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A new localization set for generalized eigenvalues

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Abstract

A new localization set for generalized eigenvalues is obtained. It is shown that the new set is tighter than that in (Numer. Linear Algebra Appl. 16:883-898, 2009). Numerical examples are given to verify the corresponding results.

MSC: 15A45; 15A48

Keywords: generalized eigenvalue; inclusion set; matrix pencil

1 Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all complex matrices of order n. For the matrices $A, B \in \mathbb{C}^{n\times n}$, we call the family of matrices $A - zB$ a matrix pencil, which is parameterized by the complex number z. Next, we regard a matrix pencil $A - zB$ as a matrix pair (A, B) [1]. A matrix pair (A, B) is called regular if $\det(A - zB) \not\equiv 0$, and otherwise singular. A complex number λ is called a generalized eigenvalue of (A, B), if

$$\det(A - \lambda B) = 0.$$

Furthermore, we call a nonzero vector $x \in \mathbb{C}^n$ a generalized eigenvector of (A, B) associated with λ if

$$Ax = \lambda Bx.$$

Let $\sigma(A, B) = \{\lambda \in \mathbb{C} : \det(A - \lambda B) = 0\}$ denote the generalized spectrum of (A, B). Clearly, if B is an identity matrix, then $\sigma(A, B)$ reduces to the spectrum of A, *i.e.* $\sigma(A, B) = \sigma(A)$. When B is nonsingular, $\sigma(A, B)$ is equivalent to the spectrum of $B^{-1}A$, that is,

$$\sigma(A, B) = \sigma\left(B^{-1}A\right).$$

So, in this case, (A, B) has n generalized eigenvalues. Moreover, if B is singular, then the degree of the characteristic polynomial $\det(A - \lambda B)$ is $d < n$, so the number of generalized eigenvalues of the matrix pair (A, B) is d, and, by convention, the remaining $n - d$ eigenvalues are ∞ [1, 2].

We now list some notation which will be used in the following. Let $N = \{1, 2, \dots, n\}$. Given two matrices $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{n\times n}$, we denote

$$r_i(A) = \sum_{\substack{k\in N,\\ k\neq i}} |a_{ik}|, \qquad r_i^j(A) = \sum_{\substack{k\in N,\\ k\neq i,j}} |a_{ik}|,$$

$$R_i(A,B,z) = \sum_{\substack{k\in N,\\ k\neq i}} |a_{ik} - zb_{ik}|, \qquad R_i^j(A,B,z) = \sum_{\substack{k\in N,\\ k\neq i,j}} |a_{ik} - zb_{ik}|,$$

$$\Gamma_i(A,B) = \bigl\{z \in \mathbb{C} : |a_{ii} - zb_{ii}| \le R_i(A,B,z)\bigr\},$$

and

$$\begin{aligned}\Phi_{ij}(A,B) = \bigl\{z \in \mathbb{C} : &\bigl|(a_{ii} - zb_{ii})(a_{jj} - zb_{jj}) - (a_{ij} - zb_{ij})(a_{ji} - zb_{ji})\bigr| \\ &\le |a_{jj} - zb_{jj}|R_i^j(A,B,z) + |a_{ij} - zb_{ij}|R_j^i(A,B,z)\bigr\}.\end{aligned}$$

The generalized eigenvalue problem arises in many scientific applications; see [3–5]. Many researchers are interested in the localization of all generalized eigenvalues of a matrix pair; see [1, 2, 6, 7]. In [1], Kostić *et al.* provided the following Geršgorin-type theorem of the generalized eigenvalue problem.

Theorem 1 ([1], Theorem 7) *Let $A, B \in \mathbb{C}^{n\times n}$, $n \ge 2$ and (A,B) be a regular matrix pair. Then*

$$\sigma(A,B) \subseteq \Gamma(A,B) = \bigcup_{i\in N} \Gamma_i(A,B).$$

Here, $\Gamma(A,B)$ is called the generalized Geršgorin set of a matrix pair (A,B) and $\Gamma_i(A,B)$ the ith generalized Geršgorin set. As showed in [1], $\Gamma(A,B)$ is a compact set in the complex plane if and only if B is strictly diagonally dominant (*SDD*) [8]. When B is not *SDD*, $\Gamma(A,B)$ may be an unbounded set or the entire complex plane (see Theorem 2).

