Wan *Journal of Inequalities and Applications* (2017) 2017:104
DOI 10.1186/s13660-017-1381-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Further result on Dirichlet-Sch type inequality and its application

Liquan Wan*

*Correspondence:
liq.wanjun@foxmail.com
School of Accounting, Henan University of Economics and Law, Zhengzhou, 450046, China

Abstract

In this paper we deal with a theoretical question raised in connection with the application of Dirichlet-Sch type inequality, obtained by Huang (Int. Math. J. 27(02):1650009, 2016), which has been already applied to obtain multiplicity results for boundary value problems in several recent papers. We also discuss a particular case of it in more detail. As an application, we deduce the least harmonic majorant and log-concavity of extended subharmonic functions.

Keywords: Dirichlet-Sch type inequality; harmonic function; Schrödinger PWB solution

1 Introduction

Let Γ be the subset of the upper half unit sphere. The set $\mathbf{R}_+ \times \Gamma$ in $\mathbf{R}^n$ is called a cone. We denote it by $\mathfrak{C}_n(\Gamma)$, where $\Gamma \subset \mathbf{S}_1$. The sets $I \times \Gamma$ and $I \times \partial\Gamma$ with an interval on $\mathbf{R}$ are denoted by $\mathfrak{C}_n(\Gamma; I)$ and $\mathfrak{S}_n(\Gamma; I)$, respectively. We denote $\mathfrak{C}_n(\Gamma) \cap S_R$ and $\mathfrak{S}_n(\Gamma; (0, +\infty))$ by $\mathfrak{S}_n(\Gamma; R)$ and $\mathfrak{S}_n(\Gamma)$, respectively.

Furthermore, we denote by $d\sigma$ (resp. dS_R) the $(n-1)$-dimensional volume elements induced by the Euclidean metric on $\partial\mathfrak{C}_n(\Gamma)$ (resp. S_R) and by dw the elements of the Euclidean volume in $\mathbf{R}^n$.

It is well known (see, *e.g.*, [2], p.41) that

$$\begin{aligned} &\Delta^*\varphi(\Theta) + \lambda\varphi(\Theta) = 0 \quad \text{in } \Gamma, \\ &\varphi(\Theta) = 0 \quad \text{on } \partial\Gamma, \end{aligned} \tag{1.1}$$

where Δ^* is the Laplace-Beltrami operator. We denote the least positive eigenvalue of this boundary value problem (1.1) by λ and the normalized positive eigenfunction corresponding to λ by $\varphi(\Theta)$, $\int_\Gamma \varphi^2(\Theta)\, dS_1 = 1$.

We remark that the function $r^{\aleph^\pm}\varphi(\Theta)$ is harmonic in $\mathfrak{C}_n(\Gamma)$, belongs to the class $C^2(\mathfrak{C}_n(\Gamma)\backslash\{O\})$ and vanishes on $\mathfrak{S}_n(\Gamma)$, where

$$2\aleph^\pm = -n + 2 \pm \sqrt{(n-2)^2 + 4\lambda}.$$

For simplicity we shall write χ instead of $\aleph^+ - \aleph^-$.

© The Author(s) 2017. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

For simplicity we shall assume that the boundary of the domain Γ is twice continuously differentiable, $\varphi \in C^2(\overline{\Gamma})$ and $\frac{\partial \varphi}{\partial n} > 0$ on $\partial\Gamma$. Then (see [3], p.7-8)

$$\operatorname{dist}(\Theta, \partial\Gamma) \approx \varphi(\Theta), \tag{1.2}$$

where $\Theta \in \Gamma$.

Let $\delta(P) = \operatorname{dist}(P, \partial\mathfrak{C}_n(\Gamma))$, we have

$$\varphi(\Theta) \approx \delta(P), \tag{1.3}$$

for any $P = (1, \Theta) \in \Gamma$ (see [4]).

Let $u(r, \Theta)$ be a function on $\mathfrak{C}_n(\Gamma)$. For any given $r \in \mathbf{R}_+$, the integral

$$\int_\Gamma u(r, \Theta)\varphi(\Theta)\, dS_1,$$

is denoted by $\mathcal{N}_u(r)$, when it exists. The finite or infinite limits

$$\lim_{r\to\infty} r^{-\aleph^+}\mathcal{N}_u(r) \quad \text{and} \quad \lim_{r\to 0} r^{-\aleph^-}\mathcal{N}_u(r)$$

are denoted by $\mathcal{U}_u$ and $\mathcal{V}_u$, respectively, when they exist.

