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Research Article
On Janowski Starlike Functions

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For analytic functions $f(z)$ in the open unit disc $\mathbb{U}$ with $f(0) = 0$ and $f'(0) = 1$, applying the fractional calculus for $f(z)$, a new fractional operator $D^{\lambda} f(z)$ is introduced. Further, a new subclass $\mathcal{S}^*_{\lambda}(A,B)$ consisting of $f(z)$ associated with Janowski function is defined. The objective of the present paper is to discuss some interesting properties of the class $\mathcal{S}^*_{\lambda}(A,B)$.

1. Introduction and preliminaries

Let Ω be the class of analytic functions $w(z)$ in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$. For arbitrary fixed real numbers A and B which satisfy $-1 \leq B < A \leq 1$, we say that $p(z)$ belongs to the class $\mathcal{P}(A,B)$ if

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.1}$$

is analytic in $\mathbb{U}$ and $p(z)$ is given by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}) \tag{1.2}$$

for some $w(z) \in \Omega$. This class, $\mathcal{P}(A,B)$, was first introduced by Janowski [1]. Therefore, we call $f(z)$ in the class $\mathcal{P}(A,B)$ Janowski functions. Further, let $\mathcal{A}$ be class of functions

$f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.3}$$

which are analytic in $\mathbb{U}$. We recall here the following definitions of the fractional calculus (fractional integrals and fractional derivatives) given by Owa [2, 3] (also by Srivastava and Owa [4]).

Definition 1.1. The fractional integral of order λ is defined, for $f(z) \in \mathcal{A}$, by

$$\mathrm{D}_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0), \tag{1.4}$$

where the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2. The fractional derivative of order λ is defined, for $f(z) \in \mathcal{A}$, by

$$\mathrm{D}_z^{\lambda} f(z) = \frac{d}{dz}\left(\mathrm{D}_z^{\lambda-1} f(z)\right) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \le \lambda < 1), \tag{1.5}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.3. Under the hypothesis of Definition 1.2, the fractional derivative of order $(n+\lambda)$ is defined, for $f(z) \in \mathcal{A}$, by

$$\mathrm{D}_z^{\lambda+n} f(z) = \frac{d^n}{dz^n}\left(\mathrm{D}_z^{\lambda} f(z)\right) \quad (0 \le \lambda < 1,\ n \in \mathbb{N}_0 = \{0,1,2,\dots\}). \tag{1.6}$$

By means of the above definitions for the fractional calculus, we see that

$$\mathrm{D}_z^{-\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k+\lambda} \quad (\lambda > 0,\ k > 0),$$

$$\mathrm{D}_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda} \quad (0 \le \lambda < 1,\ k > 0),$$

$$\mathrm{D}_z^{n+\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-n-\lambda)} z^{k-n-\lambda} \quad (0 \le \lambda < 1,\ k > 0,\ n \in \mathbb{N}_0,\ k-n \neq -1,-2,-3,\dots). \tag{1.7}$$

Therefore, we conclude that for any real λ,

$$\mathrm{D}_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda} \quad (k > 0,\ k-\lambda \neq -1,-2,-3,\dots). \tag{1.8}$$

With the definitions of the fractional calculus, we introduce the fractional operator $D^{\lambda} f(z)$, for $f(z) \in \mathcal{A}$, by

$$D^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n \quad (\lambda \neq 2,3,4,\dots). \tag{1.9}$$

If $\lambda = 1$, then

$$D^1 f(z) = D f(z) = z f'(z) \tag{1.10}$$

and if $\lambda \neq 2,3,4,\dots$ and $\alpha \neq 2,3,4,\dots$, then

$$\begin{aligned} D^{\alpha}(D^{\lambda} f(z)) = D^{\lambda}(D^{\alpha} f(z)) &= z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(2-\alpha)(\Gamma(n+1))^2}{\Gamma(n+1-\lambda)\Gamma(n+1-\alpha)} a_n z^n, \\ D(D^{\lambda} f(z)) = z(D^{\lambda} f(z))' &= \Gamma(2-\lambda) z^{\lambda}(\lambda D_z^{\lambda} f(z) + z D_z^{\lambda+1} f(z)). \end{aligned} \tag{1.11}$$

Let $\mathcal{S}_{\lambda}^{*}(A,B)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ satisfying

$$\frac{z(D^{\lambda} f(z))'}{D^{\lambda} f(z)} = p(z) \quad (\lambda \neq 2,3,4,\dots) \tag{1.12}$$

for some $p(z) \in \mathcal{P}(A,B)$. Note that (1.12) is equivalent to

$$\lambda + \frac{z D_z^{\lambda+1} f(z)}{D_z^{\lambda} f(z)} = p(z) \quad (\lambda \neq 2,3,4,\dots). \tag{1.13}$$