Theorem 2 ([1], Theorem 8) *Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$. Then the following statements hold*:

(i) *Let $i \in N$ be such that, for at least one $j \in N$, $b_{ij} \neq 0$. Then $\Gamma_i(A,B)$ is an unbounded set in the complex plane if and only if $|b_{ii}| \le r_i(B)$.*

(ii) *$\Gamma(A,B)$ is a compact set in the complex plane if and only if B is SDD, that is, $|b_{ii}| > r_i(B)$.*

(iii) *If there is an index $i \in N$ such that both $b_{ii} = 0$ and*

$$|a_{ii}| \le \sum_{\substack{k\in\beta(i),\\ k\neq i}} |a_{ik}|,$$

where $\beta(i) = \{j \in N : b_{ij} = 0\}$, then $\Gamma_i(A,B)$, and consequently $\Gamma(A,B)$, is the entire complex plane.

Recently, in [2], Nakatsukasa presented a different Geršgorin-type theorem to estimate all generalized eigenvalues of a matrix pair (A,B) for the case that the ith row of either

A (or B) is *SDD* for any $i \in N$. Although the set obtained by Nakatsukasa is simpler to compute than that in Theorem 1, the set is not tighter than that in Theorem 1 in general.

In this paper, we research the generalized eigenvalue localization for a regular matrix pair (A, B) without the restrictive assumption that the ith row of either A (or B) is *SDD* for any $i \in N$. By considering $Ax = \lambda Bx$ and using the triangle inequality, we give a new inclusion set for generalized eigenvalues, and then prove that this set is tighter than that in Theorem 1 (Theorem 7 of [1]). Numerical examples are given to verify the corresponding results.

2 Main results

In this section, a set is provided to locate all the generalized eigenvalue of a matrix pair. Next we compare the set obtained with the generalized Geršgorin set in Theorem 1.

2.1 A new generalized eigenvalue localization set

Theorem 3 *Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{n\times n}$, with $n \geq 2$ and (A, B) be a regular matrix pair. Then*

$$\sigma(A,B) \subseteq \Phi(A,B) = \bigcup_{\substack{i,j=1,\\ i\neq j}}^{n} \left\{ \Phi_{ij}(A,B) \cap \Phi_{ji}(A,B) \right\}.$$

Proof For any $\lambda \in \sigma(A, B)$, let $0 \neq x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$ be an associated generalized eigenvector, *i.e.*,

$$Ax = \lambda Bx. \tag{1}$$

Without loss of generality, let

$$|x_p| \geq |x_q| \geq \max\left\{|x_i| : i \in N, i \neq p, q\right\}.$$

Then $x_p \neq 0$.

(i) If $x_q \neq 0$, then from Equality (1), we have

$$a_{pp}x_p + a_{pq}x_q + \sum_{\substack{k\in N,\\ k\neq p,q}} a_{pk}x_k = \lambda b_{pp}x_p + \lambda b_{pq}x_q + \lambda \sum_{\substack{k\in N,\\ k\neq p,q}} b_{pk}x_k$$

and

$$a_{qq}x_q + a_{qp}x_p + \sum_{\substack{k\in N,\\ k\neq q,p}} a_{qk}x_k = \lambda b_{qq}x_q + \lambda b_{qp}x_p + \lambda \sum_{\substack{k\in N,\\ k\neq q,p}} b_{qk}x_k,$$

equivalently,

$$(a_{pp} - \lambda b_{pp})x_p + (a_{pq} - \lambda b_{pq})x_q = -\sum_{\substack{k\in N,\\ k\neq p,q}} (a_{pk} - \lambda b_{pk})x_k \tag{2}$$

and

$$(a_{qq} - \lambda b_{qq})x_q + (a_{qp} - \lambda b_{qp})x_p = -\sum_{\substack{k\in N,\\ k\neq q,p}} (a_{qk} - \lambda b_{qk})x_k. \tag{3}$$

