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A new Z-eigenvalue localization set for tensors

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Abstract
A new Z-eigenvalue localization set for tensors is given and proved to be tighter than those in the work of Wang *et al.* (Discrete Contin. Dyn. Syst., Ser. B 22(1):187-198, 2017). Based on this set, a sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is obtained. Finally, numerical examples are given to verify the theoretical results.

MSC: 15A18; 15A69

Keywords: Z-eigenvalue; localization set; nonnegative tensors; spectral radius; weakly symmetric

1 Introduction

For a positive integer n, $n \geq 2$, N denotes the set $\{1,2,\ldots,n\}$. $\mathbb{C}$ ($\mathbb{R}$) denotes the set of all complex (real) numbers. We call $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ a real tensor of order m dimension n, denoted by $\mathbb{R}^{[m,n]}$, if

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{R},$$

where $i_j \in N$ for $j = 1,2,\ldots,m$. $\mathcal{A}$ is called nonnegative if $a_{i_1 i_2 \cdots i_m} \geq 0$. $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called symmetric [2] if

$$a_{i_1 \cdots i_m} = a_{\pi(i_1 \cdots i_m)}, \quad \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called weakly symmetric [3] if the associated homogeneous polynomial

$$\mathcal{A}x^m = \sum_{i_1,i_2,\ldots,i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$$

satisfies $\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}$. It is shown in [3] that a symmetric tensor is necessarily weakly symmetric, but the converse is not true in general.

Given a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, if there are $\lambda \in \mathbb{C}$ and $x = (x_1, x_2 \cdots, x_n)^T \in \mathbb{C}^n \backslash \{0\}$ such that

$$\mathcal{A}x^{m-1} = \lambda x \quad \text{and} \quad x^T x = 1,$$

then λ is called an E-eigenvalue of $\mathcal{A}$ and x an E-eigenvector of $\mathcal{A}$ associated with λ, where $\mathcal{A}x^{m-1}$ is an n dimension vector whose ith component is

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m \in N} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

If λ and x are all real, then λ is called a Z-eigenvalue of $\mathcal{A}$ and x a Z-eigenvector of $\mathcal{A}$ associated with λ; for details, see [2, 4].

Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. We define the Z-spectrum of $\mathcal{A}$, denoted $\sigma(\mathcal{A})$ to be the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq 0$, then the Z-spectral radius [3] of $\mathcal{A}$, denoted $\varrho(\mathcal{A})$, is defined as

$$\varrho(\mathcal{A}) := \sup\bigl\{|\lambda| : \lambda \in \sigma(\mathcal{A})\bigr\}.$$

Recently, much literature has focused on locating all Z-eigenvalues of tensors and bounding the Z-spectral radius of nonnegative tensors in [1, 5–10]. It is well known that one can use eigenvalue inclusion sets to obtain the lower and upper bounds of the spectral radius of nonnegative tensors; for details, see [1, 11–14]. Therefore, the main aim of this paper is to give a tighter Z-eigenvalue inclusion set for tensors, and use it to obtain a sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors.

In 2017, Wang *et al.* [1] established the following Geršgorin-type Z-eigenvalue inclusion theorem for tensors.

Theorem 1 ([1], Theorem 3.1) *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i \in N} \mathcal{K}_i(\mathcal{A}),$$

where

$$\mathcal{K}_i(\mathcal{A}) = \bigl\{z \in \mathbb{C} : |z| \leq R_i(\mathcal{A})\bigr\}, \qquad R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m \in N} |a_{ii_2\cdots i_m}|.$$

To get a tighter Z-eigenvalue inclusion set than $\mathcal{K}(\mathcal{A})$, Wang *et al.* [1] gave the following Brauer-type Z-eigenvalue localization set for tensors.

