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Padé approximant related to the Wallis formula

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Abstract

Based on the Padé approximation method, in this paper we determine the coefficients a_j and b_j such that

$$\pi = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \left\{\frac{n^k + a_1 n^{k-1} + \cdots + a_k}{n^{k+1} + b_1 n^k + \cdots + b_{k+1}} + O\left(\frac{1}{n^{2k+3}}\right)\right\}, \quad n \to \infty,$$

where $k \geq 0$ is any given integer. Based on the obtained result, we establish a more accurate formula for approximating π, which refines some known results.

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1 Introduction

It is well known that the number π satisfies the following inequalities:

$$\frac{2}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 < \pi < \frac{1}{n}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2, \quad n \in \mathbb{N} := \{1, 2, 3, \ldots\}, \tag{1.1}$$

where

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!, \qquad (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

This result is due to Wallis (see [1]).

Based on a basic theorem in mathematical statistics concerning unbiased estimators with minimum variance, Gurland [1] yielded a closer approximation to π than that afforded by (1.1), namely,

$$\frac{4n+3}{(2n+1)^2}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 < \pi < \frac{4}{4n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2, \quad n \in \mathbb{N}. \tag{1.2}$$

By using (1.2), Brutman [2] and Falaleev [3] established estimates of the Landau constants.

Mortici [4], Theorem 2, improved Gurland's result (1.2) and obtained the following double inequality:

$$\begin{aligned}&\left(\frac{n+\frac{1}{4}}{n^2+\frac{1}{2}n+\frac{3}{32}}+\frac{9}{2{,}048n^5}-\frac{45}{8{,}192n^6}\right)\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\\&\quad<\pi<\left(\frac{n+\frac{1}{4}}{n^2+\frac{1}{2}n+\frac{3}{32}}+\frac{9}{2{,}048n^5}\right)\left(\frac{(2n)!!}{(2n-1)!!}\right)^2,\quad n\in\mathbb{N}.\end{aligned}\tag{1.3}$$

We see from (1.3) that

$$\pi=\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\left\{\frac{n+\frac{1}{4}}{n^2+\frac{1}{2}n+\frac{3}{32}}+O\left(\frac{1}{n^5}\right)\right\},\quad n\to\infty.\tag{1.4}$$

Based on the Padé approximation method, in this paper we develop the approximation formula (1.4) to produce a general result. More precisely, we determine the coefficients a_j and b_j such that

$$\pi=\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\left\{\frac{n^k+a_1n^{k-1}+\cdots+a_k}{n^{k+1}+b_1n^k+\cdots+b_{k+1}}+O\left(\frac{1}{n^{2k+3}}\right)\right\},\quad n\to\infty,\tag{1.5}$$

where $k\geq 0$ is any given integer. Based on the obtained result, we establish a more accurate formula for approximating π, which refines some known results.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2 Lemmas

Euler's gamma function $\Gamma(x)$ is one of the most important functions in mathematical analysis and has applications in diverse areas. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\Gamma'(x)/\Gamma(x)$, is called the psi (or digamma) function.

The following lemmas are required in the sequel.

Lemma 2.1 ([5]) *Let $r\neq 0$ be a given real number and $\ell\geq 0$ be a given integer. The following asymptotic expansion holds*:

$$\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\sim\sqrt{x}\left(1+\sum_{j=1}^{\infty}\frac{p_j}{x^j}\right)^{x^{\ell}/r},\quad x\to\infty,\tag{2.1}$$

with the coefficients $p_j\equiv p_j(\ell,r)$ ($j\in\mathbb{N}$) given by

$$p_j=\sum\frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{(2^2-1)B_2}{1\cdot 1\cdot 2^2}\right)^{k_1}\left(\frac{(2^4-1)B_4}{2\cdot 3\cdot 2^4}\right)^{k_2}\cdots\left(\frac{(2^{2j}-1)B_{2j}}{j(2j-1)2^{2j}}\right)^{k_j},\tag{2.2}$$

where B_j are the Bernoulli numbers summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1+(3+\ell)k_2+\cdots+(2j+\ell-1)k_j=j.$$