Remark 1 A function $g(t)$ on $(0, \infty)$ is $\mathbb{A}_{d_1,d_2}$-convex if and only if $g(t)t^{d_2}$ is a convex function of t^d ($d = d_1 + d_2$) on $(0, \infty)$, or, equivalently, if and only if $g(t)t^{-d_1}$ is a convex function of t^{-d} on $(0, \infty)$.

Remark 2 $\mathcal{N}_u(r)$ is a $\mathbb{A}_{\aleph^+,\gamma-1}$-convex on $(0, \infty)$, where u is a subharmonic function on $\mathfrak{C}_n(\Gamma)$ such that

$$\limsup_{P\in\mathfrak{C}_n(\Gamma), P\to Q\in\partial\mathfrak{C}_n(\Gamma)} u(P) \le c, \tag{1.4}$$

where c is a nonnegative number (see [5]).

The function

$$\mathbb{P}_{\mathfrak{C}_n(\Gamma)}(P, Q) = \frac{\partial \mathbb{G}_{\mathfrak{C}_n(\Gamma)}(P, Q)}{\partial n_Q}$$

is called the ordinary Poisson kernel, where $\mathbb{G}_{\mathfrak{C}_n(\Gamma)}$ is the Green function.

The Poisson integral of g relative to $\mathfrak{C}_n(\Gamma)$ is defined by

$$\mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[g](P) = \frac{1}{c_n}\int_{\mathfrak{S}_n(\Gamma)} \mathbb{P}_{\mathfrak{C}_n(\Gamma)}(P, Q)g(Q)\, d\sigma,$$

where g is a continuous function on $\partial\mathfrak{C}_n(\Gamma)$ and $\frac{\partial}{\partial n_Q}$ denotes the differentiation at Q along the inward normal into $\mathfrak{C}_n(\Gamma)$.

We set functions f satisfying

$$\int_{\mathfrak{S}_n(\Gamma)} \frac{|f(t, \Phi)|^p}{1 + t^\gamma}\, d\sigma < \infty, \tag{1.5}$$

where $-1 < p < +\infty$ and

$$\frac{-\aleph^+ - n + 2}{p} < \gamma < \frac{-\aleph^+ - n + 2}{p} + n - 1.$$

Let $-1 < p < +\infty$. we denote $\mathcal{A}_\Gamma$ the class of all measurable functions $g(t,\Phi)$ ($Q = (t,\Phi) = (Y, y_n) \in \mathfrak{C}_n(\Gamma)$) satisfying the following inequality:

$$\int_{\mathfrak{C}_n(\Gamma)} \frac{|g(t,\Phi)|^{p-1}\varphi}{1 + t^{\gamma-3}}\, dw < \infty$$

and the class $\mathcal{B}_\Gamma$, consists of all measurable functions $h(t,\Phi)$ ($(t,\Phi) = (Y, y_n) \in \mathfrak{S}_n(\Gamma)$) satisfying

$$\int_{\mathfrak{S}_n(\Gamma)} \frac{|h(t,\Phi)|^q}{1 + t^\gamma} \frac{\partial\varphi}{\partial n}\, d\sigma < \infty,$$

where $q > 0$.

We will also consider the class of all continuous functions $u(t,\Phi)$ ($(t,\Phi) \in \overline{\mathfrak{C}_n(\Gamma)}$) harmonic in $\mathfrak{C}_n(\Gamma)$ with $u^+(t,\Phi) \in \mathcal{A}_\Gamma$ ($(t,\Phi) \in \mathfrak{C}_n(\Gamma)$) and $u^+(t,\Phi) \in \mathcal{B}_\Gamma$ ($(t,\Phi) \in \mathfrak{S}_n(\Gamma)$) is denoted by $\mathcal{C}_\Gamma$ (see [6]).

In 2015, Jiang, Hou and Peixoto-de-Büyükkurt (see [7]) obtained the following result.

Theorem A *Let g be a measurable function on ∂T_n such that*

$$\int_{\partial T_n} \big(1 + |Q|\big)^{2-n} \big|g(Q)\big|\, dQ < \infty.$$

Then the harmonic function $\mathbb{PI}_{T_n}[g]$ satisfies $\mathbb{PI}_{T_n}[g](P) = o(r^2 \sec^{n-3}\theta_1)$ as $r \to \infty$ in T_n.

