

Research Article

A Multidimensional Functional Equation Having Quadratic Forms as Solutions

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We obtain the general solution and the stability of the m -variable quadratic functional equation $f(x_1 + y_1, \dots, x_m + y_m) + f(x_1 - y_1, \dots, x_m - y_m) = 2f(x_1, \dots, x_m) + 2f(y_1, \dots, y_m)$. The quadratic form $f(x_1, \dots, x_m) = \sum_{1 \leq i \leq j \leq m} a_{ij} x_i x_j$ is a solution of the given functional equation.

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1. Introduction

In this paper, let X and Y be real vector spaces. A mapping f is called a *quadratic form* if there exist $a_{ij} \in \mathbb{R}$ ($1 \leq i \leq j \leq m$) such that

$$f(x_1, \dots, x_m) = \sum_{1 \leq i \leq j \leq m} a_{ij} x_i x_j \quad (1.1)$$

for all $x_1, \dots, x_m \in X$.

For a mapping $f : X^m \rightarrow Y$, consider the m -variable quadratic functional equation

$$f(x_1 + y_1, \dots, x_m + y_m) + f(x_1 - y_1, \dots, x_m - y_m) = 2f(x_1, \dots, x_m) + 2f(y_1, \dots, y_m). \quad (1.2)$$

When $X = Y = \mathbb{R}$, the quadratic form $f : \mathbb{R}^m \rightarrow \mathbb{R}$ given by

$$f(x_1, \dots, x_m) = \sum_{1 \leq i \leq j \leq m} a_{ij} x_i x_j \quad (1.3)$$

is a solution of (1.2).

For a mapping $g : X \rightarrow Y$, consider the quadratic functional equation

$$g(x + y) + g(x - y) = 2g(x) + 2g(y). \tag{1.4}$$

In 1989, Aczél [1] proposed the solution of (1.4). Later, many different quadratic functional equations were solved by numerous authors [2–6].

In this paper, we investigate the relation between (1.2) and (1.4). And we find out the general solution and the generalized Hyers-Ulam stability of (1.2).

2. Results

The m -variable quadratic functional equation (1.2) induces the quadratic functional equation (1.4) as follows.

THEOREM 2.1. *Let $f : X^m \rightarrow Y$ be a mapping satisfying (1.2) and let $g : X \rightarrow Y$ be the mapping given by*

$$g(x) := f(x, \dots, x) \tag{2.1}$$

for all $x \in X$, then g satisfies (1.4).

Proof. By (1.2) and (2.1),

$$\begin{aligned} g(x + y) + g(x - y) &= f(x + y, \dots, x + y) + f(x - y, \dots, x - y) \\ &= 2f(x, \dots, x) + 2f(y, \dots, y) = 2g(x) + 2g(y) \end{aligned} \tag{2.2}$$

for all $x, y \in X$. □

The quadratic functional equation (1.4) induces the m -variable quadratic functional equation (1.2) with an additional condition.

THEOREM 2.2. *Let $a_{ij} \in \mathbb{R}$ ($1 \leq i \leq j \leq m$) and $g : X \rightarrow Y$ be a mapping satisfying (1.4). If $f : X^m \rightarrow Y$ is the mapping given by*

$$f(x_1, \dots, x_m) := \sum_{i=1}^m a_{ii}g(x_i) + \frac{1}{4} \sum_{1 \leq i < j \leq m} a_{ij}[g(x_i + x_j) - g(x_i - x_j)] \tag{2.3}$$

for all $x_1, \dots, x_m \in X$, then f satisfies (1.2). Furthermore, (2.1) holds if