Solving for x_p and x_q in (2) and (3), we obtain

$$\begin{aligned}&\big((a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big)x_p\\ &\quad = -(a_{qq} - \lambda b_{qq})\sum_{\substack{k\in N,\\ k\neq p,q}} (a_{pk} - \lambda b_{pk})x_k + (a_{pq} - \lambda b_{pq})\sum_{\substack{k\in N,\\ k\neq q,p}} (a_{qk} - \lambda b_{qk})x_k\end{aligned} \tag{4}$$

and

$$\begin{aligned}&\big((a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big)x_q\\ &\quad = -(a_{pp} - \lambda b_{pp})\sum_{\substack{k\in N,\\ k\neq q,p}} (a_{qk} - \lambda b_{qk})x_k + (a_{qp} - \lambda b_{qp})\sum_{\substack{k\in N,\\ k\neq p,q}} (a_{pk} - \lambda b_{pk})x_k.\end{aligned} \tag{5}$$

Taking absolute values of (4) and (5) and using the triangle inequality yield

$$\begin{aligned}&\big|(a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big||x_p|\\ &\quad \leq |a_{qq} - \lambda b_{qq}|\sum_{\substack{k\in N,\\ k\neq p,q}} |a_{pk} - \lambda b_{pk}||x_k| + |a_{pq} - \lambda b_{pq}|\sum_{\substack{k\in N,\\ k\neq q,p}} |a_{qk} - \lambda b_{qk}||x_k|\end{aligned}$$

and

$$\begin{aligned}&\big|(a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big||x_q|\\ &\quad \leq |a_{pp} - \lambda b_{pp}|\sum_{\substack{k\in N,\\ k\neq q,p}} |a_{qk} - \lambda b_{qk}||x_k| + |a_{qp} - \lambda b_{qp}|\sum_{\substack{k\in N,\\ k\neq p,q}} |a_{pk} - \lambda b_{pk}||x_k|.\end{aligned}$$

Since $x_p \neq 0$ and $x_q \neq 0$ are, in absolute value, the largest and second largest components of x, respectively, we divide through by their absolute values to obtain

$$\begin{aligned}&\big|(a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big|\\ &\quad \leq |a_{qq} - \lambda b_{qq}|R_p^q(A,B,\lambda) + |a_{pq} - \lambda b_{pq}|R_p^q(A,B,\lambda)\end{aligned}$$

and

$$\begin{aligned}&\big|(a_{pp} - \lambda b_{pp})(a_{qq} - \lambda b_{qq}) - (a_{pq} - \lambda b_{pq})(a_{qp} - \lambda b_{qp})\big|\\ &\quad \leq |a_{pp} - \lambda b_{pp}|R_q^p(A,B,\lambda) + |a_{qp} - \lambda b_{qp}|R_p^q(A,B,\lambda).\end{aligned}$$

Hence,

$$\lambda \in \Big(\Phi_{pq}(A,B) \cap \Phi_{qp}(A,B)\Big) \subseteq \Phi(A,B).$$

(ii) If $x_q = 0$, then x_p is the only nonzero entry of x. From equality (1), we have

$$A\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_p \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1p}x_p \\ \vdots \\ a_{p-1,p}x_p \\ a_{pp}x_p \\ a_{p+1,p}x_p \\ \vdots \\ a_{np}x_p \end{pmatrix} = \lambda \begin{pmatrix} b_{1p}x_p \\ \vdots \\ b_{p-1,p}x_p \\ b_{pp}x_p \\ b_{p+1,p}x_p \\ \vdots \\ b_{np}x_p \end{pmatrix},$$

which implies that, for any $i \in N$, $a_{ip} = \lambda b_{ip}$, *i.e.*, $a_{ip} - \lambda b_{ip} = 0$. Hence for any $i \in N$, $i \neq p$,

$$\lambda \in \left(\Phi_{pi}(A,B) \cap \Phi_{ip}(A,B)\right) \subseteq \Phi(A,B).$$

From (i) and (ii), $\sigma(A,B) \subseteq \Phi(A,B)$. The proof is completed. □

Since the matrix pairs (A,B) and (A^T,B^T) have the same generalized eigenvalues, we can obtain a theorem by applying Theorem 3 to (A^T,B^T).