Recently, Wang, Huang and N. Yamini (see [8]) generalized Theorem A to the conical case.

Theorem B *Let g be a continuous function on $\partial\mathfrak{C}_n(\Gamma)$ satisfying (1.5) with $p = q = 1$ and $\gamma = \aleph^+ + 1 - \aleph^-$. Then*

$$\mathcal{U}_{\mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[g]} = \mathcal{U}_{\mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[|g|]} = 0.$$

The remainder of the paper is organized as follows: in Section 2, we shall give our main theorem; in Section 3, some necessary lemmas are given; in Section 4, we shall prove the main result.

2 Main result

In this section, we give the main result of this paper.

Our main aim is to give a least harmonic majorant of a nonnegative subharmonic function on $\mathfrak{C}_n(\Gamma)$.

Theorem 1 *Let u be a function subharmonic in $\mathfrak{C}_n(\Gamma)$ and u' be the restriction of u to $\partial\mathfrak{C}_n(\Gamma)$. If u' satisfy (1.5) and $-1 \leq \mathcal{U}_u \leq 1$ then*

$$u(P) \leq h_u(P) \tag{2.1}$$

for any $P = (r,\Theta) \in \mathfrak{C}_n(\Gamma)$, *where* $h_u(P)$ *is the least harmonic majorant of* u *on* $\mathfrak{C}_n(\Gamma)$ *and has the following expression*:

$$h_u(P) = \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}\bigl[u'\bigr](P) + \mathcal{V}_u r^{\aleph^-}\varphi(\Theta) + \mathcal{U}_u r^{\aleph^+}\varphi(\Theta).$$

Remark 3 Theorem 1 solves a theoretical question raised in connection with the application of Dirichlet-Sch type inequality, obtained by Huang (see [1]), which has been already applied to obtain multiplicity results for boundary value problems in several recent papers.

3 Main lemmas

In order to prove our main result, we need the following lemmas.

Lemma 1 (see [1]) *Let* u *be a function subharmonic on* $\mathfrak{C}_n(\Gamma)$ *satisfying* (1.4). *Then the limit* $\mathcal{U}_u$ $(-1 < \mathcal{U}_u \leq 1)$ *exists*.

Lemma 2 *Let* u *be a function subharmonic on* $\mathfrak{C}_n(\Gamma)$ *satisfying* (1.4) *and*

$$\mathcal{U}_{u^+} \leq 1 \quad \textit{and} \quad \mathcal{U}_{u^+} < +\infty. \tag{3.1}$$

Then

$$u(r,\Theta) \leq \mathcal{V}_{u^+} r^{\aleph^-}\varphi(\Theta) + \mathcal{U}_{u^+} r^{\aleph^+}\varphi(\Theta). \tag{3.2}$$

Proof Take any $(r,\Theta) \in \mathfrak{C}_n(\Gamma)$ and any pair of numbers τ_1, τ_2 $(0 < \tau_1 < r < \tau_2 < +\infty)$. We define a boundary function on $\partial\mathfrak{C}_n(\Gamma;(\tau_1,\tau_2))$ by

$$\nu(r,\Theta) = \begin{cases} u(\tau_i,\Theta) & \text{on } \{\tau_i\} \times \Gamma \ (i = 1,2), \\ 0 & \text{on } [\tau_1,\tau_2] \times \partial\Gamma. \end{cases}$$

If we denote Schrödinger PWB solution of the Dirichlet-Sch problem on $\mathfrak{C}_n(\Gamma;(\tau_1,\tau_2))$ with ν by $H_\nu((r,\Theta);\mathfrak{C}_n(\Gamma;(\tau_1,\tau_2)))$, then we have

$$\begin{aligned} u(r,\Theta) &\leq H_\nu\bigl((r,\Theta);\mathfrak{C}_n\bigl(\Gamma;(\tau_1,\tau_2)\bigr)\bigr) \\ &\leq \int_\Gamma u^+(\tau_1,\Theta)\frac{\partial\mathbb{G}_{\mathfrak{C}_n(\Gamma;(\tau_1,\tau_2))}((\tau_1,\Phi),(r,\Theta))}{\partial R}\tau_1^{n-1}\,dS_1 \\ &\quad - \int_\Gamma u^+(\tau_2,\Theta)\frac{\partial\mathbb{G}_{\mathfrak{C}_n(\Gamma;(\tau_1,\tau_2))}((\tau_2,\Phi),(r,\Theta))}{\partial R}\tau_2^{n-1}\,dS_1, \end{aligned}$$