Theorem 2 ([1], Theorem 3.2) *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}) = \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \mathcal{L}_{i,j}(\mathcal{A}),$$

where

$$\mathcal{L}_{i,j}(\mathcal{A}) = \bigl\{z \in \mathbb{C} : \bigl(|z| - \bigl(R_i(\mathcal{A}) - |a_{ij\cdots j}|\bigr)\bigr)|z| \leq |a_{ij\cdots j}| R_j(\mathcal{A})\bigr\}.$$

In this paper, we continue this research on the Z-eigenvalue localization problem for tensors and its applications. We give a new Z-eigenvalue inclusion set for tensors and prove that the new set is tighter than those in Theorem 1 and Theorem 2. As an application of this set, we obtain a new upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors, which is sharper than some existing upper bounds.

2 Main results

In this section, we give a new Z-eigenvalue localization set for tensors, and establish the comparison between this set with those in Theorem 1 and Theorem 2. For simplification, we denote

$$\Delta_j = \{(i_2, i_3, \ldots, i_m) : i_k = j \text{ for some } k \in \{2, \ldots, m\}, \text{where } j, i_2, \ldots, i_m \in N\},$$
$$\overline{\Delta}_j = \{(i_2, i_3, \ldots, i_m) : i_k \neq j \text{ for any } k \in \{2, \ldots, m\}, \text{where } j, i_2, \ldots, i_m \in N\}.$$

For $\forall i, j \in N, j \neq i$, let

$$r_i^{\Delta_j}(\mathcal{A}) = \sum_{(i_2,\ldots,i_m)\in\Delta_j} |a_{ii_2\cdots i_m}|, \qquad r_i^{\overline{\Delta}_j}(\mathcal{A}) = \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_j} |a_{ii_2\cdots i_m}|.$$

Then $R_i(\mathcal{A}) = r_i^{\Delta_j}(\mathcal{A}) + r_i^{\overline{\Delta}_j}(\mathcal{A})$.

Theorem 3 *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Psi(\mathcal{A}) = \bigcup_{i\in N} \bigcap_{j\in N, j\neq i} \Psi_{i,j}(\mathcal{A}),$$

where

$$\Psi_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z| - r_i^{\overline{\Delta}_j}(\mathcal{A}))|z| \leq r_i^{\Delta_j}(\mathcal{A}) R_j(\mathcal{A})\}.$$

Proof Let λ be a Z-eigenvalue of $\mathcal{A}$ with corresponding Z-eigenvector $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$, *i.e.*,

$$\mathcal{A}x^{m-1} = \lambda x, \quad \text{and} \quad \|x\|_2 = 1. \tag{1}$$

Assume $|x_t| = \max_{i\in N} |x_i|$, then $0 < |x_t|^{m-1} \leq |x_t| \leq 1$. For $\forall j \in N$, $j \neq t$, from (1), we have

$$\lambda x_t = \sum_{(i_2,\ldots,i_m)\in\Delta_j} a_{ti_2\cdots i_m} x_{i_2} \cdots x_{i_m} + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_j} a_{ti_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Taking the modulus in the above equation and using the triangle inequality give

$$\begin{aligned} |\lambda||x_t| &\leq \sum_{(i_2,\ldots,i_m)\in\Delta_j} |a_{ti_2\cdots i_m}||x_{i_2}| \cdots |x_{i_m}| + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_j} |a_{ti_2\cdots i_m}||x_{i_2}| \cdots |x_{i_m}| \\ &\leq \sum_{(i_2,\ldots,i_m)\in\Delta_j} |a_{ti_2\cdots i_m}||x_j| + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_j} |a_{ti_2\cdots i_m}||x_t| \\ &= r_t^{\Delta_j}(\mathcal{A})|x_j| + r_t^{\overline{\Delta}_j}(\mathcal{A})|x_t|, \end{aligned}$$

i.e.,

$$(|\lambda| - r_t^{\overline{\Delta}_j}(\mathcal{A}))|x_t| \leq r_t^{\Delta_j}(\mathcal{A})|x_j|. \tag{2}$$