Theorem 4 *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $B = (b_{ij}) \in \mathbb{C}^{n\times n}$, with $n \geq 2$, and (A^T,B^T) be a regular matrix pair. Then*

$$\sigma(A,B) \subseteq \Phi\left(A^T,B^T\right).$$

Remark 1 If B is an identity matrix, then Theorems 3 and 4 reduce to the corresponding results of [9].

Remark 2 When all entries of the ith and jth rows of the matrix B are zero, then

$$\Phi_{ij}(A,B) = \left\{z \in \mathbb{C} : |a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{jj}|r_i^j(A) + |a_{ij}|r_j^i(A)\right\}$$

and

$$\Phi_{ji}(A,B) = \left\{z \in \mathbb{C} : |a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{ii}|r_j^i(A) + |a_{ji}|r_i^j(A)\right\}.$$

Hence, if

$$|a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{jj}|r_i^j(A) + |a_{ij}|r_j^i(A) \tag{6}$$

and

$$|a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{ii}|r_j^i(A) + |a_{ji}|r_i^j(A), \tag{7}$$

then

$$\Phi_{ij}(A,B) \cap \Phi_{ji}(A,B) = \mathbb{C},$$

otherwise,

$$\Phi_{ij}(A,B) \cap \Phi_{ji}(A,B) = \emptyset.$$

Moreover, when inequalities (6) and (7) hold, the matrix B is singular, and $\det(A - zB)$ has degree less than n. As we are considering regular matrix pairs, the degree of the polynomial $\det(A - zB)$ has to be at least one; thus, at least one of the sets $\Phi_{ij}(A,B) \cap \Phi_{ji}(A,B)$ has to be nonempty, implying that the set $\Phi(A,B)$ of a regular matrix pair is always nonempty.

We now establish the following properties of the set $\Phi(A,B)$.

Theorem 5 *Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{n\times n}$, with $n \geq 2$ and (A,B) be a regular matrix pair. Then the set $\Phi_{ij}(A,B) \cap \Phi_{ji}(A,B)$ contains zero if and only if inequalities* (6) *and* (7) *hold.*

Proof The conclusion follows directly from putting $z = 0$ in the inequalities of $\Phi_{ij}(A,B)$ and $\Phi_{ji}(A,B)$. □

Theorem 6 *Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{n\times n}$, with $n \geq 2$ and (A,B) be a regular matrix pair. If there exist $i,j \in N$, $i \neq j$, such that*

$$b_{ii} = b_{jj} = b_{ij} = b_{ji} = 0,$$
$$|a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{jj}| \sum_{\substack{k\in\beta(i),\\ k\neq i,j}} |a_{ik}| + |a_{ij}| \sum_{\substack{k\in\beta(j),\\ k\neq j,i}} |a_{jk}|,$$

and

$$|a_{ii}a_{jj} - a_{ij}a_{ji}| \leq |a_{ii}| \sum_{\substack{k\in\beta(j),\\ k\neq j,i}} |a_{jk}| + |a_{ji}| \sum_{\substack{k\in\beta(i),\\ k\neq i,j}} |a_{ik}|,$$

where $\beta(i) = \{k \in N : b_{ik} = 0\}$, then $\Phi_{ij}(A,B) \cap \Phi_{ji}(A,B)$, and consequently $\Phi(A,B)$ is the entire complex plane.

Proof The conclusion follows directly from the definitions of $\Phi_{ij}(A,B)$ and $\Phi_{ji}(A,B)$. □

2.2 Comparison with the generalized Geršgorin set

We now compare the set in Theorem 3 with the generalized Geršgorin set in Theorem 1. First, we observe two examples in which the generalized Geršgorin set is an unbounded set or the entire complex plane.