which shows that (3.2) holds from (3.1). □

Lemma 3 *Let* g *be a locally integrable function on* $\partial\mathfrak{C}_n(\Gamma)$ *satisfying* (1.5) *and* u *be a subharmonic function on* $\mathfrak{C}_n(\Gamma)$ *satisfying*

$$-1 \leq \liminf_{P\in\mathfrak{C}_n(\Gamma),P\to Q\in\partial\mathfrak{C}_n(\Gamma)}\bigl\{u(P) - \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[g](P)\bigr\} \leq 1 \tag{3.3}$$

and

$$\liminf_{P\in\mathfrak{C}_n(\Gamma),P\to Q\in\partial\mathfrak{C}_n(\Gamma)}\bigl\{u^+(P) - \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}\bigl[|g|\bigr](P)\bigr\} \leq 0. \tag{3.4}$$

Then the limits $\mathcal{U}_u$ *and* $\mathcal{V}_{u^+}$ $(-\infty < \mathcal{U}_u \le 1, 0 \le \mathcal{U}_{u^+} \le +\infty)$ *exist, and if* (3.1) *is satisfied, then*

$$u(P) \le \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[g](P) + \mathcal{V}_{u^+} r^{\aleph^-} \varphi(\Theta) + \mathcal{U}_{u^+} r^{\aleph^+} \varphi(\Theta) \tag{3.5}$$

for any $P = (r, \Theta) \in \mathfrak{C}_n(\Gamma)$.

Proof Put

$$U(P) = u(P) - \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[g](P) \quad \text{and} \quad U'(P) = u^+(P) - \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}\big[|g|\big](P)$$

on $\mathfrak{C}_n(\Gamma)$. From (3.3) and (3.4) we have

$$\liminf_{P\in\mathfrak{C}_n(\Gamma), P\to Q} U(P) \le -1 \quad \text{and} \quad \liminf_{P\in\mathfrak{C}_n(\Gamma), P\to Q} U'(P) \le -1.$$

Hence it follows from Lemma 1 that the limits $\mathcal{U}_U$ and $\mathcal{V}_{U'}$ $(-1 < \mathcal{U}_U \le 1, 0 \le \mathcal{V}_{U'} \le 1)$ exist. So Theorem B gives the existence of the limits $\mathcal{U}_u$, $\mathcal{V}_{u^+}$,

$$\mathcal{U}_U = \mathcal{V}_u \quad \text{and} \quad \mathcal{U}_{U'} = \mathcal{V}_{u^+}. \tag{3.6}$$

Since $0 \le U^+(P) \le u^+(P) + (\mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[g])^-(P)$ on $\mathfrak{C}_n(\Gamma)$, it also follows from Theorem B and (3.1) that

$$\mathcal{V}_{U^+} \le \mathcal{V}_{u^+} < \infty,$$

which together with Lemma 2 gives the conclusion. □

Lemma 4 *Let g be a lower semi-continuous function on* $\partial\mathfrak{C}_n(\Gamma)$ *satisfying* (1.5) *and u be a superharmonic function on* $\mathfrak{C}_n(\Gamma)$ *such that*

$$\liminf_{P\in\mathfrak{C}_n(\Gamma), P\to Q} u(P) \le g(Q) + c \tag{3.7}$$

for any $Q \in \partial\mathfrak{C}_n(\Gamma)$ *and c is a positive number. Then the limit* $\mathcal{U}_u$ $(-1 \le \mathcal{U}_u \le +1)$ *exists, and if* $\mathcal{U}_u < +\infty$, *then*

$$u(P) \le \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[g](P) + \mathcal{V}_u r^{\aleph^-} \varphi(\Theta) + \mathcal{U}_u r^{\aleph^+} \varphi(\Theta)$$

for any $P = (r, \Theta) \in \mathfrak{C}_n(\Gamma)$.

