delta^2}||p_{n+1} - p_n||^2,
\end{aligned} \tag{3.10}$$

which implies

$$\|x_{n+1}-x_n\| \le \frac{b}{\sqrt{k+1.5-\mu^2\gamma^2(1+(1/n))^2}}\|p_{n+1}-p_n\|, \tag{3.11}$$

where $b=\tau/\delta$.

Since N is r,t-Lipschitz continuous with respect to the first, second arguments, respectively, S, T are α,β-Lipschitz continuous, respectively, and g is ξ-Lipschitz continuous, by (3.4), we obtain

$$\begin{aligned}
\|p_{n+2}-p_{n+1}\| &= \|(1-\lambda)p_{n+1}+\lambda[g(x_{n+1})-\rho N(w_{n+1},y_{n+1})+\rho f] \\
&\qquad +\lambda e_{n+1}-\{(1-\lambda)p_n+\lambda[g(x_n)-\rho N(w_n,y_n)+\rho f]+\lambda e_n\}\| \\
&\le (1-\lambda)\|p_{n+1}-p_n\|+\lambda\|g(x_{n+1})-g(x_n)\| \\
&\quad +\lambda\rho(\|N(w_{n+1},y_{n+1})-N(w_n,y_{n+1})\|+\|N(w_n,y_{n+1})-N(w_n,y_n)\|) \\
&\quad +\lambda\|e_{n+1}-e_n\| \le (1-\lambda)\|p_{n+1}-p_n\| \\
&\quad +\lambda\left[\xi+\rho\left(1+\frac{1}{n}\right)(r\alpha+t\beta)\right]\|x_{n+1}-x_n\|+\lambda\|e_{n+1}-e_n\|.
\end{aligned} \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\begin{aligned}
\|p_{n+2}-p_{n+1}\| &\le \left(1-\lambda+\frac{\lambda b[\xi+\rho(1+(1/n))(r\alpha+t\beta)]}{\sqrt{k+1.5-\mu^2\gamma^2(1+(1/n))^2}}\right)\|p_{n+1}-p_n\|+\lambda\|e_{n+1}-e_n\| \\
&= [1-\lambda(1-h_n)]\|p_{n+1}-p_n\|+\lambda\|e_{n+1}-e_n\| \\
&= \theta_n\|p_{n+1}-p_n\|+\lambda\|e_{n+1}-e_n\|,
\end{aligned} \tag{3.13}$$

where

$$\theta_n = 1-\lambda(1-h_n), \quad h_n = \frac{b[\xi+\rho(1+(1/n))(r\alpha+t\beta)]}{\sqrt{k+1.5-\mu^2\gamma^2(1+(1/n))^2}}. \tag{3.14}$$

Letting

$$\theta = 1-\lambda(1-h), \quad h = \frac{b[\xi+\rho(r\alpha+t\beta)]}{\sqrt{k+1.5-\mu^2\gamma^2}}, \tag{3.15}$$

we know that $h_n \to h$ and $\theta_n \to \theta$ as $n \to \infty$. It follows from (3.7) and $0<\lambda\le 1$ that $0<h<1$ and $0<\theta<1$, and so there exists a positive number $\theta_* \in (0,1)$ such that $\theta_n < \theta_*$ for all $n \geq N$. Therefore, for all $n \geq N$, by (3.13), we now know that

$$\begin{aligned}
||p_{n+2} - p_{n+1}|| &\leq \theta_* ||p_{n+1} - p_n|| + \lambda ||e_{n+1} - e_n|| \\
&\leq \theta_* (\theta_* ||p_n - p_{n-1}|| + \lambda ||e_n - e_{n-1}||) + \lambda ||e_{n+1} - e_n|| \\
&= \theta_*^2 ||p_n - p_{n-1}|| + \lambda \theta_* ||e_n - e_{n-1}|| + \lambda ||e_{n+1} - e_n|| \\
&\leq \cdots \leq \theta_*^{n+1-N} ||p_{N+1} - p_N|| + \sum_{i=1}^{n+1-N} \theta_*^{i-1} \lambda ||e_{n+1-(i-1)} - e_{n+1-i}||,
\end{aligned} \tag{3.16}$$