In particular, setting $(\ell, r) = (0, -2)$ in (2.1) yields

$$x\left(\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right)^2 \sim 1 + \sum_{j=1}^{\infty} \frac{c_j}{x^j}, \quad x \to \infty, \tag{2.3}$$

where the coefficients $c_j \equiv p_j(0, -2)$ ($j \in \mathbb{N}$) are given by

$$c_j = \sum \frac{(-2)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{(2^2-1)B_2}{1\cdot 1\cdot 2^2}\right)^{k_1}\left(\frac{(2^4-1)B_4}{2\cdot 3\cdot 2^4}\right)^{k_2}\cdots\left(\frac{(2^{2j}-1)B_{2j}}{j(2j-1)2^{2j}}\right)^{k_j}, \tag{2.4}$$

summed over all nonnegative integers k_j satisfying the equation

$$k_1 + 3k_2 + \cdots + (2j-1)k_j = j.$$

Lemma 2.2 ([5]) *Let $m, n \in \mathbb{N}$. Then, for $x > 0$,*

$$\begin{aligned} \sum_{j=1}^{2m}\left(1-\frac{1}{2^{2j}}\right)\frac{2B_{2j}}{(2j)!}\frac{(2j+n-2)!}{x^{2j+n-1}} &< (-1)^n\left(\psi^{(n-1)}(x+1) - \psi^{(n-1)}\left(x+\frac{1}{2}\right)\right) + \frac{(n-1)!}{2x^n} \\ &< \sum_{j=1}^{2m-1}\left(1-\frac{1}{2^{2j}}\right)\frac{2B_{2j}}{(2j)!}\frac{(2j+n-2)!}{x^{2j+n-1}}. \end{aligned} \tag{2.5}$$

In particular, we have

$$U(x) < \psi(x+1) - \psi\left(x+\frac{1}{2}\right) < V(x), \tag{2.6}$$

where

$$\begin{aligned} V(x) = {}& \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2{,}048x^8} - \frac{31}{2{,}048x^{10}} + \frac{691}{16{,}384x^{12}} \\ & - \frac{5{,}461}{32{,}768x^{14}} + \frac{929{,}569}{1{,}048{,}576x^{16}} \end{aligned}$$

and

$$U(x) = V(x) - \frac{3{,}202{,}291}{524{,}288x^{18}}.$$

For our later use, we introduce Padé approximant (see [6–11]). Let f be a formal power series

$$f(t) = c_0 + c_1 t + c_2 t^2 + \cdots. \tag{2.7}$$

The Padé approximation of order (p, q) of the function f is the rational function, denoted by

$$[p/q]_f(t) = \frac{\sum_{j=0}^{p} a_j t^j}{1 + \sum_{j=1}^{q} b_j t^j}, \tag{2.8}$$

where $p \geq 0$ and $q \geq 1$ are two given integers, the coefficients a_j and b_j are given by (see [6–8, 10, 11])

$$\begin{cases} a_0 = c_0, \\ a_1 = c_0 b_1 + c_1, \\ a_2 = c_0 b_2 + c_1 b_1 + c_2, \\ \vdots \\ a_p = c_0 b_p + \cdots + c_{p-1} b_1 + c_p, \\ 0 = c_{p+1} + c_p b_1 + \cdots + c_{p-q+1} b_q, \\ \vdots \\ 0 = c_{p+q} + c_{p+q-1} b_1 + \cdots + c_p b_q, \end{cases} \tag{2.9}$$

and the following holds:

$$[p/q]_f(t) - f(t) = O\bigl(t^{p+q+1}\bigr). \tag{2.10}$$

Thus, the first $p + q + 1$ coefficients of the series expansion of $[p/q]_f$ are identical to those of f. Moreover, we have (see [9])

$$[p/q]_f(t) = \frac{\begin{vmatrix} t^q f_{p-q}(t) & t^{q-1} f_{p-q+1}(t) & \cdots & f_p(t) \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} t^q & t^{q-1} & \cdots & 1 \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}, \tag{2.11}$$

with $f_n(x) = c_0 + c_1 x + \cdots + c_n x^n$, the nth partial sum of the series f in (2.7).