If $|x_j| = 0$, by $|x_t| > 0$, we have $|\lambda| - r_t^{\overline{\Delta}_j}(\mathcal{A}) \leq 0$. Then

$$\big(|\lambda| - r_t^{\overline{\Delta}_j}(\mathcal{A})\big)|\lambda| \leq 0 \leq r_t^{\Delta_j}(\mathcal{A})R_j(\mathcal{A}).$$

Obviously, $\lambda \in \Psi_{t,j}(\mathcal{A})$. Otherwise, $|x_j| > 0$. From (1), we have

$$|\lambda||x_j| \leq \sum_{i_2,\dots,i_m \in N} |a_{ji_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}| \leq \sum_{i_2,\dots,i_m \in N} |a_{ji_2\cdots i_m}||x_t|^{m-1} \leq R_j(\mathcal{A})|x_t|. \tag{3}$$

Multiplying (2) with (3) and noting that $|x_t||x_j| > 0$, we have

$$\big(|\lambda| - r_t^{\overline{\Delta}_j}(\mathcal{A})\big)|\lambda| \leq r_t^{\Delta_j}(\mathcal{A})R_j(\mathcal{A}),$$

which implies that $\lambda \in \Psi_{t,j}(\mathcal{A})$. From the arbitrariness of j, we have $\lambda \in \bigcap_{j\in N, j\neq t} \Psi_{t,j}(\mathcal{A})$. Furthermore, we have $\lambda \in \bigcup_{i\in N}\bigcap_{j\in N, j\neq i} \Psi_{i,j}(\mathcal{A})$. □

Next, a comparison theorem is given for Theorem 1, Theorem 2 and Theorem 3.

Theorem 4 *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then*

$$\Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}).$$

Proof By Corollary 3.1 in [1], $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$ holds. Here, we only prove $\Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$. Let $z \in \Psi(\mathcal{A})$. Then there exists $i \in N$, such that $z \in \Psi_{i,j}(\mathcal{A})$, $\forall j \in N, j \neq i$, that is,

$$\big(|z| - r_i^{\overline{\Delta}_j}(\mathcal{A})\big)|z| \leq r_i^{\Delta_j}(\mathcal{A})R_j(\mathcal{A}), \quad \forall j \in N, j \neq i. \tag{4}$$

Next, we divide our subject in two cases to prove $\Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$.

Case I: If $r_i^{\Delta_j}(\mathcal{A})R_j(\mathcal{A}) = 0$, then we have

$$\big(|z| - \big(R_i(\mathcal{A}) - |a_{ij\cdots j}|\big)\big)|z| \leq \big(|z| - r_i^{\overline{\Delta}_j}(\mathcal{A})\big)|z| \leq r_i^{\Delta_j}(\mathcal{A})R_j(\mathcal{A}) = 0 \leq |a_{ij\cdots j}|R_j(\mathcal{A}),$$

which implies that $z \in \bigcap_{j\in N, j\neq i} \mathcal{L}_{i,j}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$ from the arbitrariness of j, consequently, $\Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$.

Case II: If $r_i^{\Delta_j}(\mathcal{A})R_j(\mathcal{A}) > 0$, then dividing both sides by $r_i^{\Delta_j}(\mathcal{A})R_j(\mathcal{A})$ in (4), we have

$$\frac{|z| - r_i^{\overline{\Delta}_j}(\mathcal{A})}{r_i^{\Delta_j}(\mathcal{A})}\frac{|z|}{R_j(\mathcal{A})} \leq 1, \tag{5}$$

which implies

$$\frac{|z| - r_i^{\overline{\Delta}_j}(\mathcal{A})}{r_i^{\Delta_j}(\mathcal{A})} \leq 1, \tag{6}$$

