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A new S-type upper bound for the largest singular value of nonnegative rectangular tensors

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Abstract

By breaking $N = \{1, 2, \ldots, n\}$ into disjoint subsets S and its complement, a new S-type upper bound for the largest singular value of nonnegative rectangular tensors is given and proved to be better than some existing ones. Numerical examples are given to verify the theoretical results.

MSC: 15A18; 15A42; 15A69

Keywords: nonnegative tensor; rectangular tensor; singular value

1 Introduction

Singular value problems of rectangular tensors have become an important topic in applied mathematics and numerical multilinear algebra, and it has a wide range of practical applications, such as the strong ellipticity condition problem in solid mechanics [1, 2] and the entanglement problem in quantum physics [3, 4].

Let $\mathbb{R}$ (respectively, $\mathbb{C}$) be the real (respectively, complex) field. Assume that p, q, m, n are positive integers, $m, n \geq 2$, $l = p + q$, and $N = \{1, 2, \ldots, n\}$. A real (p, q)th order $m \times n$ dimensional rectangular tensor (or simply a real rectangular tensor) $\mathcal{A}$ is defined as follows:

$$\mathcal{A} = (a_{i_1 \cdots i_p j_1 \cdots j_q}), \quad a_{i_1 \cdots i_p j_1 \cdots j_q} \in \mathbb{R}, 1 \leq i_1, \ldots, i_p \leq m, 1 \leq j_1, \ldots, j_q \leq n.$$

A real rectangular tensor $\mathcal{A}$ is called nonnegative if $a_{i_1 \cdots i_p j_1 \cdots j_q} \geq 0$ for $i_k = 1, \ldots, m, k = 1, \ldots, p$, and $j_v = 1, \ldots, n, v = 1, \ldots, q$.

For vectors $x = (x_1, \ldots, x_m)^{\mathrm{T}}$, $y = (y_1, \ldots, y_n)^{\mathrm{T}}$ and a real number α, let $x^{[\alpha]} = (x_1^\alpha, x_2^\alpha, \ldots, x_m^\alpha)^{\mathrm{T}}$, $y^{[\alpha]} = (y_1^\alpha, y_2^\alpha, \ldots, y_n^\alpha)^{\mathrm{T}}$, $\mathcal{A}x^{p-1}y^q$ be an m dimension real vector whose ith component is

$$\left(\mathcal{A}x^{p-1}y^q\right)_i = \sum_{i_2, \ldots, i_p = 1}^{m} \sum_{j_1, \ldots, j_q = 1}^{n} a_{i i_2 \cdots i_p j_1 \cdots j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q},$$

and $\mathcal{A}x^p y^{q-1}$ be an n dimension real vector whose jth component is

$$\left(\mathcal{A}x^p y^{q-1}\right)_j = \sum_{i_1,\dots,i_p=1}^{m} \sum_{j_2,\dots,j_q=1}^{n} a_{i_1\cdots i_p j j_2\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q}.$$

If $\lambda \in \mathbb{C}$, $x \in \mathbb{C}^m \backslash \{0\}$, and $y \in \mathbb{C}^n \backslash \{0\}$ are solutions of

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda x^{[l-1]}, \\ \mathcal{A}x^p y^{q-1} = \lambda y^{[l-1]}, \end{cases}$$

then we say that λ is a singular value of $\mathcal{A}$, x and y are a left and a right eigenvectors of $\mathcal{A}$, associated with λ. If $\lambda \in \mathbb{R}, x \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$, then we say that λ is an H-singular value of $\mathcal{A}$, x and y are a left and a right H-eigenvectors of $\mathcal{A}$, associated with H-singular value λ [5]. Here,

$$\lambda_0 = \max\left\{|\lambda| : \lambda \text{ is a singular value of } \mathcal{A}\right\}$$

is called the largest singular value [6].

The definition of singular values for tensors was first introduced in [7]. Note here that when l is even, the definitions in [5] is the same as in [7], and when l is odd, the definition in [5] is slightly different from that in [7], but parallel to the definition of eigenvalues of square matrices [8]; see [5] for details.

Recently, many people focus on bounding the largest singular value for nonnegative rectangular tensors [6, 9, 10]. For convenience, we first give some notation. Given a nonempty proper subset S of N, we denote

$$\begin{aligned}
\Delta^N &:= \left\{(i_2,\dots,i_p,j_1,\dots,j_q) : i_2,\dots,i_p,j_1,\dots,j_q \in N\right\}, \\
\Delta^S &:= \left\{(i_2,\dots,i_p,j_1,\dots,j_q) : i_2,\dots,i_p,j_1,\dots,j_q \in S\right\}, \\
\Omega^N &:= \left\{(i_1,\dots,i_p,j_2,\dots,j_q) : i_1,\dots,i_p,j_2,\dots,j_q \in N\right\}, \\
\Omega^S &:= \left\{(i_1,\dots,i_p,j_2,\dots,j_q) : i_1,\dots,i_p,j_2,\dots,j_q \in S\right\},
\end{aligned}$$

and then

$$\overline{\Delta^S} = \Delta^N \backslash \Delta^S, \qquad \overline{\Omega^S} = \Omega^N \backslash \Omega^S.$$