3 Main results

Let

$$f(x) = x \left(\frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \right)^2. \tag{3.1}$$

It follows from (2.3) that, as $x \to \infty$,

$$\begin{aligned} f(x) \sim \sum_{j=0}^{\infty} \frac{c_j}{x^j} = 1 - \frac{1}{4x} + \frac{1}{32x^2} + \frac{1}{128x^3} - \frac{5}{2{,}048x^4} - \frac{23}{8{,}192x^5} + \frac{53}{65{,}536x^6} \\ + \frac{593}{262{,}144x^7} - \cdots, \end{aligned} \tag{3.2}$$

with the coefficients c_j given by (2.4). In what follows, the function f is given in (3.1).

Based on the Padé approximation method, we now give a derivation of formula (1.4). To this end, we consider

$$[1/2]_f(x) = \frac{\sum_{j=0}^{1} a_j x^{-j}}{1 + \sum_{j=1}^{2} b_j x^{-j}}.$$

Noting that

$$c_0 = 1, \qquad c_1 = -\frac{1}{4}, \qquad c_2 = \frac{1}{32}, \qquad c_3 = \frac{1}{128}$$

holds, we have, by (2.9),

$$\begin{cases} a_0 = 1, \\ a_1 = b_1 - \frac{1}{4}, \\ 0 = \frac{1}{32} - \frac{1}{4}b_1 + b_2, \\ 0 = \frac{1}{128} + \frac{1}{32}b_1 - \frac{1}{4}b_2, \end{cases}$$

that is,

$$a_0 = 1, \qquad a_1 = \frac{1}{4}, \qquad b_1 = \frac{1}{2}, \qquad b_2 = \frac{3}{32}.$$

We thus obtain that

$$[1/2]_f(x) = \frac{1 + \frac{1}{4x}}{1 + \frac{1}{2x} + \frac{3}{32x^2}}, \tag{3.3}$$

and we have, by (2.10),

$$x\left(\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right)^2 - \frac{1 + \frac{1}{4x}}{1 + \frac{1}{2x} + \frac{3}{32x^2}} = O\left(\frac{1}{x^4}\right), \quad x \to \infty. \tag{3.4}$$

Noting that

$$\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} = \sqrt{\pi} \cdot \frac{(2n-1)!!}{(2n)!!}, \quad n \in \mathbb{N} \text{ (the Wallis ratio)} \tag{3.5}$$

holds, replacing x by n in (3.4) yields (1.4).

From the Padé approximation method introduced in Section 2 and the asymptotic expansion (3.2), we obtain a general result given by Theorem 3.1. As a consequence, we obtain (1.5).

Theorem 3.1 *The Padé approximation of order* (p,q) *of the asymptotic formula of the function* $f(x) = x(\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)})^2$ (*at the point* $x = \infty$) *is the following rational function*:

$$[p/q]_f(x) = \frac{1 + \sum_{j=1}^{p} a_j x^{-j}}{1 + \sum_{j=1}^{q} b_j x^{-j}} = x\left(\frac{x^p + a_1 x^{p-1} + \cdots + a_p}{x^q + b_1 x^{q-1} + \cdots + b_q}\right), \tag{3.6}$$

where $p \geq 0$ *and* $q \geq 1$ *are two given integers and* $q = p + 1$ (*an empty sum is understood to be zero*), *the coefficients* a_j *and* b_j *are given by*

$$\begin{cases} a_1 = b_1 + c_1, \\ a_2 = b_2 + c_1 b_1 + c_2, \\ \vdots \\ a_p = b_p + \cdots + c_{p-1} b_1 + c_p, \\ 0 = c_{p+1} + c_p b_1 + \cdots + c_{p-q+1} b_q, \\ \vdots \\ 0 = c_{p+q} + c_{p+q-1} b_1 + \cdots + c_p b_q, \end{cases} \tag{3.7}$$

and c_j *is given in* (2.4), *and the following holds*:

$$f(x) - [p/q]_f(x) = O\left(\frac{1}{x^{p+q+1}}\right), \quad x \to \infty. \tag{3.8}$$

Moreover, we have

$$[p/q]_f(x) = \frac{\begin{vmatrix} \frac{1}{x^q} f_{p-q}(x) & \frac{1}{x^{q-1}} f_{p-q+1}(t) & \cdots & f_p(t) \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^q} & \frac{1}{x^{q-1}} & \cdots & 1 \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}, \tag{3.9}$$

with $f_n(x) = \sum_{j=0}^{n} \frac{c_j}{x^j}$, *the nth partial sum of the asymptotic series* (3.2).