or

$$\frac{|z|}{R_j(\mathcal{A})} \leq 1. \tag{7}$$

Let $a = |z|$, $b = r_i^{\overline{\Delta_j}}(\mathcal{A})$, $c = r_i^{\Delta_j}(\mathcal{A}) - |a_{ij\cdots j}|$ and $d = |a_{ij\cdots j}|$. When (6) holds and $d = |a_{ij\cdots j}| > 0$, from Lemma 2.2 in [11], we have

$$\frac{|z| - (R_i(\mathcal{A}) - |a_{ij\cdots j}|)}{|a_{ij\cdots j}|} = \frac{a - (b + c)}{d} \le \frac{a - b}{c + d} = \frac{|z| - r_i^{\overline{\Delta_j}}(\mathcal{A})}{r_i^{\Delta_j}(\mathcal{A})}. \tag{8}$$

Furthermore, from (5) and (8), we have

$$\frac{|z| - (R_i(\mathcal{A}) - |a_{ij\cdots j}|)}{|a_{ij\cdots j}|} \frac{|z|}{R_j(\mathcal{A})} \le \frac{|z| - r_i^{\overline{\Delta_j}}(\mathcal{A})}{r_i^{\Delta_j}(\mathcal{A})} \frac{|z|}{R_j(\mathcal{A})} \le 1,$$

equivalently,

$$\bigl(|z| - \bigl(R_i(\mathcal{A}) - |a_{ij\cdots j}|\bigr)\bigr)|z| \le |a_{ij\cdots j}|R_j(\mathcal{A}),$$

which implies that $z \in \bigcap_{j\in N, j\neq i} \mathcal{L}_{i,j}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$ from the arbitrariness of j, consequently, $\Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$. When (6) holds and $d = |a_{ij\cdots j}| = 0$, we have

$$|z| - r_i^{\overline{\Delta_j}}(\mathcal{A}) - r_i^{\Delta_j}(\mathcal{A}) \le 0, \quad \textit{i.e.}, \quad |z| - \bigl(R_i(\mathcal{A}) - |a_{ij\cdots j}|\bigr) \le 0,$$

and furthermore

$$\bigl(|z| - \bigl(R_i(\mathcal{A}) - |a_{ij\cdots j}|\bigr)\bigr)|z| \le 0 = |a_{ij\cdots j}|R_j(\mathcal{A}).$$

This also implies $\Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$.

On the other hand, when (7) holds, we only prove $\Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$ under the case that

$$\frac{|z| - r_i^{\overline{\Delta_j}}(\mathcal{A})}{r_i^{\Delta_j}(\mathcal{A})} > 1. \tag{9}$$

From (9), we have $\frac{a}{b+c+d} = \frac{|z|}{R_i(\mathcal{A})} > 1$. When (7) holds and $|a_{ji\cdots i}| > 0$, by Lemma 2.3 in [11], we have

$$\frac{|z|}{R_i(\mathcal{A})} = \frac{a}{b + c + d} \le \frac{a - b}{c + d} = \frac{|z| - r_i^{\overline{\Delta_j}}(\mathcal{A})}{r_i^{\Delta_j}(\mathcal{A})}. \tag{10}$$

By (7), Lemma 2.2 in [11] and similar to the proof of (8), we have

$$\frac{|z| - (R_j(\mathcal{A}) - |a_{ji\cdots i}|)}{|a_{ji\cdots i}|} \le \frac{|z|}{R_j(\mathcal{A})}. \tag{11}$$

Multiplying (10) and (11), we have

$$\frac{|z| - (R_j(\mathcal{A}) - |a_{ji\cdots i}|)}{|a_{ji\cdots i}|} \frac{|z|}{R_i(\mathcal{A})} \le \frac{|z| - r_i^{\overline{\Delta_j}}(\mathcal{A})}{r_i^{\Delta_j}(\mathcal{A})} \frac{|z|}{R_j(\mathcal{A})} \le 1;$$

equivalently,

$$\big(|z| - \big(R_j(\mathcal{A}) - |a_{ji\cdots i}|\big)\big)|z| \leq |a_{ji\cdots i}|R_i(\mathcal{A}).$$