This implies that, for a nonnegative rectangular tensor $\mathcal{A} = (a_{i_1\cdots i_p j_1\cdots j_q})$, we have, for $i, j \in S$,

$$\begin{aligned}
r_i(\mathcal{A}) &= \sum_{\substack{i_2,\dots,i_p,j_1,\dots,j_q \in N \\ \delta_{i i_2\cdots i_p j_1\cdots j_q}=0}} a_{i i_2\cdots i_p j_1\cdots j_q} = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\overline{\Delta^S}}(\mathcal{A}), \quad r_i^j(\mathcal{A}) = r_i(\mathcal{A}) - a_{ij\cdots jj\cdots j}, \\
c_j(\mathcal{A}) &= \sum_{\substack{i_1,\dots,i_p,j_2,\dots,j_q \in N \\ \delta_{i_1\cdots i_p j j_2\cdots j_q}=0}} a_{i_1\cdots i_p j j_2\cdots j_q} = c_j^{\Omega^S}(\mathcal{A}) + c_j^{\overline{\Omega^S}}(\mathcal{A}), \quad c_j^i(\mathcal{A}) = c_j(\mathcal{A}) - a_{i\cdots iji\cdots i},
\end{aligned}$$

where

$$\delta_{i_1\cdots i_p j_1\cdots j_q} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_p = j_1 = \cdots = j_q, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$r_i^{\Delta^S}(\mathcal{A}) = \sum_{\substack{(i_2,\ldots,i_p,j_1,\ldots,j_q)\in\Delta^S \\ \delta_{ii_2\cdots i_p j_1\cdots j_q}=0}} a_{ii_2\cdots i_p j_1\cdots j_q}, \qquad r_i^{\overline{\Delta^S}}(\mathcal{A}) = \sum_{(i_2,\ldots,i_p,j_1,\ldots,j_q)\in\overline{\Delta^S}} a_{ii_2\cdots i_p j_1\cdots j_q},$$

$$c_j^{\Omega^S}(\mathcal{A}) = \sum_{\substack{(i_1,\ldots,i_p,j_2,\ldots,j_q)\in\Omega^S \\ \delta_{i_1\cdots i_p jj_2\cdots j_q}=0}} a_{i_1\cdots i_p jj_2\cdots j_q}, \qquad c_j^{\overline{\Omega^S}}(\mathcal{A}) = \sum_{(i_1,\ldots,i_p,j_2,\ldots,j_q)\in\overline{\Omega^S}} a_{i_1\cdots i_p jj_2\cdots j_q}.$$

In [6], Yang and Yang gave the following bound for the largest singular value of a nonnegative rectangular tensor $\mathcal{A}$.

Theorem 1 ([6], Theorem 4) *Let $\mathcal{A}$ be a (p,q)th order $m \times n$ dimensional nonnegative rectangular tensor. Then*

$$\lambda_0 \le \max_{1\le i\le m, 1\le j\le n} \{R_i(\mathcal{A}), C_j(\mathcal{A})\},$$

where

$$R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_p=1}^{m} \sum_{j_1,\ldots,j_q=1}^{n} a_{ii_2\cdots i_p j_1\cdots j_q}, \qquad C_j(\mathcal{A}) = \sum_{i_1,\ldots,i_p=1}^{m} \sum_{j_2,\ldots,j_q=1}^{n} a_{i_1\cdots i_p jj_2\cdots j_q}.$$

When $m = n$, He *et al.* [9] have given an upper bound which is lower than that in Theorem 1.

Theorem 2 ([9], Theorem 1.3) *Let $\mathcal{A}$ be a (p,q)th order $n \times n$ dimensional nonnegative rectangular tensor. Then*

$$\lambda_0 \le \Phi(\mathcal{A}) = \max\{\Phi_1(\mathcal{A}), \Phi_2(\mathcal{A}), \Phi_3(\mathcal{A}), \Phi_4(\mathcal{A})\},$$

where

$$\begin{aligned}
\Phi_1(\mathcal{A}) = \max_{i,j\in N, i\neq j} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_i^j(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j} r_j(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
\Phi_2(\mathcal{A}) = \max_{i,j\in N, i\neq j} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + c_i^j(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + c_i^j(\mathcal{A})\big)^2 + 4a_{j\cdots jij\cdots j} c_j(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
\Phi_3(\mathcal{A}) = \max_{i,j\in N, i\neq j} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_i^j(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j} c_j(\mathcal{A})\Big]^{\frac{1}{2}}\Big\},
\end{aligned}$$

$$\Phi_4(\mathcal{A}) = \max_{i,j\in N, i\neq j} \frac{1}{2}\Big\{a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + c_i^j(\mathcal{A}) + \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + c_i^j(\mathcal{A})\big)^2 + 4a_{j\cdots jij\cdots j}r_j(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}.$$

Similarly, under the condition of $m = n$, by breaking $N = \{1, 2, \ldots, n\}$ into disjoint subsets S and its complement $\bar{S}$, Zhao and Sang [10] provided an S-type upper bound for the largest singular value of nonnegative rectangular tensors.

Theorem 3 ([10], Theorem 2.2) *Let $\mathcal{A}$ be a (p,q)th order $n \times n$ dimensional nonnegative rectangular tensor, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\lambda_0 \leq U^S(\mathcal{A}) = \max\big\{U_1^S(\mathcal{A}), U_1^{\bar{S}}(\mathcal{A}), U_2^S(\mathcal{A}), U_2^{\bar{S}}(\mathcal{A})\big\},$$

where

$$\begin{aligned}
U_1^S(\mathcal{A}) = \max_{i\in S, j\in\bar{S}} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_j^{\overline{\Delta^S}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A})\big)^2 + 4\max\big\{r_i(\mathcal{A}), c_i(\mathcal{A})\big\} r_j^{\Delta^S}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
U_1^{\bar{S}}(\mathcal{A}) = \max_{i\in\bar{S}, j\in S} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\big)^2 + 4\max\big\{r_i(\mathcal{A}), c_i(\mathcal{A})\big\} r_j^{\Delta^{\bar{S}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
U_2^S(\mathcal{A}) = \max_{i\in S, j\in\bar{S}} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + c_j^{\overline{\Omega^S}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - c_j^{\overline{\Omega^S}}(\mathcal{A})\big)^2 + 4\max\big\{r_i(\mathcal{A}), c_i(\mathcal{A})\big\} c_j^{\Omega^S}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
U_2^{\bar{S}}(\mathcal{A}) = \max_{i\in\bar{S}, j\in S} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + c_j^{\overline{\Omega^{\bar{S}}}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - c_j^{\overline{\Omega^{\bar{S}}}}(\mathcal{A})\big)^2 + 4\max\big\{r_i(\mathcal{A}), c_i(\mathcal{A})\big\} c_j^{\Omega^{\bar{S}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}.
\end{aligned}$$

In this paper, we continue this research, and give a new S-type upper bound for the largest singular value of nonnegative rectangular tensors. It is proved that the new upper bound is better than those in Theorems 1-3.