which implies that for any $m > n > N$, we have

$$\begin{aligned}
||p_m - p_n|| &\leq \sum_{j=n}^{m-1} ||p_{j+1} - p_j|| \\
&\leq \sum_{j=n}^{m-1} \theta_*^{j+1-N} ||p_{N+1} - p_N|| + \sum_{j=n}^{m-1} \sum_{i=1}^{j+1-N} \theta_*^{i-1} \lambda ||e_{n+1-(i-1)} - e_{n+1-i}||.
\end{aligned} \tag{3.17}$$

Since $0 < \lambda \leq 1$ and $\theta_* \in (0,1)$, it follows from (3.8) and (3.17) that $\lim_{m,n\to\infty} \|p_m - p_n\| = 0$, and hence $\{p_n\}$ is a Cauchy sequence in X. Let $p_n \to p$ as $n \to \infty$. From (3.11), we know that sequence $\{x_n\}$ is also a Cauchy sequence in X. Let $x_n \to x$ as $n \to \infty$.

On the other hand, from the Lipschitzian continuity of S, T, G, and Algorithm 3.3, we have

$$\begin{aligned}
||w_n - w_{n+1}|| &\leq \left(1 + \frac{1}{n+1}\right) D(S(x_n), S(x_{n+1})) \leq \left(1 + \frac{1}{n+1}\right) \alpha ||x_n - x_{n+1}||, \\
||y_n - y_{n+1}|| &\leq \left(1 + \frac{1}{n+1}\right) D(T(x_n), T(x_{n+1})) \leq \left(1 + \frac{1}{n+1}\right) \beta ||x_n - x_{n+1}||, \\
||z_n - z_{n+1}|| &\leq \left(1 + \frac{1}{n+1}\right) D(G(x_n), G(x_{n+1})) \leq \left(1 + \frac{1}{n+1}\right) \gamma ||x_n - x_{n+1}||.
\end{aligned} \tag{3.18}$$

Since $\{x_n\}$ is a Cauchy sequence, from (3.18), we know that $\{w_n\}$, $\{y_n\}$, and $\{z_n\}$ are also Cauchy sequences. Let $w_n \to w$, $y_n \to y$, and $z_n \to z$ as $n \to \infty$. From Algorithm 3.3,

$$p_{n+1} = (1 - \lambda) p_n + \lambda (g(x_n) - \rho N(w_n, y_n) + \rho f) + \lambda e_n. \tag{3.19}$$

By the assumptions and $\lim_{n\to\infty} \|e_n\| = 0$, we have

$$\begin{gathered}
p = g(x) - \rho (N(w, y) - f), \\
g(x_n) = J_\rho^{M(z_n,\cdot)} p_n \Longrightarrow g(x) = J_\rho^{M(z,\cdot)} p.
\end{gathered} \tag{3.20}$$

From (3.20), we have p, x, w, y, z satisfy the implicit resolvent equation (3.3).

Now we will prove that $w \in S(x)$, $y \in T(x)$, and $z \in G(x)$. In fact, since $w_n \in S(x_n)$ and

$$\begin{aligned} d(w_n, S(x)) &\le \max\left\{ d(w_n, S(x)), \sup_{v\in S(x)} d(S(x_n), v)\right\} \\ &\le \max\left\{ \sup_{u\in S(x_n)} (u, S(x)), \sup_{v\in S(x)} d(S(x_n), v)\right\} \\ &= D(S(x_n), S(x)), \end{aligned} \tag{3.21}$$

we have

$$\begin{aligned} d(w, S(x)) &\le \|w - w_n\| + d(w_n, S(x)) \le \|w - w_n\| + D(S(x_n), S(x)) \\ &\le \|w - w_n\| + \gamma\|x_n - x\| \longrightarrow 0. \end{aligned} \tag{3.22}$$

This implies that $w \in S(x)$. Similarly, we know that $y \in T(x)$ and $z \in G(x)$. This completes the proof. □

If $S, T, G : X \to X$ are single-valued mappings, then Theorem 3.6 can be degenerated to the following theorem.