Finally, for $h(z) \in \mathcal{A}$ and $s(z) \in \mathcal{A}$, we say that $h(z)$ is subordinate to $s(z)$, denoted by $h(z) \prec s(z)$, if there exists some function $w(z) \in \Omega$ such that

$$h(z) = s(w(z)) \quad (z \in \mathbb{U}). \tag{1.14}$$

In particular, if $s(z)$ is univalent in $\mathbb{U}$, then the subordination $h(z) \prec s(z)$ is equivalent to $h(0) = s(0)$ and $h(\mathbb{U}) \subset s(\mathbb{U})$ (see [5]).

2. Main results

To discuss our problems, we need the following lemma due to Jack [6] or Miller and Mocanu [7].

Lemma 2.1. *Let $w(z)$ be a nonconstant analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_1, then one has*

$$z_1 w'(z_1) = k w(z_1), \tag{2.1}$$

where k is real and $k \geq 1$.

Next, we have the following lemma.

Lemma 2.2. *Let $f(z) \in \mathcal{A}$ and*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}. \tag{2.2}$$

Then, the following fractional differential equation:

$$\mathrm{D}_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z) \quad (\lambda \neq 2,3,4,\dots) \tag{2.3}$$

has the solution

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} b_n z^n. \tag{2.4}$$

Proof. It is easy to see that

$$\begin{aligned} \mathrm{D}_z^{\lambda} f(z) &= \frac{1}{\Gamma(2-\lambda)} z^{-\lambda} g(z) = \frac{1}{\Gamma(2-\lambda)} \left(z^{1-\lambda} + \sum_{n=2}^{\infty} b_n z^{n-\lambda} \right), \\ \mathrm{D}_z^{\lambda} f(z) &= \frac{1}{\Gamma(2-\lambda)} \left(z^{1-\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda} \right), \end{aligned} \tag{2.5}$$

which gives

$$a_n = \frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)\Gamma(n+1)} b_n. \tag{2.6}$$

This completes the proof of the lemma. □

Next, we derive the following theorem.

Theorem 2.3. *If $f(z) \in \mathcal{A}$ satisfies the condition*

$$\left(z \frac{(\mathrm{D}^{\lambda} f(z))'}{\mathrm{D}^{\lambda} f(z)} - 1 \right) \prec \begin{cases} \dfrac{(A-B)z}{1+Bz} = F_1(z), & B \neq 0, \\ Az = F_2(z), & B = 0, \end{cases} \tag{2.7}$$

for some λ ($\lambda \neq 2,3,4,\dots$), then $f(z) \in \mathcal{S}_{\lambda}^{*}(A,B)$. This result is sharp because the extremal function is the solution of the fractional differential equation

$$\mathrm{D}_z^{\lambda} f(z) = \begin{cases} \dfrac{z^{1-\lambda}}{\Gamma(2-\lambda)} (1+Bz)^{(A-B)/B}, & B \neq 0, \\ \dfrac{z^{1-\lambda}}{\Gamma(2-\lambda)} e^{Az}, & B = 0. \end{cases} \tag{2.8}$$

Proof. We define the function $w(z)$ by

$$\frac{\mathrm{D}^{\lambda} f(z)}{z} = \begin{cases} (1+Bw(z))^{(A-B)/B}, & B\neq 0, \\ e^{Aw(z)}, & B = 0. \end{cases} \tag{2.9}$$

When $(1+Bw(z))^{(A-B)/B}$ and $e^{Aw(z)}$ have the value 1 at $z=0$ (i.e., we consider the corresponding Riemann branch), then $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$, and

$$\left(z\frac{(\mathrm{D}^{\lambda} f(z))'}{\mathrm{D}^{\lambda} f(z)} - 1\right) = \begin{cases} \dfrac{(A-B)zw'(z)}{1+Bw(z)}, & B\neq 0, \\ Azw'(z), & B = 0. \end{cases} \tag{2.10}$$