Remark 3.1 Using (3.9), we can also derive (3.3). Indeed, we have

$$[1/2]_f(x) = \frac{\begin{vmatrix} \frac{1}{x^2} f_{-1}(x) & \frac{1}{x} f_0(x) & f_1(x) \\ c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} & 1 \\ c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \end{vmatrix}} = \frac{\begin{vmatrix} 0 & \frac{1}{x} & 1 - \frac{1}{4x} \\ 1 & -\frac{1}{4} & \frac{1}{32} \\ -\frac{1}{4} & \frac{1}{32} & \frac{1}{128} \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} & 1 \\ 1 & -\frac{1}{4} & \frac{1}{32} \\ -\frac{1}{4} & \frac{1}{32} & \frac{1}{128} \end{vmatrix}} = \frac{1 + \frac{1}{4x}}{1 + \frac{1}{2x} + \frac{3}{32x^2}}.$$

Replacing x by n in (3.8) applying (3.5), we obtain the following corollary.

Corollary 3.1 *As* $n \to \infty$,

$$\pi = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \left\{\frac{n^p + \sum_{j=1}^{p} a_j n^{p-j}}{n^q + \sum_{j=1}^{q} b_j n^{q-j}} + O\left(\frac{1}{n^{p+q+2}}\right)\right\}, \quad n \to \infty, \tag{3.10}$$

where $p \geq 0$ *and* $q \geq 1$ *are two given integers and* $q = p + 1$, *and the coefficients* a_j *and* b_j *are given by* (3.7).

Remark 3.2 Setting $(p,q)=(k,k+1)$ in (3.10) yields (1.5).

Setting

$$(p,q)=(4,5) \quad \text{and} \quad (p,q)=(5,6)$$

in (3.10), respectively, we find

$$\pi=\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\left\{\frac{n^4+n^3+\frac{107}{64}n^2+\frac{91}{128}n+\frac{789}{4{,}096}}{n^5+\frac{5}{4}n^4+\frac{125}{64}n^3+\frac{295}{256}n^2+\frac{1{,}689}{4{,}096}n+\frac{945}{16{,}384}}+O\left(\frac{1}{n^{11}}\right)\right\} \tag{3.11}$$

and

$$\begin{aligned}\pi&=\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\\&\quad\times\left\{\frac{n^5+\frac{5}{4}n^4+\frac{51}{16}n^3+\frac{133}{64}n^2+\frac{5{,}243}{4{,}096}n+\frac{3{,}867}{16{,}384}}{n^6+\frac{3}{2}n^5+\frac{113}{32}n^4+\frac{93}{32}n^3+\frac{7{,}729}{4{,}096}n^2+\frac{4{,}881}{8{,}192}n+\frac{10{,}395}{131{,}072}}+O\left(\frac{1}{n^{13}}\right)\right\}\end{aligned} \tag{3.12}$$

as $n\to\infty$.

Formulas (3.11) and (3.12) motivate us to establish the following theorem.

Theorem 3.2 *The following inequality holds*:

$$\begin{aligned}&\frac{x^5+\frac{5}{4}x^4+\frac{51}{16}x^3+\frac{133}{64}x^2+\frac{5{,}243}{4{,}096}x+\frac{3{,}867}{16{,}384}}{x^6+\frac{3}{2}x^5+\frac{113}{32}x^4+\frac{93}{32}x^3+\frac{7{,}729}{4{,}096}x^2+\frac{4{,}881}{8{,}192}x+\frac{10{,}395}{131{,}072}}\\&\quad<\left(\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right)^2\\&\quad<\frac{x^4+x^3+\frac{107}{64}x^2+\frac{91}{128}x+\frac{789}{4{,}096}}{x^5+\frac{5}{4}x^4+\frac{125}{64}x^3+\frac{295}{256}x^2+\frac{1{,}689}{4{,}096}x+\frac{945}{16{,}384}}.\end{aligned} \tag{3.13}$$

The left-hand side inequality holds for $x\geq 4$, while the right-hand side inequality is valid for $x\geq 3$.

