Liu *Journal of Inequalities and Applications* (2017) 2017:117
DOI 10.1186/s13660-017-1394-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Skew log-concavity of the Boros-Moll sequences

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Abstract

Let $\{T(n,k)\}_{0\leq n<\infty, 0\leq k\leq n}$ be a triangular array of numbers. We say that $T(n,k)$ is skew log-concave if for any fixed n, the sequence $\{T(n+k,k)\}_{0\leq k<\infty}$ is log-concave. In this paper, we show that the Boros-Moll sequences are almost skew log-concave.

MSC: 05A20; 05A10

Keywords: log-concavity; skew log-concavity; the Boros-Moll sequence

1 Introduction and main result

Boros and Moll [1, 2] explored a special class of Jacobi polynomials in their study of a quartic integral. They have shown that for any $a > -1$ and any nonnegative integer m,

$$\int_0^{\infty} \frac{1}{(x^4+2ax^2+1)^{m+1}}\,dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a),$$

where

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(a+1)^j(a-1)^k}{2^{3(k+j)}}. \tag{1.1}$$

Using Ramanujan's master theorem, Boros and Moll [2] derived the following formula for $P_m(a)$:

$$P_m(a) = 2^{-2m}\sum_k 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k, \tag{1.2}$$

which implies that the coefficient of a^i in $P_m(a)$ is positive for $0 \leq i \leq m$. Let $d_i(m)$ be given by

$$P_m(a) = \sum_{i=0}^{m} d_i(m) a^i. \tag{1.3}$$

The polynomial $P_m(a)$ is called the Boros-Moll polynomial, and the sequence $\{d_i(m)\}_{0\leq i\leq m}$ of the coefficients is called a Boros-Moll sequence. From (1.3), we know that $d_i(m)$ can be given by

$$d_i(m) = 2^{-2m} \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i}. \tag{1.4}$$

Some combinatorial properties of $\{d_i(m)\}_{0\le i\le m}$ have been proved. Boros and Moll [1] proved that the sequence $\{d_i(m)\}_{0\le i\le m}$ is unimodal, and the maximum element appears in the middle. Recall that a sequence $\{a_i\}_{0\le i\le m}$ of real numbers is said to be unimodal if there exists an index $0 \le j \le m$ such that

$$a_0 \le a_1 \le \cdots \le a_{j-1} \le a_j \ge a_{j+1} \ge \cdots \ge a_m$$

and $\{a_i\}_{0\le i\le m}$ is said to be log-concave if

$$a_i^2 - a_{i+1}a_{i-1} \ge 0, \quad 1 \le i \le m, \tag{1.5}$$

where $a_{-1} = a_{m+1} = 0$. Moll [2] conjectured that the sequence $\{d_i(m)\}_{0\le i\le m}$ is log-concave. Kauers and Paule [3] proved this conjecture based on recurrence relations found using a computer algebra approach. Recently, Chen and Xia [4] showed that the sequence $\{d_i(m)\}_{0\le i\le m}$ satisfies the strongly ratio monotone property which implies the log-concavity and the spiral property. They [5] also confirmed a conjecture of Moll which says that $\{i(i+1)(d_i^2(m) - d_{i-1}(m)d_{i+1}(m))\}_{1\le i\le m}$ attains its minimum at $i = m$. Chen et al. [6] proved that the Boros-Moll sequences are interlacing log-concave. Chen and Gu [7] showed that the sequence $\{d_i(m)\}_{0\le i\le m}$ satisfies the reverse ultra log-concavity. Chen and Xia [8] proved that the Boros-Moll sequences are 2-log-concave, and Xia [9] studied the concavity and convexity of the Boros-Moll sequences.

In this paper, we give a new definition, i.e., skew log-concavity. Let $\{T(n,k)\}_{0\le n<\infty, 0\le k\le n}$ be a triangular array of numbers. We say that $T(n,k)$ is skew log-concave if for any fixed n, the sequence $\{T(n+k,k)\}_{0\le k<\infty}$ is log-concave. We will show that the Boros-Moll sequences are almost skew log-concave.

The main results of this paper can be stated as follows.