2 Main results

Theorem 4 *Let $\mathcal{A}$ be a (p,q)th order $n \times n$ dimensional nonnegative rectangular tensor, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\lambda_0 \leq \Psi^S(\mathcal{A}) = \max\big\{\Psi_1^S(\mathcal{A}), \Psi_1^{\bar{S}}(\mathcal{A}), \Psi_2^S(\mathcal{A}), \Psi_2^{\bar{S}}(\mathcal{A}), \Psi_3^S(\mathcal{A}), \Psi_3^{\bar{S}}(\mathcal{A}), \Psi_4^S(\mathcal{A}), \Psi_4^{\bar{S}}(\mathcal{A})\big\},$$

where

$$\begin{aligned}
\Psi_1^S(\mathcal{A}) = \max_{i\in S, j\in\bar{S}} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_i^{\Delta^S}(\mathcal{A}) + r_j^{\overline{\Delta^S}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + r_i^{\Delta^S}(\mathcal{A}) - r_j^{\overline{\Delta^S}}(\mathcal{A})\big)^2 + 4r_i^{\overline{\Delta^S}}(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\},
\end{aligned}$$

$$\begin{aligned}
\Psi_1^{\bar{S}}(\mathcal{A}) = \max_{i\in\bar{S},j\in S} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) + r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) - r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\big)^2 + 4r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})r_j^{\Delta^{\bar{S}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
\Psi_2^{S}(\mathcal{A}) = \max_{i\in S,j\in\bar{S}} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + c_i^{\Omega^{S}}(\mathcal{A}) + c_j^{\overline{\Omega^{S}}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + c_i^{\Omega^{S}}(\mathcal{A}) - c_j^{\overline{\Omega^{S}}}(\mathcal{A})\big)^2 + 4c_i^{\overline{\Omega^{S}}}(\mathcal{A})c_j^{\Omega^{S}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
\Psi_2^{\bar{S}}(\mathcal{A}) = \max_{i\in\bar{S},j\in S} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + c_i^{\Omega^{\bar{S}}}(\mathcal{A}) + c_j^{\overline{\Omega^{\bar{S}}}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + c_i^{\Omega^{\bar{S}}}(\mathcal{A}) - c_j^{\overline{\Omega^{\bar{S}}}}(\mathcal{A})\big)^2 + 4c_i^{\overline{\Omega^{\bar{S}}}}(\mathcal{A})c_j^{\Omega^{\bar{S}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
\Psi_3^{S}(\mathcal{A}) = \max_{i\in S,j\in\bar{S}} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_i^{\Delta^{S}}(\mathcal{A}) + c_j^{\overline{\Omega^{S}}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + r_i^{\Delta^{S}}(\mathcal{A}) - c_j^{\overline{\Omega^{S}}}(\mathcal{A})\big)^2 + 4r_i^{\overline{\Delta^{S}}}(\mathcal{A})c_j^{\Omega^{S}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
\Psi_3^{\bar{S}}(\mathcal{A}) = \max_{i\in\bar{S},j\in S} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + c_i^{\Omega^{\bar{S}}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + c_i^{\Omega^{\bar{S}}}(\mathcal{A})\big)^2 + 4r_j^{\Delta^{\bar{S}}}(\mathcal{A})c_i^{\overline{\Omega^{\bar{S}}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
\Psi_4^{S}(\mathcal{A}) = \max_{i\in S,j\in\bar{S}} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_j^{\overline{\Delta^{S}}}(\mathcal{A}) + c_i^{\Omega^{S}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^{S}}}(\mathcal{A}) + c_i^{\Omega^{S}}(\mathcal{A})\big)^2 + 4r_j^{\Delta^{S}}(\mathcal{A})c_i^{\overline{\Omega^{S}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}, \\
\Psi_4^{\bar{S}}(\mathcal{A}) = \max_{i\in\bar{S},j\in S} \frac{1}{2}\Big\{& a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) + c_j^{\overline{\Omega^{\bar{S}}}}(\mathcal{A}) \\
&+ \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) - c_j^{\overline{\Omega^{\bar{S}}}}(\mathcal{A})\big)^2 + 4r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})c_j^{\Omega^{\bar{S}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}.
\end{aligned}$$

Proof Because λ_0 is the largest singular value of $\mathcal{A}$, from Theorem 2 in [6], there are nonnegative nonzero vectors $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \dots, y_n)^{\mathrm{T}}$, such that

$$\mathcal{A}x^{p-1}y^q = \lambda_0 x^{[l-1]}, \tag{1}$$

$$\mathcal{A}x^p y^{q-1} = \lambda_0 y^{[l-1]}. \tag{2}$$

Let

$$\begin{aligned}
&x_t = \max\{x_i : i \in S\}, \qquad x_h = \max\{x_i : i \in \bar{S}\}; \\
&y_f = \max\{y_i : i \in S\}, \qquad y_g = \max\{y_i : i \in \bar{S}\}; \\
&w_i = \max\{x_i, y_i\}, \quad i \in N, \qquad w_S = \max\{w_i : i \in S\}, \qquad w_{\bar{S}} = \max\{w_i : i \in \bar{S}\}.
\end{aligned}$$

Then at least one of x_t and x_h is nonzero, and at least one of y_f and y_g is nonzero. We next divide into four cases to prove.

Case I: If $w_S = x_t, w_{\bar{S}} = x_h$, then $x_t \geq y_t, x_h \geq y_h$.

