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New inclusion sets for singular values

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Abstract

In this paper, for a given matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, in terms of r_i and c_i , where $r_i = \sum_{j=1, j \neq i}^n |a_{ij}|$, $c_i = \sum_{j=1, j \neq i}^n |a_{ji}|$, some new inclusion sets for singular values of the matrix are established. It is proved that the new inclusion sets are tighter than the Geršgorin-type sets (Qi in *Linear Algebra Appl.* 56:105-119, 1984) and the Brauer-type sets (Li in *Comput. Math. Appl.* 37:9-15, 1999). A numerical experiment shows the efficiency of our new results.

MSC: 15A18; 15A57; 65F15

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1 Introduction

Singular values and the singular value decomposition play an important role in numerical analysis and many other applied fields [3–8]. First, we will use the following notations and definitions. Let $N := \{1, 2, \dots, n\}$, and assume $n \geq 2$ throughout. For a given matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, we define $a_i = |a_{ii}|$, $s_i = \max\{r_i, c_i\}$ for any $i \in N$ and $u_+ = \max\{0, u\}$, u is a real number, and where

$$r_i := \sum_{j=1, j \neq i}^n |a_{ij}|, \quad c_i := \sum_{j=1, j \neq i}^n |a_{ji}|.$$

In terms of s_i , the Geršgorin-type, Brauer-type and Ky Fan-type inclusion sets of the matrix singular values are given in [1, 2, 9, 10], we list the results as follows.

Theorem 1 *If a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, then*

(i) (Geršgorin-type, see [1]) *all singular values of A are contained in*

$$C(A) := \bigcup_{i=1}^n C_i \quad \text{with } C_i = [(a_i - s_i)_+, (a_i + s_i)] \in \mathbb{R}; \tag{1}$$

(ii) (Brauer-type, see [2]) *all singular values of A are contained in*

$$D(A) := \bigcup_{i=1}^n \bigcup_{j=1, j \neq i}^n \{z \geq 0 : |z - a_i| |z - a_j| \leq s_i s_j\}; \tag{2}$$

(iii) (*Ky Fan-type*, see [2]) let $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ be a nonnegative matrix satisfying $b_{ij} \geq \max\{|a_{ij}|, |a_{ji}|\}$ for any $i \neq j$, then all singular values of A are contained in

$$E(A) := \bigcup_{i=1}^n \{z \geq 0 : |z - a_i| \leq \rho(B) - b_{ii}\}.$$

We observe that all the results in Theorem 1 are based on the values of $s_i = \max\{r_i, c_i\}$, if $r_i \ll c_i$ or $r_i \gg c_i$, all these singular value localization sets in Theorem 1 become very crude. In this paper, we give some new singular value localization sets which are based on the values of r_i and c_i . The remainder of the paper is organized as follows. In Section 2, we give our main results. In Section 3, a numerical experiment is given to show the efficiency of our new results.

2 New inclusion sets for singular values

Based on the idea of Li in [2], we give our main results as follows.

Theorem 2 *If a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, then all singular values of A are contained in*

$$\Gamma(A) := \Gamma_1(A) \cup \Gamma_2(A),$$

where

$$\Gamma_1(A) := \bigcup_{i=1}^n \{\sigma \geq 0 : |\sigma^2 - |a_{ii}|^2| \leq |a_{ii}|r_i(A) + \sigma c_i(A)\}$$

and

$$\Gamma_2(A) := \bigcup_{i=1}^n \{\sigma \geq 0 : |\sigma^2 - |a_{ii}|^2| \leq |a_{ii}|c_i(A) + \sigma r_i(A)\}.$$

Proof Let σ be an arbitrary singular value of A . Then there exist two nonzero vectors $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ such that

$$\sigma x = A^*y \quad \text{and} \quad \sigma y = Ax. \tag{3}$$

Denote

$$|x_p| = \max\{|x_i|, 1 \leq i \leq n\}, \quad |y_q| = \max\{|y_i|, 1 \leq i \leq n\}.$$

Now, we assume that $|x_p| \leq |y_q|$, the q th equations in (3) imply

$$\sigma x_q - \bar{a}_{qq}y_q = \sum_{j=1, j \neq q}^n \bar{a}_{jq}y_j, \tag{4}$$

$$\sigma y_q - a_{qq}x_q = \sum_{j=1, j \neq q}^n a_{qj}x_j. \tag{5}$$

Solving for y_q we can get

$$(\sigma^2 - a_{qq}\bar{a}_{qq})y_q = a_{qq} \sum_{j=1, j \neq q}^n \bar{a}_{jq}y_j + \sigma \sum_{j=1, j \neq q}^n a_{qj}x_j. \tag{6}$$