Example 1 Let

$$A = (a_{ij}) = \begin{pmatrix} -1 & 1 & 0 & 0.2 \\ 0 & 1 & 0.4 & 0 \\ 0 & 0 & i & 1 \\ 0.2 & 0 & 0 & -i \end{pmatrix}, \qquad B = (b_{ij}) = \begin{pmatrix} 0.3 & 0.1 & 0.1 & 0.1 \\ 0 & -1 & 0.1 & 0.1 \\ 0 & 0 & i & 0.1 \\ 0.1 & 0 & 0 & -0.2i \end{pmatrix}.$$

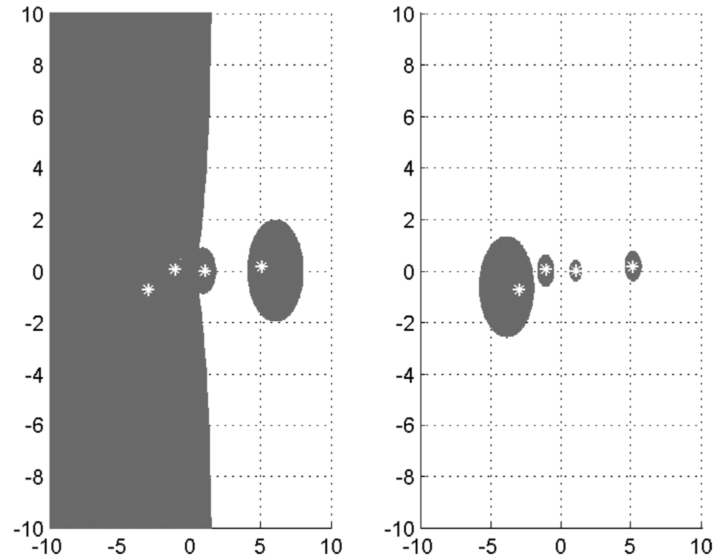

Figure 1 $\Gamma(A,B)$ **of Example 1 on the left, and** $\Phi(A,B)$ **on the right.**

It is easy to see that $b_{12} = 0.1 > 0$ and

$$|b_{11}| = \sum_{k=2,3,4} |b_{1k}| = 0.3.$$

Hence, from the part (i) of Theorem 2, we see that $\Gamma(A,B)$ is unbounded. However, the set $\Phi(A,B)$ in Theorem 3 is compact. These sets are given by Figure 1, where the actual generalized eigenvalues are plotted with asterisks. Clearly, $\Phi(A,B) \subset \Gamma(A,B)$.

Example 2 Let

$$A = (a_{ij}) = \begin{pmatrix} -1 & 1 & 0 & 0.2 \\ 0 & 1 & 0.4 & 0 \\ 0 & 0 & i & 1 \\ 0.2 & 0 & 0 & -i \end{pmatrix}, \qquad B = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0.1 & 0.1 \\ 0 & -1 & 0.1 & 0.1 \\ 0 & 0 & i & 0.1 \\ 0.1 & 0 & 0 & -0.2i \end{pmatrix}.$$

It is easy to see that $b_{11} = 0$, $\beta(1) = \{2\}$ and

$$|a_{11}| = \sum_{\substack{k\in\beta(1),\\ k\neq 1}} |a_{1k}| = |a_{12}| = 1.$$

Hence, from the part (iii) of Theorem 2, we see that $\Gamma(A,B)$ is the entire complex plane, but the set $\Phi(A,B)$ in Theorem 3 is not. $\Phi(A,B)$ is given by Figure 2, where the actual generalized eigenvalues are plotted with asterisks.

We establish their comparison in the following.