(i) If $x_h \geq x_t$, then $x_h = \max\{w_i : i \in N\}$. From (3) of Theorem 2.2 in [10], we have

$$\big(\lambda_0 - a_{h\cdots hh\cdots h} - r_h^{\overline{\Delta^S}}(\mathcal{A})\big)x_h^{l-1} \leq r_h^{\Delta^S}(\mathcal{A})x_t^{l-1}. \tag{3}$$

If $x_t = 0$, by $x_h > 0$, we have $\lambda_0 - a_{h\cdots hh\cdots h} - r_h^{\overline{\Delta^S}}(\mathcal{A}) \le 0$. Then $\lambda_0 \le a_{h\cdots hh\cdots h} + r_h^{\overline{\Delta^S}}(\mathcal{A}) \le \Psi_1^S(\mathcal{A})$. Otherwise, $x_t > 0$. From (1), we have

$$\begin{aligned}
(\lambda_0 - a_{t\cdots tt\cdots t})x_t^{l-1} &\le \lambda_0 x_t^{l-1} - a_{t\cdots tt\cdots t}x_t^{p-1}y_t^q \\
&= \sum_{\substack{(i_2,\dots,i_p,j_1,\dots,j_q)\in\Delta^S \\ \delta_{ti_2\cdots i_pj_1\cdots j_q}=0}} a_{ti_2\cdots i_pj_1\cdots j_q}x_{i_2}\cdots x_{i_p}y_{j_1}\cdots y_{j_q} \\
&\quad + \sum_{(i_2,\dots,i_p,j_1,\dots,j_q)\in\overline{\Delta^S}} a_{ti_2\cdots i_pj_1\cdots j_q}x_{i_2}\cdots x_{i_p}y_{j_1}\cdots y_{j_q} \\
&\le \sum_{\substack{(i_2,\dots,i_p,j_1,\dots,j_q)\in\Delta^S \\ \delta_{ti_2\cdots i_pj_1\cdots j_q}=0}} a_{ti_2\cdots i_pj_1\cdots j_q}x_t^{l-1} + \sum_{(i_2,\dots,i_p,j_1,\dots,j_q)\in\overline{\Delta^S}} a_{ti_2\cdots i_pj_1\cdots j_q}x_h^{l-1} \\
&= r_t^{\Delta^S}(\mathcal{A})x_t^{l-1} + r_t^{\overline{\Delta^S}}(\mathcal{A})x_h^{l-1},
\end{aligned}$$

i.e.,

$$\bigl(\lambda_0 - a_{t\cdots tt\cdots t} - r_t^{\Delta^S}(\mathcal{A})\bigr)x_t^{l-1} \le r_t^{\overline{\Delta^S}}(\mathcal{A})x_h^{l-1}. \tag{4}$$

If $\lambda_0 - a_{t\cdots tt\cdots t} - r_t^{\Delta^S}(\mathcal{A}) \le 0$, then $\lambda_0 \le a_{t\cdots tt\cdots t} + r_t^{\Delta^S}(\mathcal{A}) \le \Psi_1^S(\mathcal{A})$. If $\lambda_0 - a_{t\cdots tt\cdots t} - r_t^{\Delta^S}(\mathcal{A}) > 0$, multiplying (3) with (4) and noting that $x_t^{l-1}x_h^{l-1} > 0$, we have

$$\bigl(\lambda_0 - a_{t\cdots tt\cdots t} - r_t^{\Delta^S}(\mathcal{A})\bigr)\bigl(\lambda_0 - a_{h\cdots hh\cdots h} - r_h^{\overline{\Delta^S}}(\mathcal{A})\bigr) \le r_t^{\overline{\Delta^S}}(\mathcal{A})r_h^{\Delta^S}(\mathcal{A}). \tag{5}$$

Solving λ_0 in (5) gives

$$\begin{aligned}
\lambda_0 &\le \frac{1}{2}\Bigl\{a_{t\cdots tt\cdots t} + a_{h\cdots hh\cdots h} + r_t^{\Delta^S}(\mathcal{A}) + r_h^{\overline{\Delta^S}}(\mathcal{A}) \\
&\quad + \Bigl[\bigl(a_{t\cdots tt\cdots t} + r_t^{\Delta^S}(\mathcal{A}) - a_{h\cdots hh\cdots h} - r_h^{\overline{\Delta^S}}(\mathcal{A})\bigr)^2 + 4r_t^{\overline{\Delta^S}}(\mathcal{A})r_h^{\Delta^S}(\mathcal{A})\Bigr]^{\frac{1}{2}}\Bigr\} \\
&\le \max_{i\in S, j\in\bar{S}} \frac{1}{2}\Bigl\{a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_i^{\Delta^S}(\mathcal{A}) + r_j^{\overline{\Delta^S}}(\mathcal{A}) \\
&\quad + \Bigl[\bigl(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + r_i^{\Delta^S}(\mathcal{A}) - r_j^{\overline{\Delta^S}}(\mathcal{A})\bigr)^2 + 4r_i^{\overline{\Delta^S}}(\mathcal{A})r_j^{\Delta^S}(\mathcal{A})\Bigr]^{\frac{1}{2}}\Bigr\} \\
&= \Psi_1^S(\mathcal{A}).
\end{aligned}$$

(ii) If $x_t \ge x_h$, similar to the proof of (i), we have

$$\bigl(\lambda_0 - a_{h\cdots hh\cdots h} - r_h^{\Delta^{\bar{S}}}(\mathcal{A})\bigr)\bigl(\lambda_0 - a_{t\cdots tt\cdots t} - r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\bigr) \le r_h^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})r_t^{\Delta^{\bar{S}}}(\mathcal{A}),$$

and

$$\begin{aligned}
\lambda_0 &\le \frac{1}{2}\Bigl\{a_{h\cdots hh\cdots h} + a_{t\cdots tt\cdots t} + r_h^{\Delta^{\bar{S}}}(\mathcal{A}) + r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \\
&\quad + \Bigl[\bigl(a_{h\cdots hh\cdots h} - a_{t\cdots tt\cdots t} + r_h^{\Delta^{\bar{S}}}(\mathcal{A}) - r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\bigr)^2 + 4r_h^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})r_t^{\Delta^{\bar{S}}}(\mathcal{A})\Bigr]^{\frac{1}{2}}\Bigr\} \\
&\le \max_{i\in\bar{S}, j\in S} \frac{1}{2}\Bigl\{a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) + r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})
\end{aligned}$$

$$\begin{aligned}&+\Big[\big(a_{i\cdots ii\cdots i}-a_{j\cdots jj\cdots j}+r_i^{\Delta^{\bar S}}(\mathcal{A})-r_j^{\overline{\Delta^{\bar S}}}(\mathcal{A})\big)^2+4r_i^{\overline{\Delta^{\bar S}}}(\mathcal{A})r_j^{\Delta^{\bar S}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}\\&=\Psi_1^{\bar S}(\mathcal{A}).\end{aligned}$$