$$\sum_{1 \leq i \leq j \leq m} a_{ij} = 1. \tag{2.4}$$

Proof. By (1.4) and (2.3),

$$\begin{aligned}
& f(x_1 + y_1, \dots, x_m + y_m) + f(x_1 - y_1, \dots, x_m - y_m) \\
&= \sum_{i=1}^m a_{ii} [g(x_i + y_i) + g(x_i - y_i)] \\
&\quad + \frac{1}{4} \sum_{1 \leq i < j \leq m} a_{ij} [g(x_i + y_i + x_j + y_j) - g(x_i + y_i - x_j - y_j)] \\
&\quad + \frac{1}{4} \sum_{1 \leq i < j \leq m} a_{ij} [g(x_i - y_i + x_j - y_j) - g(x_i - y_i - x_j + y_j)] \\
&= 2 \sum_{i=1}^m a_{ii} [g(x_i) + g(y_i)] \\
&\quad + \frac{1}{4} \sum_{1 \leq i < j \leq m} a_{ij} [g(x_i + y_i + x_j + y_j) + g(x_i - y_i + x_j - y_j)] \\
&\quad - \frac{1}{4} \sum_{1 \leq i < j \leq m} a_{ij} [g(x_i + y_i - x_j - y_j) + g(x_i - y_i - x_j + y_j)] \\
&= 2 \sum_{i=1}^m a_{ii} [g(x_i) + g(y_i)] \\
&\quad + \frac{1}{2} \sum_{1 \leq i < j \leq m} a_{ij} [g(x_i + x_j) + g(y_i + y_j) - g(x_i - x_j) - g(y_i - y_j)] \\
&= 2f(x_1, \dots, x_m) + 2f(y_1, \dots, y_m)
\end{aligned} \tag{2.5}$$

for all $x_1, \dots, x_m, y_1, \dots, y_m \in X$.

Letting $x = y = 0$ and $y = x$ in (1.4), respectively,

$$g(0) = 0, \quad g(2x) = 4g(x) \tag{2.6}$$

for all $x \in X$. By (2.3) and the above two equalities,

$$f(x, \dots, x) = \sum_{i=1}^m a_{ii} g(x) + \frac{1}{4} \sum_{1 \leq i < j \leq m} a_{ij} [g(2x) - g(0)] = \sum_{1 \leq i \leq j \leq m} a_{ij} g(x) = g(x) \tag{2.7}$$

for all $x \in X$. □

Example 2.3. The function $g : \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = x^2$ satisfies (1.4). By Theorem 2.2, the quadratic form $f : \mathbb{R}^m \rightarrow \mathbb{R}$ given by

$$f(x_1, \dots, x_m) = \sum_{1 \leq i \leq j \leq m} a_{ij} x_i x_j \tag{2.8}$$

satisfies (1.2).

Example 2.4. Let $g : \mathbb{C} \rightarrow \mathbb{C}$ be the function given by $g(z) = z\bar{z}$. Then, it satisfies the quadratic functional equation (1.4). If $f : \mathbb{C}^m \rightarrow \mathbb{C}$ is the mapping given by (2.3), that is,

$$f(z_1, \dots, z_m) = \sum_{i=1}^m z_i \bar{z}_i \left(\sum_{j=1}^i a_{ji} + \frac{1}{2} \sum_{j=i+1}^m a_{ij} \right), \tag{2.9}$$

then f satisfies the m -variable quadratic functional equation (1.2).

Example 2.5. Let $M_2(\mathbb{R})$ be the real vector space of all 2×2 real matrices and $g : M_2(\mathbb{R}) \rightarrow \mathbb{R}$ the determinant function given by

$$g(A) = \det(A) \tag{2.10}$$

for all $A \in M_2(\mathbb{R})$. Then, it satisfies (1.4). Using (2.3), $f : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \rightarrow \mathbb{R}$ is given by $f(A, B) = (a_{11} + (1/2)a_{12}) \det(A) + (a_{22} + (1/2)a_{12}) \det(B)$ ($a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$). Also, f satisfies (1.2).

In the following theorem, we find out the general solution of the m -variable quadratic functional equation (1.2).

THEOREM 2.6. *A mapping $f : X^m \rightarrow Y$ satisfies (1.2) if and only if there exist symmetric biadditive mappings $S_1, \dots, S_m : X^2 \rightarrow Y$ and biadditive mappings $M_{ij} : X^2 \rightarrow Y$ ($1 \leq i < j \leq m$) such that*

$$f(x_1, \dots, x_m) = \sum_{i=1}^m S_i(x_i, x_i) + \sum_{1 \leq i < j \leq m} M_{ij}(x_i, x_j) \tag{2.11}$$

for all $x_1, \dots, x_m \in X$.