Theorem 1.1 *Let $d_i(m)$ be defined by* (1.4). *We have, for any fixed $m \ge 1$,*

$$d_i^2(m+i) > d_{i-1}(m+i-1)d_{i+1}(m+i+1), \quad i \ge 1, \tag{1.6}$$

and

$$d_i^2(i) < d_{i-1}(i-1)d_{i+1}(i+1), \quad i \ge 1. \tag{1.7}$$

2 Proof of Theorem 1.1

From (1.4), we see that $d_m(m) = 2^{-m}\binom{2m}{m}$, which implies that (1.7) holds.

By (1.4),

$$d_m(m+1) = \frac{(2m+3)(2m+1)}{2(m+1)} 2^{-m} \binom{2m}{m},$$

which yields

$$d_i^2(i+1) > d_{i-1}(i)d_{i+1}(i+2).$$

Therefore, (1.6) holds when $m = 1$.

Hence, in the following, we always assume that $m \geq 2$ and $i \geq 1$. We first recall the following three recurrence relations derived by Kauers and Paule [3]:

$$d_i(m+1) = \frac{m+i}{m+1}d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)}d_i(m), \quad 0 \leq i \leq m+1, \tag{2.1}$$

$$\begin{aligned} d_i(m+1) = {} & \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)}d_i(m) \\ & - \frac{i(i+1)}{(m+1)(m+1-i)}d_{i+1}(m), \quad 0 \leq i \leq m, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} d_i(m+2) = {} & \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)}d_i(m+1) \\ & - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)}d_i(m), \quad 0 \leq i \leq m+1. \end{aligned} \tag{2.3}$$

Now we represent the difference $d_i^2(m+i) - d_{i-1}(m+i-1)d_{i+1}(m+i+1)$ in terms of $d_i(m+i)$ and $d_i(m+i+1)$. Thanks to (2.1), (2.2) and (2.3),

$$\begin{aligned} & d_i^2(m+i) - d_{i-1}(m+i-1)d_{i+1}(m+i+1) \\ & \quad = Ad_i^2(m+i+1) + Bd_i(m+i+1)d_i(m+i) + Cd_i^2(m+i), \end{aligned} \tag{2.4}$$

where

$$A = \frac{(4m+6i+5)(m+1+i)(m+i)(m+1)^2(4m+6i-1)}{(i+1)i(4m+4i+1)(4m+4i-1)(m+2i)(m+2i-1)}, \tag{2.5}$$

$$B = -\frac{(m+1)(m+i)D}{(i+1)i(4m+4i+1)(4m+4i-1)(m+2i)(m+2i-1)}, \tag{2.6}$$

$$C = \frac{E}{4(m+2i-1)(m+2i)(4m+4i-1)(4m+4i+1)(m+1+i)i(i+1)} \tag{2.7}$$

with

$$\begin{aligned} D = {} & -15 + 400mi + 35i + 13m + 140m^2 + 292i^2 + 864mi^2 + 688m^2i \\ & + 176m^3 + 336i^3 + 64m^4 + 72i^4 + 320m^3i + 560m^2i^2 + 384mi^3, \end{aligned} \tag{2.8}$$

$$\begin{aligned} E = {} & -68mi - 45i - 45m - 66m^2 + 2i^2 + 2{,}614mi^2 + 1{,}901m^2i + 451m^3 + 1{,}164i^3 \\ & + 1{,}560m^4 + 3{,}320i^4 + 7{,}732m^3i + 14{,}176m^2i^2 + 11{,}328mi^3 + 1{,}152i^6 + 1{,}984m^5 \\ & + 3{,}392i^5 + 11{,}888m^4i + 16{,}856i^4m + 27{,}772m^3i^2 + 31{,}332m^2i^3 + 8{,}128m^5i \\ & + 23{,}040m^4i^2 + 9{,}216i^5m + 33{,}216m^3i^3 + 25{,}216m^2i^4 + 6{,}720m^5i^2 + 11{,}584m^4i^3 \\ & + 1{,}152i^6m + 11{,}072m^3i^4 + 5{,}568m^2i^5 + 2{,}048m^6i + 1{,}152m^6 + 256m^7. \end{aligned} \tag{2.9}$$