Now, it is easy to realize that the subordination (2.7) is equivalent to $|w(z)| < 1$ for all $z \in \mathbb{U}$. Indeed, assume the contrary. Then, there exists a point $z_1 \in \mathbb{D}$ such that $|w(z_1)| = 1$. Then, by Lemma 2.1, $z_1 w'(z_1) = kw(z_1)$ for some real $k \geq 1$; for such $z_1 \in \mathbb{U}$, then we have

$$\left(z_1\frac{(\mathrm{D}^{\lambda} f(z_1))'}{\mathrm{D}^{\lambda} f(z_1)} - 1\right) = \begin{cases} \dfrac{(A-B)kw(z_1)}{1+Bw(z_1)} = F_1(w(z_1)) \notin F_1(\mathbb{U}), & B\neq 0, \\ Akw(z_1) = F_2(w(z_1)) \notin F_2(\mathbb{U}), & B = 0, \end{cases} \tag{2.11}$$

but this contradicts the condition (2.7) of this theorem and so the assumption is wrong, that is, $|w(z)| < 1$ for every $z \in \mathbb{U}$. The sharpness of this result follows from the fact that

$$\begin{gathered}
\mathrm{D}_z^{\lambda} f(z) = \begin{cases} \dfrac{z^{1-\lambda}}{\Gamma(2-\lambda)}(1+Bz)^{(A-B)/B}, & B\neq 0, \\ \dfrac{z^{1-\lambda}}{\Gamma(2-\lambda)}e^{Az}, & B = 0, \end{cases} \Longrightarrow \\
\frac{\mathrm{D}^{\lambda} f(z)}{z} = \begin{cases} (1+Bz)^{(A-B)/B}, & B\neq 0, \\ e^{Az}, & B = 0, \end{cases} \Longrightarrow \\
\left(z\frac{(\mathrm{D}^{\lambda} f(z))'}{\mathrm{D}^{\lambda} f(z)} - 1\right) = \begin{cases} \dfrac{(A-B)z}{1+Bz}, & B\neq 0, \\ Az, & B = 0, \end{cases} \Longrightarrow \\
z\frac{(\mathrm{D}^{\lambda} f(z))'}{\mathrm{D}^{\lambda} f(z)} = \begin{cases} \dfrac{1+Az}{1+Bz}, & B\neq 0, \\ 1+Az, & B = 0. \end{cases}
\end{gathered} \tag{2.12}$$

□

Corollary 2.4. *If $f(z) \in \mathcal{S}^*_\lambda(A,B)$, then*

$$\begin{aligned} &\left|\left(\Gamma(2-\lambda)z^{\lambda-1}\mathrm{D}_z^\lambda f(z)\right)^{B/(A-B)}-1\right|<1, \quad B\neq 0, \\ &\left|\log\left(\Gamma(2-\lambda)z^{\lambda-1}\mathrm{D}_z^\lambda f(z)\right)^{1/A}\right|<1, \quad B=0. \end{aligned} \tag{2.13}$$

Proof. This corollary is a simple consequence of Theorem 2.3, and these inequalities are known as the Marx-Strohhacker inequalities for the class $\mathcal{S}^*_\lambda(A,B)$. □

Next, our result is contained in the following theorem.

Theorem 2.5. *If $f(z) \in \mathcal{S}^*_\lambda(A,B)$, then*

$$\begin{aligned} &\frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}(1-Br)^{(A-B)/B} \leq \left|\mathrm{D}_z^\lambda f(z)\right| \leq \frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}(1+Br)^{(A-B)/B}, \quad B\neq 0, \\ &\frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}e^{-Ar} \leq \left|\mathrm{D}_z^\lambda f(z)\right| \leq \frac{1}{\Gamma(2-\lambda)}r^{1-\lambda}e^{Ar}, \quad B=0. \end{aligned} \tag{2.14}$$

These results are sharp because extremal function is the solution of the fractional differential equation

$$\mathrm{D}_z^\lambda f(z) = \begin{cases} \dfrac{1}{\Gamma(2-\lambda)}z^{1-\lambda}(1+Bz)^{(A-B)/B}, & B\neq 0, \\ \dfrac{1}{\Gamma(2-\lambda)}z^{1-\lambda}e^{Az}, & B=0. \end{cases} \tag{2.15}$$

Proof. Janowski [1] proved that if $p(z) \in \mathcal{P}(A,B)$, then

$$\begin{aligned} &\left|p(z)-\frac{1-ABr^2}{1-B^2r^2}\right| < \frac{(A-B)r}{1-B^2r^2}, \quad B\neq 0, \\ &\left|p(z)-1\right| < Ar, \quad B=0 \ . \end{aligned} \tag{2.16}$$

Using the definition of the class $\mathcal{S}^*_\lambda(A,B)$, the inequality (2.16) can be rewritten in the form

$$\begin{aligned} &\left|z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}-\frac{1-ABr^2}{1-B^2r^2}\right| < \frac{(A-B)r}{1-B^2r^2}, \quad B\neq 0 \ , \\ &\left|z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}-1\right| < Ar, \quad B=0 \ . \end{aligned} \tag{2.17}$$