Proof It suffices to show that

$$F(x)>0 \quad \text{for } x\geq 4 \quad \text{and} \quad G(x)<0 \quad \text{for } x\geq 3,$$

where

$$F(x)=2\ln\left(\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right)-\ln\frac{x^5+\frac{5}{4}x^4+\frac{51}{16}x^3+\frac{133}{64}x^2+\frac{5{,}243}{4{,}096}x+\frac{3{,}867}{16{,}384}}{x^6+\frac{3}{2}x^5+\frac{113}{32}x^4+\frac{93}{32}x^3+\frac{7{,}729}{4{,}096}x^2+\frac{4{,}881}{8{,}192}x+\frac{10{,}395}{131{,}072}}$$

and

$$G(x)=2\ln\left(\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right)-\ln\frac{x^4+x^3+\frac{107}{64}x^2+\frac{91}{128}x+\frac{789}{4{,}096}}{x^5+\frac{5}{4}x^4+\frac{125}{64}x^3+\frac{295}{256}x^2+\frac{1{,}689}{4{,}096}x+\frac{945}{16{,}384}}.$$

Using the following asymptotic expansion (see [12]):

$$\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2 \sim \frac{1}{x}\exp\left(-\frac{1}{4x}+\frac{1}{96x^3}-\frac{1}{320x^5}+\frac{17}{7{,}168x^7}-\frac{31}{9{,}216x^9}\right.$$
$$\left.+\frac{691}{90{,}112x^{11}}-\frac{5{,}461}{212{,}992x^{13}}+\frac{929{,}569}{7{,}864{,}320x^{15}}-\cdots\right), \quad x\to\infty, \qquad (3.14)$$

we obtain that

$$\lim_{x\to\infty} F(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} G(x) = 0.$$

Differentiating $F(x)$ and applying the first inequality in (2.6), we find

$$\begin{aligned} F'(x) &= -2\left[\psi(x+1)-\psi\left(x+\frac{1}{2}\right)\right]+\frac{P_{10}(x)}{P_{11}(x)} \\ &< -2U(x)+\frac{P_{10}(x)}{P_{11}(x)} = -\frac{P_{16}(x-4)}{524{,}288x^{18}P_{11}(x)}, \end{aligned}$$

where

$$\begin{aligned} P_{10}(x) = 4\bigl(&20{,}998{,}323+301{,}244{,}208x+1{,}329{,}622{,}624x^2+3{,}532{,}111{,}872x^3 \\ &+6{,}831{,}390{,}720x^4+8{,}950{,}906{,}880x^5+9{,}510{,}060{,}032x^6 \\ &+6{,}476{,}005{,}376x^7+4{,}244{,}635{,}648x^8+1{,}342{,}177{,}280x^9+536{,}870{,}912x^{10}\bigr), \\ P_{11}(x) = \bigl(&16{,}384x^5+20{,}480x^4+52{,}224x^3+34{,}048x^2+20{,}972x+3{,}867\bigr) \\ &\times\bigl(131{,}072x^6+196{,}608x^5+462{,}848x^4+380{,}928x^3+247{,}328x^2 \\ &+78{,}096x+10{,}395\bigr) \end{aligned}$$

and

$$\begin{aligned} P_{16}(x) = {}&73{,}399{,}302{,}245{,}132{,}658{,}732{,}474+401{,}687{,}666{,}421{,}636{,}714{,}876{,}048x \\ &+882{,}663{,}824{,}965{,}187{,}436{,}960{,}169x^2 \\ &+1{,}129{,}813{,}735{,}156{,}766{,}429{,}414{,}420x^3 \\ &+975{,}385{,}167{,}000{,}268{,}446{,}720{,}384x^4 \\ &+611{,}802{,}531{,}654{,}753{,}268{,}270{,}848x^5 \\ &+290{,}696{,}674{,}545{,}996{,}984{,}221{,}376x^6 \\ &+107{,}149{,}026{,}028{,}490{,}487{,}475{,}968x^7 \\ &+31{,}018{,}031{,}026{,}615{,}120{,}693{,}760x^8 \\ &+7{,}080{,}024{,}048{,}117{,}231{,}228{,}928x^9 \\ &+1{,}270{,}066{,}473{,}244{,}063{,}756{,}800x^{10}+177{,}136{,}978{,}237{,}041{,}715{,}200x^{11} \\ &+18{,}824{,}726{,}793{,}935{,}462{,}400x^{12}+1{,}473{,}208{,}721{,}923{,}276{,}800x^{13} \end{aligned}$$