Theorem 3.7. *Let X, g, η, $N(\cdot,\cdot)$, $M(\cdot,\cdot)$ be the same as in Theorem 3.6, and let $S, T, G : X \to X$ be α, β, γ-Lipschitz continuous single-valued mappings, respectively. If conditions (3.6)–(3.8) hold, then the sequences $\{x_n\}$ generated by Algorithm 3.4 converges strongly to the unique solution x of problem (3.2).*

Proof. By Theorem 3.6, problem (3.2) has a solution $x \in X$ and $x_n \to x$ as $n \to \infty$. Now we prove that x is a unique solution of problem (3.2). Let $x^* \in X$ be another solution of problem (3.2). Then

$$g(x^*) = J_\rho^{M(G(x^*),\cdot)} m(x^*), \quad m(x^*) = g(x^*) - \rho(N(S(x^*), T(x^*)) - f). \tag{3.23}$$

We have

$$\begin{aligned} \|x - x^*\|^2 &= \|x - x^* + (g(x) - g(x^*)) - (J_\rho^{M(G(x),\cdot)} m(x) - J_\rho^{M(G(x^*),\cdot)} m(x^*))\|^2 \\ &\le \|J_\rho^{M(G(x),\cdot)} m(x) - J_\rho^{M(G(x^*),\cdot)} m(x^*)\|^2 - 2\langle g(x) - g(x^*) + x - x^*, j(x - x^*)\rangle \\ &\le (\|J_\rho^{M(G(x),\cdot)} m(x) - J_\rho^{M(G(x^*),\cdot)} m(x)\| + \|J_\rho^{M(G(x^*),\cdot)} m(x) - J_\rho^{M(G(x^*),\cdot)} m(x^*)\|)^2 \\ &\quad - 2\langle g(x) - g(x^*), j(x - x^*)\rangle - 2\langle x - x^*, j(x - x^*)\rangle \\ &\le \left(\mu\|G(x) - G(x^*)\| + \frac{\tau}{\delta}\|m(x) - m(x^*)\|\right)^2 - 2(k+1)\|x - x^*\|^2 \\ &\le 2(\mu^2\gamma^2 - k - 1)\|x - x^*\|^2 + 2\frac{\tau^2}{\delta^2}\|m_(x) - m(x^*)\|^2. \end{aligned} \tag{3.24}$$

This implies that

$$||x - x^*|| \leq \frac{b}{\sqrt{k + 1.5 - \mu^2\gamma^2}} ||m(x) - m(x^*)||, \tag{3.25}$$

where $b = \tau/\delta$. Furthermore,

$$\begin{aligned} ||m(x) - m(x^*)|| &= ||g(x) - g(x^*) - \rho(N(S(x), T(x)) - N(S(x^*), T(x^*)))|| \\ &\leq ||g(x) - g(x^*)|| + \rho\Big(||N(S(x), T(x)) - N(S(x^*), T(x))|| \\ &\qquad + ||N(S(x^*), T(x)) - N(S(x^*), T(x^*))||\Big) \\ &\leq [\xi + \rho(r\alpha + t\beta)]||x - x^*||. \end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26), we have

$$||x - x^*|| \leq \frac{b[\xi + \rho(r\alpha + t\beta)]}{\sqrt{k + 1.5 - \mu^2\gamma^2}} ||x - x^*|| = h||x - x^*||, \tag{3.27}$$

where

$$h = \frac{b[\xi + \rho(r\alpha + t\beta)]}{\sqrt{k + 1.5 - \mu^2\gamma^2}}. \tag{3.28}$$

It follows from (3.7) that $0 < h < 1$ and so $x = x^*$. This completes the proof. □

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