From (2.17), with simple calculations, we get

$$\begin{aligned} &\frac{1-(A-B)r-ABr^2}{1-B^2r^2} \leq \mathrm{Re}\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}\right) \leq \frac{1+(A-B)r-ABr^2}{1-B^2r^2}, \quad B\neq 0, \\ &1-Ar \leq \mathrm{Re}\left(z\frac{(\mathrm{D}^\lambda f(z))'}{\mathrm{D}^\lambda f(z)}\right) \leq 1+Ar, \quad B=0. \end{aligned} \tag{2.18}$$

Since

$$\operatorname{Re}\left(z\frac{(D_z^\lambda f(z))'}{D_z^\lambda f(z)}\right) = r\frac{\partial}{\partial r}\log\left|D^\lambda f(z)\right|, \tag{2.19}$$

using (2.18) and (2.19), we obtain

$$\begin{gathered}
\frac{1-(A-B)r-ABr^2}{r(1+Br)(1-Br)} \le \frac{\partial}{\partial r}\log\left|D^\lambda f(z)\right| \le \frac{1+(A-B)r-ABr^2}{r(1+Br)(1-Br)}, \quad B\neq 0, \\
\frac{1}{r}-A \le \frac{\partial}{\partial r}\log\left|D^\lambda f(z)\right| \le \frac{1}{r}+A, \quad B=0.
\end{gathered} \tag{2.20}$$

Integrating both sides of (2.20) from 0 to r and after simple calculations, we complete the proof of the theorem. □

Corollary 2.6. *Giving specific values to A and B, one obtains the distortion of the following class.*

(i) $\mathcal{S}_\lambda^*(1,-1)$,
(ii) $\mathcal{S}_\lambda^*(1-2\beta,-1)$, $0\le\beta<1$,
(iii) $\mathcal{S}_\lambda^*(1,-1+1/M)$, $M>1/2$,
(iv) $\mathcal{S}_\lambda^*(\beta,-\beta)$, $0\le\beta<1$.

Finally, we discuss the coefficient inequalities for $f(z)\in\mathcal{S}_\lambda^*(A,B)$.

Theorem 2.7. *If $f(z)\in\mathcal{S}_\lambda^*(A,B)$, then*

$$\left|a_n\right| \le \begin{cases} \dfrac{|A-B|}{(n-1)!}\dfrac{\left|\Gamma(n+1-\lambda)\right|}{\Gamma(n+1)\left|\Gamma(2-\lambda)\right|}\displaystyle\prod_{k=1}^{n-2}(k+|A-B|), & B\neq 0, \\[2ex] \dfrac{|A|}{(n-1)!}\dfrac{\left|\Gamma(n+1-\lambda)\right|}{\Gamma(n+1)\left|\Gamma(2-\lambda)\right|}\displaystyle\prod_{k=1}^{n-2}(k+|A|), & B=0. \end{cases} \tag{2.21}$$

Proof. Using the definition of the class, we can write, for $B\neq 0$,

$$\begin{aligned}
z\frac{(D^\lambda f(z))'}{D^\lambda f(z)} = p(z) \Longleftrightarrow z\big(D^\lambda f(z)\big)' &= D^\lambda f(z)p(z) \\
&\Longrightarrow z+2a_2\frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)}z^2+3a_3\frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)}z^3 \\
&\quad +\cdots+na_n\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}z^n+\cdots \\
&= (1+p_1z+\cdots+p_nz^n+\cdots) \\
&\quad \cdot\left(z+a_2\frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)}z^2+a_3\frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)}z^3+\cdots+a_n\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}z^n+\cdots\right).
\end{aligned} \tag{2.22}$$

Equaling the coefficient of z^n in both sides of (2.22), we get

$$a_n = \frac{1}{(n-1)}\frac{\Gamma(n+1-\lambda)}{\Gamma(n+1)}\sum_{k=1}^{n-1}\frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}a_k p_{n-k}, \quad a_1 \equiv 1. \tag{2.23}$$

On the other hand, if $p(z) \in \mathcal{P}(A,B)$, then $|p_n| \leq (A-B)$ (see [8]); so we obtain

$$|a_n| \leq \frac{|A-B|}{(n-1)}\frac{|\Gamma(n+1-\lambda)|}{\Gamma(n+1)}\sum_{k=1}^{n-1}\frac{\Gamma(k+1)}{|\Gamma(k+1-\lambda)|}|a_k|, \quad |a_1| \equiv 1. \tag{2.24}$$