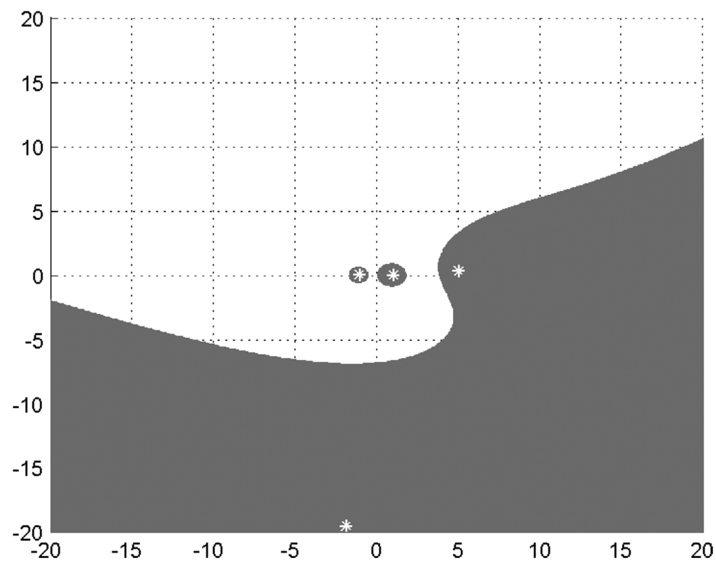

Figure 2 **$\Phi(A, B)$ of Example 2.**

Theorem 7 *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $B = (b_{ij}) \in \mathbb{C}^{n\times n}$, with $n \geq 2$ and (A, B) be a regular matrix pair. Then*

$$\Phi(A, B) \subseteq \Gamma(A, B).$$

Proof Let $z \in \Phi(A, B)$. Then there are $i, j \in N$, $i \neq j$ such that

$$z \in \Big(\Phi_{ij}(A, B) \cap \Phi_{ji}(A, B)\Big).$$

Next, we prove that

$$\Phi_{ij}(A, B) \subseteq \Big(\Gamma_i(A, B) \cup \Gamma_j(A, B)\Big) \tag{8}$$

and

$$\Phi_{ji}(A, B) \subseteq \Big(\Gamma_i(A, B) \cup \Gamma_j(A, B)\Big). \tag{9}$$

(i) For $z \in \Phi_{ij}(A, B)$, then $z \in \Gamma_i(A, B)$ or $z \notin \Gamma_i(A, B)$. If $z \in \Gamma_i(A, B)$, then (8) holds. If $z \notin \Gamma_i(A, B)$, that is,

$$|a_{ii} - zb_{ii}| > R_i(A, B, z), \tag{10}$$

then

$$\begin{aligned} &|a_{jj} - zb_{jj}|R_i^j(A, B, z) + |a_{ij} - zb_{ij}|R_i^j(A, B, z) \\ &\quad \geq \big|(a_{ii} - zb_{ii})(a_{jj} - zb_{jj}) - (a_{ij} - zb_{ij})(a_{ji} - zb_{ji})\big| \\ &\quad \geq |a_{ii} - zb_{ii}||a_{jj} - zb_{jj}| - |a_{ij} - zb_{ij}||a_{ji} - zb_{ji}|. \end{aligned} \tag{11}$$

Note that $R_i^j(A,B,z) = R_i(A,B,z) - |a_{ij} - zb_{ij}|$ and $R_j^i(A,B,z) = R_j(A,B,z) - |a_{ji} - zb_{ji}|$. Then from inequalities (10) and (11), we have

$$\begin{aligned}&|a_{jj} - zb_{jj}|\bigl(R_i(A,B,z) - |a_{ij} - zb_{ij}|\bigr) + |a_{ij} - zb_{ij}|\bigl(R_j(A,B,z) - |a_{ji} - zb_{ji}|\bigr)\\&\quad\geq |a_{jj} - zb_{jj}|R_i(A,B,z) - |a_{ij} - zb_{ij}||a_{ji} - zb_{ji}|,\end{aligned}$$

which implies that

$$|a_{ij} - zb_{ij}|R_j(A,B,z) \geq |a_{ij} - zb_{ij}||a_{jj} - zb_{jj}|. \tag{12}$$

If $a_{ij} = zb_{ij}$, then from $z \in \Phi_{ij}(A,B)$, we have

$$|a_{ii} - zb_{ii}||a_{jj} - zb_{jj}| \leq |a_{jj} - zb_{jj}|R_i^j(A,B,z) \leq |a_{jj} - zb_{jj}|R_i(A,B,z).$$

Moreover, from inequality (10), we obtain $|a_{jj} - zb_{jj}| = 0$. It is obvious that

$$z \in \Gamma_j(A,B) \subseteq \Bigl(\Gamma_i(A,B) \cup \Gamma_j(A,B)\Bigr).$$

If $a_{ij} \neq zb_{ij}$, then from inequality (12), we have

$$|a_{jj} - zb_{jj}| \leq R_j(A,B,z),$$

that is,

$$z \in \Gamma_j(A,B) \subseteq \Bigl(\Gamma_i(A,B) \cup \Gamma_j(A,B)\Bigr).$$

Hence, (8) holds.