Case II: Assume that $w_S = y_f, w_{\bar S} = y_g$. If $y_g \geq y_f$, similar to the proof of (i), we have

$$\big(\lambda_0-a_{f\cdots ff\cdots f}-c_f^{\Omega^S}(\mathcal{A})\big)\big(\lambda_0-a_{g\cdots gg\cdots g}-c_g^{\overline{\Omega^S}}(\mathcal{A})\big)\leq c_f^{\overline{\Omega^S}}(\mathcal{A})c_g^{\Omega^S}(\mathcal{A}),$$

and

$$\begin{aligned}\lambda_0&\leq\frac{1}{2}\Big\{a_{f\cdots ff\cdots f}+a_{g\cdots gg\cdots g}+c_f^{\Omega^S}(\mathcal{A})+c_g^{\overline{\Omega^S}}(\mathcal{A})\\&\quad+\Big[\big(a_{f\cdots ff\cdots f}-a_{g\cdots gg\cdots g}+c_f^{\Omega^S}(\mathcal{A})-c_g^{\overline{\Omega^S}}(\mathcal{A})\big)^2+4c_f^{\overline{\Omega^S}}(\mathcal{A})c_g^{\Omega^S}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}\\&\leq\max_{i\in S,j\in\bar S}\frac{1}{2}\Big\{a_{i\cdots ii\cdots i}+a_{j\cdots jj\cdots j}+c_i^{\Omega^S}(\mathcal{A})+c_j^{\overline{\Omega^S}}(\mathcal{A})\\&\quad+\Big[\big(a_{i\cdots ii\cdots i}-a_{j\cdots jj\cdots j}+c_i^{\Omega^S}(\mathcal{A})-c_j^{\overline{\Omega^S}}(\mathcal{A})\big)^2+4c_i^{\overline{\Omega^S}}(\mathcal{A})c_j^{\Omega^S}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}\\&=\Psi_2^S(\mathcal{A}).\end{aligned}$$

If $y_f \geq y_g$, similarly, we have

$$\big(\lambda_0-a_{g\cdots gg\cdots g}-c_g^{\Omega^{\bar S}}(\mathcal{A})\big)\big(\lambda_0-a_{f\cdots ff\cdots f}-c_f^{\overline{\Omega^{\bar S}}}(\mathcal{A})\big)\leq c_g^{\overline{\Omega^{\bar S}}}(\mathcal{A})c_f^{\Omega^{\bar S}}(\mathcal{A})$$

and

$$\begin{aligned}\lambda_0&\leq\frac{1}{2}\Big\{a_{g\cdots gg\cdots g}+a_{f\cdots ff\cdots f}+c_g^{\Omega^{\bar S}}(\mathcal{A})+c_f^{\overline{\Omega^{\bar S}}}(\mathcal{A})\\&\quad+\Big[\big(a_{g\cdots g\cdots g}-a_{f\cdots ff\cdots f}+c_g^{\Omega^{\bar S}}(\mathcal{A})-c_f^{\overline{\Omega^{\bar S}}}(\mathcal{A})\big)^2+4c_g^{\overline{\Omega^{\bar S}}}(\mathcal{A})c_f^{\Omega^{\bar S}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}\\&\leq\max_{i\in\bar S,j\in S}\frac{1}{2}\Big\{a_{i\cdots ii\cdots i}+a_{j\cdots jj\cdots j}+c_i^{\Omega^{\bar S}}(\mathcal{A})+c_j^{\overline{\Omega^{\bar S}}}(\mathcal{A})\\&\quad+\Big[\big(a_{i\cdots ii\cdots i}-a_{j\cdots jj\cdots j}+c_i^{\Omega^{\bar S}}(\mathcal{A})-c_j^{\overline{\Omega^{\bar S}}}(\mathcal{A})\big)^2+4c_i^{\overline{\Omega^{\bar S}}}(\mathcal{A})c_j^{\Omega^{\bar S}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}\\&=\Psi_2^{\bar S}(\mathcal{A}).\end{aligned}$$

Case III: Assume that $w_S = x_t, w_{\bar S} = y_g$. If $y_g \geq x_t$, similar to the proof of (i), we have

$$\big(\lambda_0-a_{t\cdots tt\cdots t}-r_t^{\Delta^S}(\mathcal{A})\big)\big(\lambda_0-a_{g\cdots gg\cdots g}-c_g^{\overline{\Omega^S}}(\mathcal{A})\big)\leq r_t^{\overline{\Delta^S}}(\mathcal{A})c_g^{\Omega^S}(\mathcal{A})$$

and

$$\begin{aligned}\lambda_0&\leq\frac{1}{2}\Big\{a_{t\cdots tt\cdots t}+a_{g\cdots gg\cdots g}+r_t^{\Delta^S}(\mathcal{A})+c_g^{\overline{\Omega^S}}(\mathcal{A})\\&\quad+\Big[\big(a_{t\cdots tt\cdots t}-a_{g\cdots gg\cdots g}+r_t^{\Delta^S}(\mathcal{A})-c_g^{\overline{\Omega^S}}(\mathcal{A})\big)^2+4r_t^{\overline{\Delta^S}}(\mathcal{A})c_g^{\Omega^S}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}\\&\leq\max_{i\in S,j\in\bar S}\frac{1}{2}\Big\{a_{i\cdots ii\cdots i}+a_{j\cdots jj\cdots j}+r_i^{\Delta^S}(\mathcal{A})+c_j^{\overline{\Omega^S}}(\mathcal{A})\\&\quad+\Big[\big(a_{i\cdots ii\cdots i}-a_{j\cdots jj\cdots j}+r_i^{\Delta^S}(\mathcal{A})-c_j^{\overline{\Omega^S}}(\mathcal{A})\big)^2+4r_i^{\overline{\Delta^S}}(\mathcal{A})c_j^{\Omega^S}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\}\\&=\Psi_3^S(\mathcal{A}).\end{aligned}$$