It is easy to check that

$$\Delta = B^2 - 4AC = \frac{(m+1)^2(m+i)F}{i(i+1)^2(4i+4m+1)^2(4i+4m-1)^2(2i+m)^2(2i+m-1)^2},$$

where

$$\begin{aligned} F ={}& 5{,}184i^8 + 19{,}008i^7m + 27{,}648i^6m^2 + 19{,}968i^5m^3 + 7{,}168i^4m^4 + 1{,}024i^3m^5 \\ &+ 6{,}912i^7 + 16{,}128i^6m + 768i^5m^2 - 33{,}024i^4m^3 - 44{,}288i^3m^4 - 26{,}880i^2m^5 \\ &- 8{,}192im^6 - 1{,}024m^7 + 5{,}184i^6 + 13{,}920i^5m + 9{,}584i^4m^2 - 5{,}936i^3m^3 \\ &- 11{,}648i^2m^4 - 5{,}888im^5 - 1{,}024m^6 + 6{,}096i^5 + 23{,}488i^4m + 35{,}600i^3m^2 \\ &+ 26{,}512i^2m^3 + 9{,}728im^4 + 1{,}408m^5 + 2{,}000i^4 + 7{,}232i^3m + 9{,}536i^2m^2 \\ &+ 5{,}360im^3 + 1{,}088m^4 - 1{,}048i^3 - 2{,}336i^2m - 1{,}728im^2 - 404m^3 \\ &- 143i^2 - 175im - 64m^2 + 40i + 20m. \end{aligned}$$

Note that A is positive. Hence, in order to prove that the right-hand side of (2.4) is positive, it suffices to prove that when Δ is nonnegative,

$$\frac{d_i(m+i+1)}{d_i(m+i)} > \frac{-B+\sqrt{\Delta}}{2A}. \tag{2.10}$$

Therefore, in the following, we assume that $\Delta \geq 0$.

Recall that Kauers and Paule [3] proved the following inequality:

$$\frac{d_i(m+1)}{d_i(m)} \geq \frac{4m^2+7m+i+3}{2(m+1)(m+1-i)}, \quad 0 \leq i \leq m.$$

Replacing m by $m+i$, we see that

$$\frac{d_i(m+i+1)}{d_i(m+i)} \geq \frac{4i^2+8im+4m^2+8i+7m+3}{2(m+1+i)(m+1)}, \quad i \geq 0. \tag{2.11}$$

It is a routine to verify that

$$\begin{aligned} &\left(A\frac{4i^2+8im+4m^2+8i+7m+3}{(m+1+i)(m+1)} + B\right)^2 - \Delta \\ &\quad = \frac{4(i+m)(m+1)^2(6i+4m+5)(6i+4m-1)G}{i(i+1)^2(4i+4m+1)^2(4i+4m-1)^2(2i+m)^2(2i+m-1)^2}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} G ={}& 28i^4m + 108i^3m^2 + 144i^2m^3 + 80im^4 + 16m^5 - 32i^4 - 66i^3m \\ &- 46i^2m^2 - 12im^3 - 32i^3 - 78i^2m - 64im^2 - 17m^3 + 2i^2 + 2im + 2i + m. \end{aligned}$$

Note that when $m \geq 2$ and $i \geq 1$, G is positive. Thus the right-hand side of (2.12) is positive. On the other hand,

$$\begin{aligned} &A\frac{4i^2+8im+4m^2+8i+7m+3}{(m+1+i)(m+1)}+B \\ &\quad=\frac{(i+m)(m+1)(-3-12i+28im+48i^2+72i^3+32im^2+96i^2m)}{(i+1)(4i+4m+1)(4i+4m-1)(2i+m)(2i+m-1)}, \end{aligned}$$

which is positive. Therefore, from (2.12), we have

$$A\frac{4i^2+8im+4m^2+8i+7m+3}{(m+1+i)(m+1)}+B>\Delta,$$

which can be rewritten as

$$\frac{4i^2+8im+4m^2+8i+7m+3}{2(m+1+i)(m+1)}>\frac{-B+\sqrt{\Delta}}{2A}. \tag{2.13}$$

From (2.11) and (2.13), we obtain (2.10) and this completes the proof.

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This work was supported by the National Science Foundation of China (11526136, 11501356).

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Received: 14 January 2017 Accepted: 2 May 2017 Published online: 18 May 2017

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