(ii) Similar to the proof of (i), we also see that, for $z \in \Phi_{ji}(A,B)$, (9) holds.

The conclusion follows from (i) and (ii). □

Since the matrix pairs (A,B) and (A^T,B^T) have the same generalized eigenvalues, we can obtain a theorem by applying Theorem 7 to (A^T,B^T).

Theorem 8 *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $B = (b_{ij}) \in \mathbb{C}^{n\times n}$, with $n \geq 2$ and (A^T,B^T) be a regular matrix pair. Then*

$$\Phi\bigl(A^T,B^T\bigr) \subseteq \Gamma\bigl(A^T,B^T\bigr).$$

Example 3 ([1], Example 1) Let

$$A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 0 & 0.2\\ 0 & -1 & 0.4 & 0\\ 0 & 0 & i & 1\\ 0.2 & 0 & 0 & -i\end{pmatrix}, \qquad B = (b_{ij}) = \begin{pmatrix} 0.5 & 0.1 & 0.1 & 0.1\\ 0 & -1 & 0.1 & 0.1\\ 0 & 0 & i & 0.1\\ 0.1 & 0 & 0 & -0.5i\end{pmatrix}.$$

It is easy to see that B is *SDD*. Hence, from the part (ii) of Theorem 2, we see that $\Gamma(A,B)$ is compact. $\Gamma(A,B)$ and $\Phi(A,B)$ are given by Figure 3, where the exact generalized eigenvalues are plotted with asterisks. Clearly, $\Phi(A,B) \subset \Gamma(A,B)$.

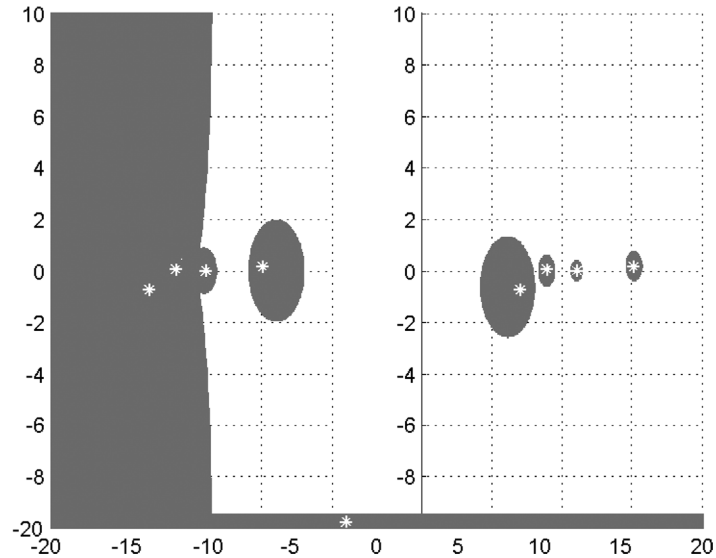

Figure 3 $\Gamma(A,B)$ **of Example 3 on the left, and** $\Phi(A,B)$ **on the right.**

Remark 3 From Examples 1, 2 and 3, we see that the set in Theorem 3 is tighter than that in Theorem 1 (Theorem 7 of [1]). In addition, note that A and B in Example 1 satisfy

$$|a_{11}| = 1 < \sum_{k=2,3,4} |a_{1k}| = 1.2$$

and

$$|b_{11}| = \sum_{k=2,3,4} |b_{1k}| = 0.3,$$

respectively. Hence, we cannot use the method in [2] to estimate the generalized eigenvalues of the matrix pair (A,B). However, the set we obtain is very compact.

3 Conclusions

In this paper, we present a new generalized eigenvalue localization set $\Phi(A,B)$, and we establish the comparison of the sets $\Phi(A,B)$ and $\Gamma(A,B)$ in Theorem 7 of [1], that is, $\Phi(A,B)$ captures all generalized eigenvalues more precisely than $\Gamma(A,B)$, which is shown by three numerical examples.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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