Proof Since $-g$ is upper semi-continuous function in $\partial\mathfrak{C}_n(\Gamma)$, it follows from [8], p.3, that

$$\liminf_{P\in\mathfrak{C}_n(\Gamma), P\to Q} \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[g](P) \ge g(Q) - c. \tag{3.8}$$

We see from (3.7) and (3.8) that

$$-1 \le \limsup_{P\in\mathfrak{C}_n(\Gamma), P\to Q} \big\{u(P) - \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[g](P)\big\} \le 1,$$

which gives (3.3). Since g and u are positive, (3.4) also holds. Lemma 4 is proved. □

Lemma 5 *Let u be a subharmonic function in $\overline{\mathfrak{C}_n(\Gamma)}$ such that $u' = u|\partial\mathfrak{C}_n(\Gamma)$ satisfies* (1.5). *Then $\mathbb{PI}_{\mathfrak{C}_n(\Gamma)}[u'](P) \leq h(P)$ on $\mathfrak{C}_n(\Gamma)$, where $h(P)$ is the any harmonic majorant of u on $\mathfrak{C}_n(\Gamma)$.*

Proof Take any $P' = (r', \Theta') \in \mathfrak{C}_n(\Gamma)$. Let ϵ be any positive number. In the same way as in the proof of Lemma 2, we can choose R such that

$$\int_{\mathfrak{S}_n(\Gamma;(R,\infty))} \mathbb{P}_{\mathfrak{C}_n(\Gamma)}\big(P', Q\big)u'(Q)\,d\sigma < \frac{\epsilon}{2}. \tag{3.9}$$

Further, take an integer j ($j > R$) such that (see [7])

$$\int_{\mathfrak{S}_n(\Gamma;(0,R))} \frac{\partial\Gamma_j(P', Q)}{\partial n_Q} u'(Q)\,d\sigma < \frac{\epsilon}{2}. \tag{3.10}$$

Since

$$\int_{\mathfrak{S}_n(\Gamma;(0,R))} \frac{\partial\mathbb{G}_{\mathfrak{C}_n(\Gamma;(0,j))}(P, Q)}{\partial n_Q} u'(Q)\,d\sigma \leq H_u\big(P; \mathfrak{C}_n\big(\Gamma; (0, j)\big)\big)$$

for any $P \in \mathfrak{C}_n(\Gamma; (0, j))$, we have from (3.9) and (3.10)

$$\begin{aligned}
&\mathbb{PI}_{\mathfrak{C}_n(\Gamma)}\big[u'\big]\big(P'\big) - H_u\big(P'; \mathfrak{C}_n\big(\Gamma; (0, j)\big)\big) \\
&\quad\leq \int_{\mathfrak{S}_n(\Gamma;(0,R))} \frac{\partial\Gamma_j(P', Q)}{\partial n_Q} u'(Q)\,d\sigma \\
&\qquad + \frac{1}{c_n}\int_{\mathfrak{S}_n(\Gamma;(R,\infty))} \mathbb{P}_{\mathfrak{C}_n(\Gamma)}\big(P', Q\big)u'(Q)\,d\sigma \\
&\quad< \epsilon.
\end{aligned} \tag{3.11}$$

Here note that $H_u(P; \mathfrak{C}_n(\Gamma; (0, j)))$ is the least harmonic majorant of u on $\mathfrak{C}_n(\Gamma; (0, j))$ (see [9], Theorem 3.15). If h is a harmonic majorant of u on $\mathfrak{C}_n(\Gamma)$, then

$$H_u\big(P'; \mathfrak{C}_n\big(\Gamma; (0, j)\big)\big) \leq h\big(P'\big).$$

Thus we obtain from (3.11)

$$\mathbb{PI}_{\mathfrak{C}_n(\Gamma)}\big[u'\big]\big(P'\big) < h\big(P'\big) + \epsilon,$$

which gives the conclusion of Lemma 5. □

4 Proof of Theorem 1

Let P be any point of $\mathfrak{C}_n(\Gamma)$ and ϵ be any positive number. By the Vitali-Carathéodory theorem with respect to the Schrödinger operator (see [10], p.56), there exists a lower semi-continuous function $g'(Q)$ on $\partial\mathfrak{C}_n(\Gamma)$ such that

$$u'(Q) \leq g'(Q) \tag{4.1}$$

and

$$\mathbb{PI}_{\mathfrak{C}_n(\Gamma)}\big[g'\big](P) < \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}\big[u'\big](P) + \epsilon. \tag{4.2}$$

Since

$$\lim_{P\in\mathfrak{C}_n(\Gamma),P\to Q} u(P) \leq u'(Q) \leq g'(Q)$$

for any $Q \in \partial\mathfrak{C}_n(\Gamma)$ from (4.1), it follows from [1], Lemma 2.1, that the limits $\mathcal{U}_u$ and $\mathcal{U}_u$ exist, and if $-1 \leq \mathcal{U}_u < 1$ and $-1 \leq \mathcal{V}_u < 1$, then

$$u(P) \leq \mathbb{PI}_{\mathfrak{C}_n(\Gamma)}\big[g'\big](P) + \mathcal{V}_u r^{\aleph^-}\varphi(\Theta) + \mathcal{U}_u r^{\aleph^+}\varphi(\Theta). \tag{4.3}$$

Hence we see from (4.2) and (4.3) that (2.1) holds.