If $x_t \geq y_g$, similarly, we have

$$\big(\lambda_0 - a_{g\cdots gg\cdots g} - c_g^{\Omega^{\bar{S}}}(\mathcal{A})\big)\big(\lambda_0 - a_{t\cdots tt\cdots t} - r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\big) \leq c_g^{\overline{\Omega^{\bar{S}}}}(\mathcal{A}) r_t^{\Delta^{\bar{S}}}(\mathcal{A})$$

and

$$\begin{aligned}
\lambda_0 &\leq \frac{1}{2}\Big\{a_{t\cdots tt\cdots t} + a_{g\cdots gg\cdots g} + r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + c_g^{\Omega^{\bar{S}}}(\mathcal{A}) \\
&\quad + \Big[\big(a_{t\cdots tt\cdots t} - a_{g\cdots gg\cdots g} + r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) - c_g^{\Omega^{\bar{S}}}(\mathcal{A})\big)^2 + 4 r_t^{\Delta^{\bar{S}}}(\mathcal{A}) c_g^{\overline{\Omega^{\bar{S}}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\} \\
&\leq \max_{i\in\bar{S}, j\in S} \frac{1}{2}\Big\{a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + c_i^{\Omega^{\bar{S}}}(\mathcal{A}) \\
&\quad + \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) + c_i^{\Omega^{\bar{S}}}(\mathcal{A})\big)^2 + 4 r_j^{\Delta^{\bar{S}}}(\mathcal{A}) c_i^{\overline{\Omega^{\bar{S}}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\} \\
&= \Psi_3^{\bar{S}}(\mathcal{A}).
\end{aligned}$$

Case IV: Assume that $w_S = y_f, w_{\bar{S}} = x_h$. If $x_h \geq y_f$, similar to the proof of (i), we have

$$\big(\lambda_0 - a_{f\cdots ff\cdots f} - c_f^{\Omega^{S}}(\mathcal{A})\big)\big(\lambda_0 - a_{h\cdots hh\cdots h} - r_h^{\overline{\Delta^{S}}}(\mathcal{A})\big) \leq c_f^{\overline{\Omega^{S}}}(\mathcal{A}) r_h^{\Delta^{S}}(\mathcal{A})$$

and

$$\begin{aligned}
\lambda_0 &\leq \frac{1}{2}\Big\{a_{f\cdots ff\cdots f} + a_{h\cdots hh\cdots h} + r_h^{\overline{\Delta^{S}}}(\mathcal{A}) + c_f^{\Omega^{S}}(\mathcal{A}) \\
&\quad + \Big[\big(a_{f\cdots ff\cdots f} - a_{h\cdots hh\cdots h} - r_h^{\overline{\Delta^{S}}}(\mathcal{A}) + c_f^{\Omega^{S}}(\mathcal{A})\big)^2 + 4 c_f^{\overline{\Omega^{S}}}(\mathcal{A}) r_h^{\Delta^{S}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\} \\
&\leq \max_{i\in S, j\in\bar{S}} \frac{1}{2}\Big\{a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_j^{\overline{\Delta^{S}}}(\mathcal{A}) + c_i^{\Omega^{S}}(\mathcal{A}) \\
&\quad + \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^{S}}}(\mathcal{A}) + c_i^{\Omega^{S}}(\mathcal{A})\big)^2 + 4 c_i^{\overline{\Omega^{S}}}(\mathcal{A}) r_j^{\Delta^{S}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\} \\
&= \Psi_4^{S}(\mathcal{A}).
\end{aligned}$$

If $y_f \geq x_h$, similarly, we have

$$\big(\lambda_0 - a_{h\cdots hh\cdots h} - r_h^{\Delta^{\bar{S}}}(\mathcal{A})\big)\big(\lambda_0 - a_{f\cdots ff\cdots f} - c_f^{\overline{\Omega^{\bar{S}}}}(\mathcal{A})\big) \leq r_h^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) c_f^{\Omega^{\bar{S}}}(\mathcal{A})$$

and

$$\begin{aligned}
\lambda_0 &\leq \frac{1}{2}\Big\{a_{h\cdots hh\cdots h} + a_{f\cdots ff\cdots f} + r_h^{\Delta^{\bar{S}}}(\mathcal{A}) + c_f^{\overline{\Omega^{\bar{S}}}}(\mathcal{A}) \\
&\quad + \Big[\big(a_{h\cdots hh\cdots h} - a_{f\cdots ff\cdots f} + r_h^{\Delta^{\bar{S}}}(\mathcal{A}) - c_f^{\overline{\Omega^{\bar{S}}}}(\mathcal{A})\big)^2 + 4 r_h^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) c_f^{\Omega^{\bar{S}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\} \\
&\leq \max_{i\in\bar{S}, j\in S} \frac{1}{2}\Big\{a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) + c_j^{\overline{\Omega^{\bar{S}}}}(\mathcal{A}) \\
&\quad + \Big[\big(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) - c_j^{\overline{\Omega^{\bar{S}}}}(\mathcal{A})\big)^2 + 4 r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) c_j^{\Omega^{\bar{S}}}(\mathcal{A})\Big]^{\frac{1}{2}}\Big\} \\
&= \Psi_4^{\bar{S}}(\mathcal{A}).
\end{aligned}$$

The conclusion follows from Cases I, II, III and IV. □

We next give the following comparison theorem for these upper bounds in Theorems 1-4.

Theorem 5 *Let $\mathcal{A}$ be a (p,q)th order $n \times n$ dimensional nonnegative rectangular tensor, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\Psi^S(\mathcal{A}) \leq U^S(\mathcal{A}) \leq \Phi(\mathcal{A}) \leq \max_{i,j\in N}\bigl\{R_i(\mathcal{A}), C_j(\mathcal{A})\bigr\}.$$

Proof I. By Remark 2.2 in [9], $\Phi(\mathcal{A}) \leq \max_{i,j\in N}\{R_i(\mathcal{A}), C_j(\mathcal{A})\}$ holds.