$$+ 80{,}051{,}720{,}723{,}251{,}200x^{14} + 2{,}698{,}074{,}228{,}326{,}400x^{15} + 42{,}489{,}357{,}926{,}400x^{16}.$$

Hence, $F'(x) < 0$ for $x \geq 4$, and we have

$$F(x) > \lim_{t\to\infty} F(t) = 0, \quad x \geq 4.$$

Differentiating $G(x)$ and applying the second inequality in (2.6), we find

$$\begin{aligned} G'(x) &= -2\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right)\right] + \frac{4P_8(x)}{P_9(x)} > -2V(x) + \frac{4P_8(x)}{P_9(x)} \\ &= \frac{P_{14}(x-3)}{524{,}288x^{16}P_9(x)}, \end{aligned}$$

where

$$\begin{aligned} P_8(x) &= 16{,}777{,}216x^8 + 33{,}554{,}432x^7 + 72{,}351{,}744x^6 + 79{,}167{,}488x^5 + 75{,}583{,}488x^4 \\ &\quad + 45{,}043{,}712x^3 + 18{,}211{,}328x^2 + 4{,}212{,}480x + 644{,}661, \\ P_9(x) &= \left(4{,}096x^4 + 4{,}096x^3 + 6{,}848x^2 + 2{,}912x + 789\right) \\ &\quad \times \left(16{,}384x^5 + 20{,}480x^4 + 32{,}000x^3 + 18{,}880x^2 + 6{,}756x + 945\right) \end{aligned}$$

and

$$\begin{aligned} P_{14}(x) &= 427{,}884{,}340{,}806{,}856{,}575 + 5{,}508{,}337{,}280{,}234{,}438{,}700x \\ &\quad + 16{,}278{,}641{,}070{,}340{,}979{,}232x^2 \\ &\quad + 25{,}110{,}186{,}749{,}213{,}013{,}376x^3 + 25{,}009{,}399{,}125{,}661{,}680{,}960x^4 \\ &\quad + 17{,}642{,}792{,}222{,}808{,}253{,}696x^5 \\ &\quad + 9{,}230{,}356{,}959{,}310{,}493{,}184x^6 + 3{,}661{,}094{,}552{,}739{,}530{,}752x^7 \\ &\quad + 1{,}108{,}535{,}832{,}992{,}448{,}000x^8 \\ &\quad + 255{,}024{,}028{,}762{,}675{,}200x^9 + 43{,}854{,}087{,}132{,}979{,}200x^{10} \\ &\quad + 5{,}462{,}018{,}666{,}496{,}000x^{11} \\ &\quad + 465{,}495{,}496{,}704{,}000x^{12} + 24{,}287{,}993{,}856{,}000x^{13} \\ &\quad + 585{,}252{,}864{,}000x^{14}. \end{aligned}$$

Hence, $G'(x) > 0$ for $x \geq 3$, and we have

$$G(x) < \lim_{t\to\infty} G(t) = 0, \quad x \geq 3.$$

The proof is complete. □

Corollary 3.2 *For $n \in \mathbb{N}$,*

$$a_n < \pi < b_n, \tag{3.15}$$

where

$$a_n = \frac{n^5 + \frac{5}{4}n^4 + \frac{51}{16}n^3 + \frac{133}{64}n^2 + \frac{5{,}243}{4{,}096}n + \frac{3{,}867}{16{,}384}}{n^6 + \frac{3}{2}n^5 + \frac{113}{32}n^4 + \frac{93}{32}n^3 + \frac{7{,}729}{4{,}096}n^2 + \frac{4{,}881}{8{,}192}n + \frac{10{,}395}{131{,}072}}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \tag{3.16}$$

and

$$b_n = \frac{n^4 + n^3 + \frac{107}{64}n^2 + \frac{91}{128}n + \frac{789}{4{,}096}}{n^5 + \frac{5}{4}n^4 + \frac{125}{64}n^3 + \frac{295}{256}n^2 + \frac{1{,}689}{4{,}096}n + \frac{945}{16{,}384}}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2. \tag{3.17}$$