Proof. We first assume that f is a solution of (1.2). Define $f_1, \dots, f_m : X \rightarrow Y$ by $f_1(x) := f(x, 0, \dots, 0), \dots, f_m(x) := f(0, \dots, 0, x)$ for all $x \in X$. One can easily verify that f_1, \dots, f_m are quadratic. By [1], there exist symmetric biadditive mappings $S_1, \dots, S_m : X^2 \rightarrow Y$ such that $f_1(x) = S_1(x, x), \dots, f_m(x) = S_m(x, x)$ for all $x \in X$. Define $M_{ij} : X^2 \rightarrow Y$ by

$$M_{ij}(x, y) := f(0, \dots, 0, x, 0, \dots, 0, y, 0, \dots, 0) - f(0, \dots, 0, x, 0, \dots, 0, 0, \dots, 0) - f(0, \dots, 0, 0, 0, \dots, 0, y, 0, \dots, 0) \tag{2.12}$$

for all i, j with $1 \leq i < j \leq m$ and all $x, y \in X$. On the right-hand side of (2.12), x and y are the i th and the j th components, respectively. Then, M_{ij} are biadditive for all i, j with

$1 \leq i < j \leq m$. Indeed, by (1.2) and (2.12), we obtain

$$\begin{aligned}
& M_{ij}(x_1 + x_2, y) \\
&= f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, y, 0, \dots, 0) - f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, 0, 0, \dots, 0) \\
&\quad - f(0, \dots, 0, 0, 0, \dots, 0, y, 0, \dots, 0) \\
&= f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, y, 0, \dots, 0) \\
&\quad - \frac{1}{2}[f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, y, 0, \dots, 0) + f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, -y, 0, \dots, 0)] \\
&= \frac{1}{2}[f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, y, 0, \dots, 0) - f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, -y, 0, \dots, 0)] \\
&= f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, y, 0, \dots, 0) \\
&\quad - \frac{1}{2}[f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, y, 0, \dots, 0) + f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, -y, 0, \dots, 0)] \\
&= f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, y, 0, \dots, 0) - f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, 0, 0, \dots, 0) \\
&\quad - f(0, \dots, 0, 0, 0, \dots, 0, y, 0, \dots, 0) \\
&= \frac{1}{2}[2f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, y, 0, \dots, 0) + 2f(0, \dots, 0, 0, 0, \dots, 0, y, 0, \dots, 0) \\
&\quad - 2f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, 0, 0, \dots, 0)] - 2f(0, \dots, 0, 0, 0, \dots, 0, y, 0, \dots, 0) \\
&= \frac{1}{2}[f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, 2y, 0, \dots, 0) - f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, 0, 0, \dots, 0)] \\
&\quad - 2f(0, \dots, 0, 0, 0, \dots, 0, y, 0, \dots, 0) \\
&= \frac{1}{2}[f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, 2y, 0, \dots, 0) + f(0, \dots, 0, x_1 - x_2, 0, \dots, 0, 0, 0, \dots, 0)] \\
&\quad - \frac{1}{2}[f(0, \dots, 0, x_1 + x_2, 0, \dots, 0, 0, 0, \dots, 0) + f(0, \dots, 0, x_1 - x_2, 0, \dots, 0, 0, 0, \dots, 0)] \\
&\quad - 2f(0, \dots, 0, 0, 0, \dots, 0, y, 0, \dots, 0) \\
&= f(0, \dots, 0, x_1, 0, \dots, 0, y, 0, \dots, 0) + f(0, \dots, 0, x_2, 0, \dots, 0, y, 0, \dots, 0) \\
&\quad - f(0, \dots, 0, x_1, 0, \dots, 0, 0, 0, \dots, 0) - f(0, \dots, 0, x_2, 0, \dots, 0, 0, 0, \dots, 0) \\
&\quad - 2f(0, \dots, 0, 0, 0, \dots, 0, y, 0, \dots, 0) \\
&= f(0, \dots, 0, x_1, 0, \dots, 0, y, 0, \dots, 0) \\
&\quad - [f(0, \dots, 0, x_1, 0, \dots, 0, 0, 0, \dots, 0) + f(0, \dots, 0, 0, 0, \dots, 0, y, 0, \dots, 0)] \\
&\quad + f(0, \dots, 0, x_2, 0, \dots, 0, y, 0, \dots, 0) \\
&\quad - [f(0, \dots, 0, x_2, 0, \dots, 0, 0, 0, \dots, 0) + f(0, \dots, 0, 0, 0, \dots, 0, y, 0, \dots, 0)] \\
&= M_{ij}(0, \dots, 0, x_1, 0, \dots, 0, y, 0, \dots, 0) + M_{ij}(0, \dots, 0, x_2, 0, \dots, 0, y, 0, \dots, 0)
\end{aligned}$$

(2.13)

for all $x_1, x_2, y \in X$. Similarly,

$$\begin{aligned}
 &M_{ij}(0, \dots, 0, x, 0, \dots, 0, y_1 + y_2, 0, \dots, 0) \\
 &= M_{ij}(0, \dots, 0, x, 0, \dots, 0, y_1, 0, \dots, 0) + M_{ij}(0, \dots, 0, x, 0, \dots, 0, y_2, 0, \dots, 0)
 \end{aligned} \tag{2.14}$$

for all $x, y_1, y_2 \in X$.