II. Next, we prove $U^S(\mathcal{A}) \leq \Phi(\mathcal{A})$. Here, we only prove $U_1^S(\mathcal{A}) \leq \Phi(\mathcal{A})$. Similarly, we can prove $U_1^{\bar{S}}(\mathcal{A}), U_2^S(\mathcal{A}), U_2^{\bar{S}}(\mathcal{A}) \leq \Phi(\mathcal{A})$, respectively.

(i) Suppose that

$$\begin{aligned} U_1^S(\mathcal{A}) = \max_{i\in S, j\in \bar{S}} \frac{1}{2}\bigl\{&a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_j^{\overline{\Delta^S}}(\mathcal{A}) \\ &+ \bigl[\bigl(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A})\bigr)^2 + 4r_i(\mathcal{A})r_j^{\Delta^S}(\mathcal{A})\bigr]^{\frac{1}{2}}\bigr\}. \end{aligned}$$

From the proof of Theorem 2.2 in [10], we can see that the bound $U_1^S(\mathcal{A})$ is obtained by solving λ_0 from

$$(\lambda_0 - a_{i\cdots ii\cdots i})\bigl(\lambda_0 - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A})\bigr) \leq r_i(\mathcal{A})r_j^{\Delta^S}(\mathcal{A}). \tag{6}$$

From the proof of Theorem 1.3 in [9], we can see that the bound

$$\begin{aligned} \Phi_1(\mathcal{A}) = \max_{i,j\in N, i\neq j} \frac{1}{2}\bigl\{&a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_j^i(\mathcal{A}) \\ &+ \bigl[\bigl(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - r_j^i(\mathcal{A})\bigr)^2 + 4a_{ji\cdots ii\cdots i}r_i(\mathcal{A})\bigr]^{\frac{1}{2}}\bigr\} \end{aligned}$$

is obtained by solving λ_0 from

$$(\lambda_0 - a_{i\cdots ii\cdots i})\bigl(\lambda_0 - a_{j\cdots jj\cdots j} - r_j^i(\mathcal{A})\bigr) \leq a_{ji\cdots ii\cdots i}r_i(\mathcal{A}). \tag{7}$$

Taking $i \in S, j \in \bar{S}$ in (7), by the proof of Theorem 6 in [11], we know that if λ_0 satisfies (6), then λ_0 satisfies (7), which implies that

$$\begin{aligned} \Phi_1(\mathcal{A}) \geq \max_{i\in S, j\in \bar{S}} \frac{1}{2}\bigl\{&a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_j^i(\mathcal{A}) \\ &+ \bigl[\bigl(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - r_j^i(\mathcal{A})\bigr)^2 + 4a_{ji\cdots ii\cdots i}r_i(\mathcal{A})\bigr]^{\frac{1}{2}}\bigr\} \\ \geq{}& U_1^S(\mathcal{A}). \end{aligned}$$

Obviously, $U_1^S(\mathcal{A}) \leq \Phi(\mathcal{A})$.

(ii) Suppose that

$$\begin{aligned} U_1^S(\mathcal{A}) = \max_{i\in S, j\in \bar{S}} \frac{1}{2}\bigl\{&a_{i\cdots ii\cdots i} + a_{j\cdots jj\cdots j} + r_j^{\overline{\Delta^S}}(\mathcal{A}) \\ &+ \bigl[\bigl(a_{i\cdots ii\cdots i} - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A})\bigr)^2 + 4c_i(\mathcal{A})r_j^{\Delta^S}(\mathcal{A})\bigr]^{\frac{1}{2}}\bigr\}. \end{aligned}$$

Similar to the proof of (i), we can obtain $U_1^S(\mathcal{A}) \leq \Phi_3(\mathcal{A}) \leq \Phi(\mathcal{A})$.

III. Finally, we prove that $\Psi^S(\mathcal{A}) \leq U^S(\mathcal{A})$. Here, we only prove $\Psi_1^S(\mathcal{A}) \leq U^S(\mathcal{A})$. Similarly, we can prove $\Psi_1^{\bar{S}}(\mathcal{A}), \Psi_2^S(\mathcal{A}), \Psi_2^{\bar{S}}(\mathcal{A}), \Psi_3^S(\mathcal{A}), \Psi_3^{\bar{S}}(\mathcal{A}), \Psi_4^S(\mathcal{A}), \Psi_4^{\bar{S}}(\mathcal{A}) \leq U^S(\mathcal{A})$, respectively.

Let $i \in S$ and $j \in \bar{S}$. From the proof of Theorem 4, we can see that the bound $\Psi_1^S(\mathcal{A})$ is obtained by solving λ_0 from

$$\bigl(\lambda_0 - a_{i\cdots ii\cdots i} - r_i^{\Delta^S}(\mathcal{A})\bigr)\bigl(\lambda_0 - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A})\bigr) \leq r_i^{\overline{\Delta^S}}(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A}). \tag{8}$$

(i) Suppose that $r_i^{\overline{\Delta^S}}(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A}) = 0$. If $\lambda_0 - a_{i\cdots ii\cdots i} - r_i^{\Delta^S}(\mathcal{A}) > 0$, *i.e.*, $\lambda_0 > a_{i\cdots ii\cdots i} + r_i^{\Delta^S}(\mathcal{A})$, then $\lambda_0 - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A}) \leq 0$, and for any $i \in S$,

$$(\lambda_0 - a_{i\cdots ii\cdots i})\bigl(\lambda_0 - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A})\bigr) \leq 0 \leq r_i(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A}).$$

That is to say, if λ_0 satisfies (8), then λ_0 satisfies (6), which implies that $\Psi_1^S(\mathcal{A}) \leq U_1^S(\mathcal{A}) \leq U^S(\mathcal{A})$.