This implies $z \in \bigcap_{i\in N, i\neq j} \mathcal{L}_{j,i}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$ and $\Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$ from the arbitrariness of i. When (7) holds and $|a_{ji\cdots i}| = 0$, we can obtain

$$|z| - R_j(\mathcal{A}) \leq 0, \quad \textit{i.e.}, \quad |z| - \big(R_j(\mathcal{A}) - |a_{ji\cdots i}|\big) \leq 0$$

and

$$\big(|z| - \big(R_j(\mathcal{A}) - |a_{ji\cdots i}|\big)\big)|z| \leq 0 = |a_{ji\cdots i}|R_i(\mathcal{A}).$$

This also implies $\Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$. The conclusion follows from Case I and Case II. □

Remark 1 Theorem 4 shows that the set $\Psi(\mathcal{A})$ in Theorem 3 is tighter than $\mathcal{K}(\mathcal{A})$ in Theorem 1 and $\mathcal{L}(\mathcal{A})$ in Theorem 2, that is, $\Psi(\mathcal{A})$ can capture all Z-eigenvalues of $\mathcal{A}$ more precisely than $\mathcal{K}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$.

Now, we give an example to show that $\Psi(\mathcal{A})$ is tighter than $\mathcal{K}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$.

Example 1 Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be a symmetric tensor defined by

$$a_{1222} = 1, \qquad a_{2222} = 2, \quad \text{and} \quad a_{ijkl} = 0 \quad \text{elsewhere.}$$

By computation, we see that all the Z-eigenvalues of $\mathcal{A}$ are -0.5000, 0 and 2.7000. By Theorem 1, we have

$$\begin{aligned}\mathcal{K}(\mathcal{A}) &= \mathcal{K}_1(\mathcal{A}) \cup \mathcal{K}_2(\mathcal{A}) = \big\{z \in \mathbb{C} : |z| \leq 1\big\} \cup \big\{z \in \mathbb{C} : |z| \leq 5\big\} \\ &= \big\{z \in \mathbb{C} : |z| \leq 5\big\}.\end{aligned}$$

By Theorem 2, we have

$$\begin{aligned}\mathcal{L}(\mathcal{A}) &= \mathcal{L}_{1,2}(\mathcal{A}) \cup \mathcal{L}_{2,1}(\mathcal{A}) = \big\{z \in \mathbb{C} : |z| \leq 2.2361\big\} \cup \big\{z \in \mathbb{C} : |z| \leq 5\big\} \\ &= \big\{z \in \mathbb{C} : |z| \leq 5\big\}.\end{aligned}$$

By Theorem 3, we have

$$\begin{aligned}\Psi(\mathcal{A}) &= \Psi_{1,2}(\mathcal{A}) \cup \Psi_{2,1}(\mathcal{A}) = \big\{z \in \mathbb{C} : |z| \leq 2.2361\big\} \cup \big\{z \in \mathbb{C} : |z| \leq 3\big\} \\ &= \big\{z \in \mathbb{C} : |z| \leq 3\big\}.\end{aligned}$$

The Z-eigenvalue inclusion sets $\mathcal{K}(\mathcal{A})$, $\mathcal{L}(\mathcal{A})$, $\Psi(\mathcal{A})$ and the exact Z-eigenvalues are drawn in Figure 1, where $\mathcal{K}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$ are represented by blue dashed boundary, $\Psi(\mathcal{A})$ is represented by red solid boundary and the exact eigenvalues are plotted by '+', respectively. It is easy to see $\sigma(\mathcal{A}) \subseteq \Psi(\mathcal{A}) \subset \mathcal{L}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$, that is, $\Psi(\mathcal{A})$ can capture all Z-eigenvalues of $\mathcal{A}$ more precisely than $\mathcal{L}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$.