Proof Noting that (3.5) holds, we see by (3.13) that the left-hand side of (3.15) holds for $n \geq 4$, while the right-hand side of (3.15) is valid for $n \geq 3$. Elementary calculations show that the left-hand side of (3.15) is also valid for $n = 1, 2$ and 3, and the right-hand side of (3.15) is valid for $n = 1$ and 2. The proof is complete. □

4 Comparison

Recently, Lin [12] improved Mortici's result (1.3) and obtained the following inequalities:

$$\lambda_n < \pi < \mu_n \tag{4.1}$$

and

$$\delta_n < \pi < \omega_n, \tag{4.2}$$

where

$$\lambda_n = \left(1 + \frac{1}{4n} - \frac{3}{32n^2} + \frac{3}{128n^3} + \frac{3}{2{,}048n^4} - \frac{33}{8{,}192n^5} - \frac{39}{65{,}536n^6}\right) \times \frac{2}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2, \tag{4.3}$$

$$\mu_n = \left(1 + \frac{1}{4n} - \frac{3}{32n^2} + \frac{3}{128n^3} + \frac{3}{2{,}048n^4}\right)\frac{2}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2, \tag{4.4}$$

$$\delta_n = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{1}{n}\exp\left(-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7{,}168n^7} - \frac{31}{9{,}216n^9}\right), \tag{4.5}$$

$$\omega_n = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{1}{n}\exp\left(-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7{,}168n^7}\right). \tag{4.6}$$

Direct computation yields

$$\begin{aligned} a_n - \lambda_n &= \frac{3(7{,}634{,}944n^5 + 12{,}928{,}000n^4 + 18{,}895{,}616n^3 + 9{,}755{,}072n^2 + 1{,}930{,}008n + 135{,}135)}{32{,}768n^6(2n+1)(131{,}072n^6 + 196{,}608n^5 + 462{,}848n^4 + 380{,}928n^3 + 247{,}328n^2 + 78{,}096n + 10{,}395)} \\ &\quad \times \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 > 0 \end{aligned}$$

Table 1 Comparison between inequalities (3.15) and (4.2)

n	$a_n - \delta_n$	$\omega_n - b_n$
1	6.673798×10^{-3}	3.789512×10^{-3}
10	2.264856×10^{-13}	9.947434×10^{-12}
100	2.398663×10^{-24}	1.051407×10^{-20}
1,000	2.408054×10^{-35}	1.056218×10^{-29}
10,000	2.408948×10^{-46}	1.056690×10^{-38}

and

$$\begin{aligned} b_n - \mu_n &= -\frac{3(45{,}056n^4 + 62{,}976n^3 + 66{,}496n^2 + 21{,}876n + 945)}{1{,}024n^4(2n+1)(16{,}384n^5 + 20{,}480n^4 + 32{,}000n^3 + 18{,}880n^2 + 6{,}756n + 945)} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \\ &< 0. \end{aligned}$$

Hence, (3.15) improves (4.1).

The following numerical computations (see Table 1) would show that $\delta_n < a_n$ and $b_n < \omega_n$ for $n \in \mathbb{N}$. That is to say, inequalities (3.15) are sharper than inequalities (4.2).

In fact, we have

$$\begin{aligned} \lambda_n &= \pi + O\left(\frac{1}{n^7}\right), & \mu_n &= \pi + O\left(\frac{1}{n^5}\right), \\ \delta_n &= \pi + O\left(\frac{1}{n^{11}}\right), & \omega_n &= \pi + O\left(\frac{1}{n^9}\right), \\ a_n &= \pi + O\left(\frac{1}{n^{12}}\right), & b_n &= \pi + O\left(\frac{1}{n^{10}}\right). \end{aligned}$$

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Authors' contributions
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