If $\lambda_0 - a_{i\cdots ii\cdots i} - r_i^{\Delta^S}(\mathcal{A}) \leq 0$, then $\lambda_0 - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A}) \geq 0$, *i.e.*, $\lambda_0 \geq a_{j\cdots jj\cdots j} + r_j^{\overline{\Delta^S}}(\mathcal{A})$. From (3), we can obtain $\lambda_0 - a_{j\cdots jj\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A}) \leq r_j^{\Delta^S}(\mathcal{A})$, *i.e.*,

$$\lambda_0 - a_{j\cdots jj\cdots j} \leq r_j(\mathcal{A}). \tag{9}$$

By $\lambda_0 - a_{i\cdots ii\cdots i} - r_i^{\Delta^S}(\mathcal{A}) \leq 0 \leq r_i^{\overline{\Delta^S}}(\mathcal{A})$, *i.e.*, $\lambda_0 - a_{i\cdots ii\cdots i} \leq r_i(\mathcal{A})$, we have

$$\lambda_0 - a_{i\cdots ii\cdots i} - r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \leq r_i^{\Delta^{\bar{S}}}(\mathcal{A}). \tag{10}$$

Multiplying (9) with (10), we can obtain

$$(\lambda_0 - a_{j\cdots jj\cdots j})\bigl(\lambda_0 - a_{i\cdots ii\cdots i} - r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})\bigr) \leq r_i^{\Delta^{\bar{S}}}(\mathcal{A}) r_j(\mathcal{A}), \tag{11}$$

which implies that if λ_0 satisfies (8), then λ_0 satisfies (6), consequently, $\Psi_1^S(\mathcal{A}) \leq U_1^{\bar{S}}(\mathcal{A}) \leq U^S(\mathcal{A})$.

(ii) Suppose that $r_i^{\overline{\Delta^S}}(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A}) > 0$. Then dividing (8) by $r_i^{\overline{\Delta^S}}(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A})$, we have

$$\frac{(\lambda_0 - a_{i\cdots i} - r_i^{\Delta^S}(\mathcal{A}))}{r_i^{\overline{\Delta^S}}(\mathcal{A})} \frac{(\lambda_0 - a_{j\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A}))}{r_j^{\Delta^S}(\mathcal{A})} \leq 1. \tag{12}$$

Furthermore, if $\frac{\lambda_0 - a_{i\cdots i} - r_i^{\Delta^S}(\mathcal{A})}{r_i^{\overline{\Delta^S}}(\mathcal{A})} \geq 1$, then by Lemma 2.3 in [12] and (12), we have

$$\frac{(\lambda_0 - a_{i\cdots i})}{r_i(\mathcal{A})} \frac{(\lambda_0 - a_{j\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A}))}{r_j^{\Delta^S}(\mathcal{A})} \leq \frac{(\lambda_0 - a_{i\cdots i} - r_i^{\Delta^S}(\mathcal{A}))}{r_i^{\overline{\Delta^S}}(\mathcal{A})} \frac{(\lambda_0 - a_{j\cdots j} - r_j^{\overline{\Delta^S}}(\mathcal{A}))}{r_j^{\Delta^S}(\mathcal{A})} \leq 1.$$

Thus, (6) holds, which implies that if λ_0 satisfies (8), then λ_0 satisfies (6), consequently, $\Psi_1^S(\mathcal{A}) \leq U_1^S(\mathcal{A})$. And if $\frac{\lambda_0 - a_{i\cdots i} - r_i^{\Delta^S}(\mathcal{A})}{r_i^{\overline{\Delta^S}}(\mathcal{A})} \leq 1$, then (10) holds, which leads to (11) from (9). This implies that if λ_0 satisfies (8), then λ_0 satisfies (6), consequently, $\Psi_1^S(\mathcal{A}) \leq U_1^{\bar{S}}(\mathcal{A}) \leq U^S(\mathcal{A})$. The conclusion follows immediately from what we have proved. □

3 Numerical examples

Example 1 Let $\mathcal{A} = (a_{ijkl})$ be a $(2,2)$th order 3×3 dimensional nonnegative rectangular tensor with entries defined as follows:

$$A(:,:,1,1) = \begin{bmatrix} 6 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad A(:,:,2,1) = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix},$$

$$A(:,:,3,1) = \begin{bmatrix} 3 & 0 & 3 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$A(:,:,1,2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \qquad A(:,:,2,2) = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

$$A(:,:,3,2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix},$$

$$A(:,:,1,3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \qquad A(:,:,2,3) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

$$A(:,:,3,3) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

By Theorem 1, we have

$$\lambda_0 \le 33.$$

By Theorem 2, we have

$$\lambda_0 \le 32.8924.$$

Taking $S = \{1,2\}, \bar{S} = \{3\}$, by Theorem 3, we have

$$\lambda_0 \le 32.0540;$$

by Theorem 4, we have

$$\lambda_0 \le 30.0965.$$

In fact, $\lambda_0 = 29.8830$. This example shows that the upper bound in Theorem 4 is smaller than those in Theorems 1-3.

Example 2 Let $\mathcal{A} = (a_{ijkl})$ be a $(2,2)$th order 2×2 dimensional nonnegative rectangular tensor with entries defined as follows:

$$a_{1111} = a_{1112} = a_{1222} = a_{2112} = a_{2121} = a_{2221} = 1,$$

the other $a_{ijkl} = 0$. By Theorem 4, we have

$$\lambda_0 \leq 3.$$

In fact, $\lambda_0 = 3$. This example shows that the upper bound in Theorem 4 is sharp.

4 Conclusions

In this paper, a new S-type upper bound $\Psi^S(\mathcal{A})$ of the largest singular value for a nonnegative rectangular tensor $\mathcal{A}$ with $m = n$ is obtained by breaking N into disjoint subsets S and its complement. It is proved that the bound $\Psi^S(\mathcal{A})$ is better than those in [6, 9, 10].

Note here that when $n = 2$, $\Phi(\mathcal{A}) = U^S(\mathcal{A}) = \Psi^S(\mathcal{A})$, and when $n \geq 3$, $\Phi(\mathcal{A}) \geq U^S(\mathcal{A}) \geq \Psi^S(\mathcal{A})$ always holds. How to pick S to make $\Psi^S(\mathcal{A})$ as small as possible is an interesting problem, but difficult when n is large. We will research this problem in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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