Conversely, we assume that there exist symmetric biadditive mappings $S_1, \dots, S_m: X^2 \rightarrow Y$ and biadditive mappings $M_{ij}: X^2 \rightarrow Y$ ($1 \leq i < j \leq m$) such that

$$f(x_1, \dots, x_m) = \sum_{i=1}^m S_i(x_i, x_i) + \sum_{1 \leq i < j \leq m} M_{ij}(x_i, x_j) \tag{2.15}$$

for all $x_1, \dots, x_m \in X$. Since M_{ij} ($1 \leq i < j \leq m$) are biadditive and S_1, \dots, S_m are symmetric biadditive,

$$\begin{aligned}
 &f(x_1 + y_1, \dots, x_m + y_m) + f(x_1 - y_1, \dots, x_m - y_m) \\
 &= \sum_{i=1}^m S_i(x_i + y_i, x_i + y_i) + \sum_{1 \leq i < j \leq m} M_{ij}(x_i + y_i, x_j + y_j) \\
 &\quad + \sum_{i=1}^m S_i(x_i - y_i, x_i - y_i) + \sum_{1 \leq i < j \leq m} M_{ij}(x_i - y_i, x_j - y_j) \\
 &= \sum_{i=1}^m [S_i(x_i, x_i) + 2S_i(x_i, y_i) + S_i(y_i, y_i)] \\
 &\quad + \sum_{1 \leq i < j \leq m} [M_{ij}(x_i, x_j) + M_{ij}(x_i, y_j) + M_{ij}(y_i, x_j) + M_{ij}(y_i, y_j)] \\
 &\quad + \sum_{i=1}^m [S_i(x_i, x_i) - 2S_i(x_i, y_i) + S_i(y_i, y_i)] \\
 &\quad + \sum_{1 \leq i < j \leq m} [M_{ij}(x_i, x_j) - M_{ij}(x_i, y_j) - M_{ij}(y_i, x_j) + M_{ij}(y_i, y_j)] \\
 &= 2 \left[\sum_{i=1}^m S_i(x_i, x_i) + \sum_{1 \leq i < j \leq m} M_{ij}(x_i, x_j) \right] \\
 &\quad + 2 \left[\sum_{i=1}^m S_i(y_i, y_i) + \sum_{1 \leq i < j \leq m} M_{ij}(y_i, y_j) \right] \\
 &= 2f(x_1, \dots, x_m) + 2f(y_1, \dots, y_m)
 \end{aligned} \tag{2.16}$$

for all $x_1, \dots, x_m, y_1, \dots, y_m \in X$. □

Let Y be complete and let $\varphi: X^{2m} \rightarrow [0, \infty)$ be a function satisfying

$$\tilde{\varphi}(x_1, \dots, x_m, y_1, \dots, y_m) := \sum_{j=0}^{\infty} \frac{1}{4^{j+1}} \varphi(2^j x_1, \dots, 2^j x_m, 2^j y_1, \dots, 2^j y_m) < \infty \tag{2.17}$$

for all $x_1, \dots, x_m, y_1, \dots, y_m \in X$.

THEOREM 2.7. Let $f : X^m \rightarrow Y$ be a mapping such that

$$\begin{aligned} & \|f(x_1 + y_1, \dots, x_m + y_m) + f(x_1 - y_1, \dots, x_m - y_m) \\ & - 2f(x_1, \dots, x_m) - 2f(y_1, \dots, y_m)\| \leq \varphi(x_1, \dots, x_m, y_1, \dots, y_m) \end{aligned} \quad (2.18)$$

for all $x_1, \dots, x_m, y_1, \dots, y_m \in X$. Then, there exists a unique m -variable quadratic mapping $F : X^m \rightarrow Y$ such that

$$\|f(x_1, \dots, x_m) - F(x_1, \dots, x_m)\| \leq \tilde{\varphi}(x_1, \dots, x_m, x_1, \dots, x_m) \quad (2.19)$$

for all $x_1, \dots, x_m \in X$. The mapping F is given by

$$F(x_1, \dots, x_m) := \lim_{j \rightarrow \infty} \frac{1}{4^j} f(2^j x_1, \dots, 2^j x_m) \quad (2.20)$$

for all $x_1, \dots, x_m \in X$.