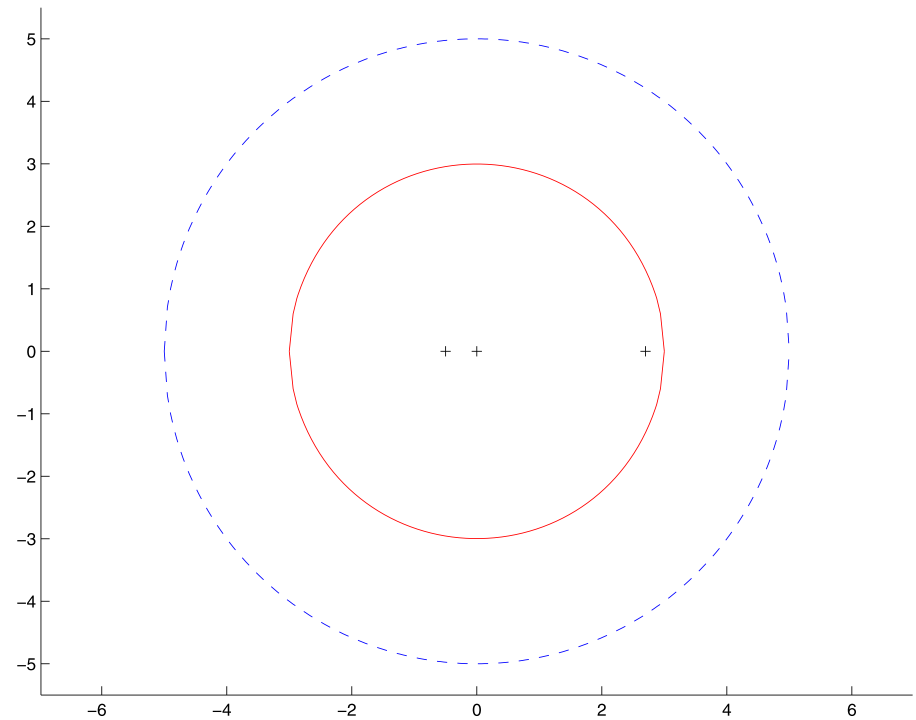

Figure 1 **Comparisons of $\mathcal{K}(\mathcal{A})$, $\mathcal{L}(\mathcal{A})$ and $\Psi(\mathcal{A})$.**

3 A new upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors

As an application of the results in Section 2, we in this section give a new upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors.

Theorem 5 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor. Then*

$$\varrho(\mathcal{A}) \le \max_{i\in N} \min_{j\in N, j\neq i} \Phi_{i,j}(\mathcal{A}),$$

where

$$\Phi_{i,j}(\mathcal{A}) = \frac{1}{2}\left\{ r_i^{\overline{\Delta}_j}(\mathcal{A}) + \sqrt{\left(r_i^{\overline{\Delta}_j}(\mathcal{A})\right)^2 + 4 r_i^{\Delta_j}(\mathcal{A}) R_j(\mathcal{A})}\right\}.$$

Proof From Lemma 4.4 in [1], we know that $\varrho(\mathcal{A})$ is the largest Z-eigenvalue of $\mathcal{A}$. It follows from Theorem 3 that there exists $i \in N$ such that

$$\left(\varrho(\mathcal{A}) - r_i^{\overline{\Delta}_j}(\mathcal{A})\right)\varrho(\mathcal{A}) \le r_i^{\Delta_j}(\mathcal{A}) R_j(\mathcal{A}), \quad \forall j \in N, j \neq i. \tag{12}$$

Solving $\varrho(\mathcal{A})$ in (12) gives

$$\varrho(\mathcal{A}) \le \frac{1}{2}\left\{ r_i^{\overline{\Delta}_j}(\mathcal{A}) + \sqrt{\left(r_i^{\overline{\Delta}_j}(\mathcal{A})\right)^2 + 4 r_i^{\Delta_j}(\mathcal{A}) R_j(\mathcal{A})}\right\} = \Phi_{i,j}(\mathcal{A}).$$