Using the induction method form (2.24), we obtain,

$$\begin{aligned}
|a_2| &\leq \frac{|A-B|}{1}\frac{|\Gamma(3-\lambda)|\Gamma(2)}{\Gamma(3)|\Gamma(2-\lambda)|}, \quad \text{for } n=2,\\
|a_3| &\leq \frac{|A-B|}{2}\frac{|\Gamma(4-\lambda)|\Gamma(2)}{\Gamma(4)|\Gamma(2-\lambda)|}\left(1+\frac{|A-B|}{1}\right), \quad \text{for } n=3,\\
|a_4| &\leq \frac{|A-B|}{3}\frac{|\Gamma(5-\lambda)|\Gamma(2)}{\Gamma(5)|\Gamma(2-\lambda)|}\left(1+\frac{|A-B|}{1}\right)\left(1+\frac{|A-B|}{2}\right),\ldots, \quad \text{for } n=4,\\
|a_n| &\leq \frac{|A-B|}{(n-1)!}\frac{|\Gamma(n+1-\lambda)|}{\Gamma(n+1)|\Gamma(2-\lambda)|}\prod_{k=1}^{n-2}(k+|A-B|).
\end{aligned} \tag{2.25}$$

□

Remark 2.8. One considers the extremal function $f(z)$ defined by

$$D_z^\lambda f(z) = \begin{cases} \dfrac{z^{1-\lambda}}{\Gamma(2-\lambda)}(1+Bz)^{(A-B)/B}, & B \neq 0,\\[2ex] \dfrac{z^{1-\lambda}}{\Gamma(2-\lambda)}e^{Az}, & B=0, \end{cases} \tag{2.26}$$

in Theorems 2.3 and 2.5.

If $B=0$, then

$$\begin{aligned}
D_z^\lambda f(z) &= \frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}e^{Az}\\
&= \frac{1}{\Gamma(2-\lambda)}\left(z^{1-\lambda}+\sum_{n=2}^{\infty}\frac{A^{n-1}}{(n-1)!}z^{n-\lambda}\right),\\
D_z^\lambda f(z) &= \frac{1}{\Gamma(2-\lambda)}\left(z^{1-\lambda}+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)}a_n z^{n-\lambda}\right),
\end{aligned} \tag{2.27}$$

which gives

$$a_n = \frac{A^{n-1}\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)n!(n-1)!}. \tag{2.28}$$

If $B\neq 0$, then

$$\mathrm{D}_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda}(1+Bz)^{(A-B)/B}$$

$$= \frac{1}{\Gamma(2-\lambda)} \left(z^{1-\lambda} + \sum_{n=2}^{\infty} \begin{pmatrix} \frac{A-B}{B} \\ n-1 \end{pmatrix} B^{n-1} z^{n-\lambda} \right), \tag{2.29}$$

$$\mathrm{D}_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} \left(z^{1-\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^{n-\lambda} \right),$$

which gives

$$a_n = \begin{pmatrix} \frac{A-B}{B} \\ n-1 \end{pmatrix} B^{n-1} \frac{\Gamma(n+1-\lambda)}{n!\Gamma(2-\lambda)}$$

$$= \frac{(A-B)(A-2B)(A-3B)\cdots(A-(n-1)B)\Gamma(n+1-\lambda)}{n!\Gamma(2-\lambda)} \tag{2.30}$$

$$= \frac{(2-\lambda)_{n-1}}{(1)_n} \left(\prod_{j=1}^{n-1} (A-jB) \right),$$

where $(a)_n$ denotes the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1 & (n=0,\ a\neq 0), \\ a(a+1)(a+2)\cdots(a+n-1) & (n=1,2,3,\dots), \end{cases} \tag{2.31}$$

so

$$\frac{\Gamma(n+1-\lambda)}{\Gamma(2-\lambda)} = (n-\lambda)(n-\lambda-1)(n-\lambda-2)\cdots(2-\lambda) = (2-\lambda)_{n-1}. \tag{2.32}$$

We note that, by giving specific values to A and B, we obtain the distortion and coefficient inequalities for the classes $\mathcal{S}_\lambda^*(1,-1)$, $\mathcal{S}_\lambda^*(1,0)$, $\mathcal{S}_\lambda^*(\beta,-\beta)$ $(0 \le \beta < 1)$, $\mathcal{S}_\lambda^*(1,-1+1/M)$ $(M > 1/2)$, and $\mathcal{S}_\lambda^*(1-2\beta,-1)$ $(0 \le \beta < 1)$.

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