Next we call the least harmonic majorant of u on $\mathfrak{C}_n(\Gamma)$: $h_u(P)$. Set $h''(P)$ is a Schrödinger harmonic function in $\mathfrak{C}_n(\Gamma)$ such that (see [7])

$$u(P) \leq h''(P) + \epsilon. \tag{4.4}$$

Put

$$h^*(P) = h_u(P) - h''(P) \quad \text{on } \mathfrak{C}_n(\Gamma).$$

It is easy to see that

$$h^*(P) \leq h_u(P).$$

It follows from Theorem B that $\mathcal{V}_{h^{*+}} < +\infty$. Further, from Lemma 5 we see that

$$\limsup_{P\in\mathfrak{C}_n(\Gamma),P\to Q} h^*(P) = \liminf_{P\in\mathfrak{C}_n(\Gamma),P\to Q}\big\{\mathbb{PI}_{\mathfrak{C}_n(\Gamma)}\big[u'\big](P) - h''(P)\big\} \leq -1.$$

From Theorem B and (4.4) we know

$$\mathcal{V}_{h^*} = \mathcal{V}_{h_u} - \mathcal{V}_{h''} = \mathcal{V}_u - \mathcal{U}_{h''} \leq \mathcal{U}_u - \mathcal{U}_u = 0.$$

We see from Lemma 2 that $-1 \leq h^*(P) \leq \epsilon$ on $\mathfrak{C}_n(\Gamma)$, which shows that $h_u(P)$ is the least harmonic majorant in $\mathfrak{C}_n(\Gamma)$. Theorem 1 is proved.

5 Conclusion

In this article, we dealt with a theoretical question raised in connection with the application of Dirichlet-Sch type inequality. Additionally, we discussed a particular case of it in more detail. As applications, we deduced the least harmonic majorant and log-concavity of extended subharmonic functions.

Competing interests
The author declares that he has no competing interests.

Acknowledgements
I would like to thank the referees for their constructive suggestions and useful comments which resulted in an improved version of this paper. This work was supported by the 2015 Universities Philosophy Social Sciences Innovation Team of Henan Province: the Institutional Arrangements for Mixed Ownership Reform of Stateowned Enterprises in Henan Province (No. 2015-CXTD-09).

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 4 March 2017 Accepted: 21 April 2017 Published online: 08 May 2017

References

1. Huang, J: Growth property at infinity of the maximum modulus with respect to the Schrödinger operator. Int. Math. J. **27**(02), 1650009 (2016)
2. Rosenblum, G, Solomyak, M, Shubin, M: Spectral Theory of Differential Operators. VINITI, Moscow (1989)
3. Miranda, C: Partial Differential Equations of Elliptic Type. Springer, Berlin (1970)
4. Courant, R, Hilbert, D: Methods of Mathematical Physics, vol. 1. Interscience, New York (1953)
5. Wanby, G: Convexity of means and growth of certain subharmonic functions in an n-dimensional cone. Ark. Mat. **21**, 29-43 (1983)
6. Li, W, Zaprawa, M: Applications of the generalized Dirichlet integral inequality to the Neumann problem with fast-growing continuous data. J. Inequal. Appl. **2016**, 250 (2016)
7. Jiang, Z, Hou, L, Peixoto-de-Büyükkurt, C: Growth property at infinity of harmonic functions. J. Inequal. Appl. **2015**, 401 (2015)
8. Wang, J, Huang, B, Yamini, N: An augmented Riesz decomposition method for sharp estimates of certain boundary value problem. Bound. Value Probl. **2016**, 156 (2016)
9. Hayman, WH, Kennedy, PB: Subharmonic Functions, vol. 1. Academic Press, London (1976)
10. Helms, LL: Introduction to Potential Theory. Wiley-Interscience, New York (1969)

Submit your manuscript to a SpringerOpen journal and benefit from:

► Convenient online submission
► Rigorous peer review
► Immediate publication on acceptance
► Open access: articles freely available online
► High visibility within the field
► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com