Proof. Letting $y_1 = x_1, \dots, y_m = x_m$ in (2.18), we have

$$\left\| f(x_1, \dots, x_m) - \frac{1}{4} [f(0, \dots, 0) + f(2x_1, \dots, 2x_m)] \right\| \leq \frac{1}{4} \varphi(x_1, \dots, x_m, x_1, \dots, x_m) \quad (2.21)$$

for all $x_1, \dots, x_m \in X$. Thus, we obtain

$$\begin{aligned} & \left\| \frac{1}{4^j} f(2^j x_1, \dots, 2^j x_m) - \frac{1}{4^{j+1}} [f(0, \dots, 0) + f(2^{j+1} x_1, \dots, 2^{j+1} x_m)] \right\| \\ & \leq \frac{1}{4^{j+1}} \varphi(2^j x_1, \dots, 2^j x_m, 2^j x_1, \dots, 2^j x_m) \end{aligned} \quad (2.22)$$

for all $x_1, \dots, x_m \in X$ and all j . For given integers l, n ($0 \leq l < n$), we get

$$\begin{aligned} & \left\| \frac{1}{4^l} f(2^l x_1, \dots, 2^l x_m) - \frac{1}{4^n} [f(0, \dots, 0) + f(2^n x_1, \dots, 2^n x_m)] \right\| \\ & \sum_{j=l}^{n-1} \frac{1}{4^{j+1}} \varphi(2^j x_1, \dots, 2^j x_m, 2^j x_1, \dots, 2^j x_m) \end{aligned} \quad (2.23)$$

for all $x_1, \dots, x_m \in X$. By (2.23), the sequence $\{(1/4^j)f(2^j x_1, \dots, 2^j x_m)\}$ is a Cauchy sequence for all $x_1, \dots, x_m \in X$. Since Y is complete, the sequence $\{(1/4^j)f(2^j x_1, \dots, 2^j x_m)\}$ converges for all $x_1, \dots, x_m \in X$. Define $F : X^m \rightarrow Y$ by

$$F(x_1, \dots, x_m) := \lim_{j \rightarrow \infty} \frac{1}{4^j} f(2^j x_1, \dots, 2^j x_m) \quad (2.24)$$

for all $x_1, \dots, x_m \in X$. By (2.18), we have

$$\begin{aligned} & \left\| \frac{1}{4^j} f(2^j(x_1 + y_1), \dots, 2^j(x_m + y_m)) + \frac{1}{4^j} f(2^j(x_1 - y_1), \dots, 2^j(x_m - y_m)) \right. \\ & \left. - \frac{2}{4^j} f(2^j x_1, \dots, 2^j x_m) - \frac{2}{4^j} f(2^j y_1, \dots, 2^j y_m) \right\| \leq \frac{1}{4^j} \varphi(2^j x_1, \dots, 2^j x_m, 2^j y_1, \dots, 2^j y_m) \end{aligned} \quad (2.25)$$

for all $x_1, \dots, x_m, y_1, \dots, y_m \in X$ and all j . Letting $j \rightarrow \infty$ and using (2.17), we see that F satisfies (1.2). Setting $l = 0$ and taking $n \rightarrow \infty$ in (2.23), one can obtain the inequality (2.19). If $G: X^m \rightarrow Y$ is another m -variable quadratic mapping satisfying (2.19), we obtain

$$\begin{aligned} & \|F(x_1, \dots, x_m) - G(x_1, \dots, x_m)\| \\ &= \frac{1}{4^n} \|F(2^n x_1, \dots, 2^n x_m) - G(2^n x_1, \dots, 2^n x_m)\| \\ &\leq \frac{1}{4^n} \|F(2^n x_1, \dots, 2^n x_m) - f(2^n x_1, \dots, 2^n x_m)\| \\ &\quad + \frac{1}{4^n} \|f(2^n x_1, \dots, 2^n x_m) - G(2^n x_1, \dots, 2^n x_m)\| \\ &\leq \frac{2}{4^n} \tilde{\varphi}(2^n x_1, \dots, 2^n x_m, 2^n x_1, \dots, 2^n x_m) \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned} \quad (2.26)$$

for all $x_1, \dots, x_m \in X$. Hence, the mapping F is the unique m -variable quadratic mapping, as desired. \square

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