From the arbitrariness of j, we have $\varrho(\mathcal{A}) \le \min_{j\in N, j\neq i} \Phi_{i,j}(\mathcal{A})$. Furthermore, $\varrho(\mathcal{A}) \le \max_{i\in N} \min_{j\in N, j\neq i} \Phi_{i,j}(\mathcal{A})$. □

By Theorem 4, Theorem 4.5 and Corollary 4.1 in [1], the following comparison theorem can be derived easily.

Theorem 6 *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ *be a weakly symmetric nonnegative tensor. Then the upper bound in Theorem* 5 *is sharper than those in Theorem* 4.5 *of* [1] *and Corollary* 4.5 *of* [5], *that is*,

$$\begin{aligned}\varrho(\mathcal{A}) &\le \max_{i\in N}\min_{j\in N, j\neq i}\Phi_{i,j}(\mathcal{A})\\ &\le \max_{i\in N}\min_{j\in N, j\neq i}\frac{1}{2}\Big\{R_i(\mathcal{A}) - a_{ij\cdots j} + \sqrt{\big(R_i(\mathcal{A}) - a_{ij\cdots j}\big)^2 + 4a_{ij\cdots j}R_j(\mathcal{A})}\Big\}\\ &\le \max_{i\in N} R_i(\mathcal{A}).\end{aligned}$$

Finally, we show that the upper bound in Theorem 5 is sharper than those in [1, 5–8, 10] by the following example.

Example 2 Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,3]}$ with the entries defined as follows:

$$\mathcal{A}(:,:,1) = \begin{pmatrix} 3 & 3 & 0\\ 3 & 2 & 2.5\\ 0.5 & 2.5 & 0\end{pmatrix}, \qquad \mathcal{A}(:,:,2) = \begin{pmatrix} 3 & 2 & 2\\ 2 & 0 & 3\\ 2.5 & 3 & 1\end{pmatrix},$$
$$\mathcal{A}(:,:,3) = \begin{pmatrix} 1 & 3 & 0\\ 2.5 & 3 & 1\\ 0 & 1 & 0\end{pmatrix}.$$

It is not difficult to verify that $\mathcal{A}$ is a weakly symmetric nonnegative tensor. By both Corollary 4.5 of [5] and Theorem 3.3 of [6], we have

$$\varrho(\mathcal{A}) \le 19.$$

By Theorem 3.5 of [7], we have

$$\varrho(\mathcal{A}) \le 18.6788.$$

By Theorem 4.6 of [1], we have

$$\varrho(\mathcal{A}) \le 18.6603.$$

By both Theorem 4.5 of [1] and Theorem 6 of [8], we have

$$\varrho(\mathcal{A}) \le 18.5656.$$

By Theorem 4.7 of [1], we have

$$\varrho(\mathcal{A}) \le 18.3417.$$

By Theorem 2.9 of [10], we have

$$\varrho(\mathcal{A}) \le 17.2063.$$

By Theorem 5, we obtain

$$\varrho(\mathcal{A}) \le 15.2580,$$

which shows that the upper bound in Theorem 5 is sharper.

4 Conclusions

In this paper, we present a new Z-eigenvalue localization set $\Psi(\mathcal{A})$ and prove that this set is tighter than those in [1]. As an application, we obtain a new upper bound $\max_{i\in N}\min_{j\in N,j\neq i}\Phi_{i,j}(\mathcal{A})$ for the Z-spectral radius of weakly symmetric nonnegative tensors, and we show that this bound is sharper than those in [1, 5–8, 10] in some cases by a numerical example.

Competing interests
The author declares that they have no competing interests.

Author's contributions
The author read and approved the final manuscript.

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