Taking the absolute value on both sides of the equation and using the triangle inequality yield

$$|\sigma^2 - a_{qq}|^2 |y_q| \leq |a_{qq}| \sum_{j=1, j \neq q}^n |\bar{a}_{jq}| |y_j| + \sigma \sum_{j=1, j \neq q}^n |a_{qj}| |x_j|. \tag{7}$$

Then we can get

$$|\sigma^2 - |a_{qq}|| \leq |a_{qq}|c_q(A) + \sigma r_q(A).$$

Similarly, if $|y_q| \leq |x_p|$, we can get

$$|\sigma^2 - |a_{pp}|| \leq |a_{pp}|r_p(A) + \sigma c_p(A).$$

Thus, we complete the proof. □

Remark 1 Since

$$|a_{ii}|r_i(A) + \sigma c_i(A) \leq (|a_{ii}| + \sigma)s_i$$

and

$$|a_{ii}|c_i(A) + \sigma r_i(A) \leq (|a_{ii}| + \sigma)s_i,$$

the results in Theorem 2 are always better than the results in Theorem 1(i).

Theorem 3 *If a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, then all singular values of A are contained in*

$$\Omega(A) := \Omega_1(A) \cup \Omega_2(A) \cup \Omega_3(A),$$

where

$$\begin{aligned} \Omega_1(A) &:= \bigcup_{i \neq j} \{ \sigma \geq 0 : \\ &\quad |\sigma^2 - |a_{ii}|^2| |\sigma^2 - |a_{jj}|^2| \leq (|a_{ii}|r_i(A) + \sigma c_i(A))(|a_{jj}|r_j(A) + \sigma c_j(A)) \}, \\ \Omega_2(A) &:= \bigcup_{i \neq j} \{ \sigma \geq 0 : \\ &\quad |\sigma^2 - |a_{ii}|^2| |\sigma^2 - |a_{jj}|^2| \leq (|a_{ii}|c_i(A) + \sigma r_i(A))(|a_{jj}|c_j(A) + \sigma r_j(A)) \}, \\ \Omega_3(A) &:= \bigcup_{i \neq j} \{ \sigma \geq 0 : \\ &\quad |\sigma^2 - |a_{ii}|^2| |\sigma^2 - |a_{jj}|^2| \leq (|a_{ii}|c_i(A) + \sigma r_i(A))(|a_{jj}|c_j(A) + \sigma r_j(A)) \} \end{aligned}$$

and

$$\Omega_q(A) := \bigcup_{i \neq j} \{ \sigma \geq 0 : |\sigma^2 - |a_{ii}|^2| |\sigma^2 - |a_{jj}|^2| \leq (|a_{ii}|r_i(A) + \sigma c_i(A))(|a_{jj}|c_j(A) + \sigma r_j(A)) \}.$$

Proof Let σ be an arbitrary singular value of A . Then there exist two nonzero vectors $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ such that

$$\sigma x = A^*y \quad \text{and} \quad \sigma y = Ax. \tag{8}$$

Denote $\omega_i = \max\{|x_i|, |y_i|\}$. Let q be an index such that $\omega_q = \max\{\omega_i, i \in N\}$. Obviously, $\omega_q \neq 0$. Let p be an index such that $\omega_p = \max\{\omega_i, i \in N, i \neq q\}$.

Case I: We suppose $\omega_q = |x_q|, \omega_p = |x_p|$, similar to the proof of Theorem 2, the q th equations in (8) imply

$$\begin{aligned} |\sigma^2 - |a_{qq}|^2| \omega_q &\leq |a_{qq}| \sum_{j=1, j \neq q}^n |a_{qj}| |y_j| + \sigma \sum_{j=1, j \neq q}^n |a_{jq}| |x_j| \\ &\leq \left(|a_{qq}| \sum_{j=1, j \neq q}^n |a_{qj}| + \sigma \sum_{j=1, j \neq q}^n |a_{jq}| \right) \omega_p. \end{aligned} \tag{9}$$

Similarly, the p th equations in (8) imply

$$|\sigma^2 - |a_{pp}|^2| \omega_p \leq \left(|a_{pp}| \sum_{j=1, j \neq p}^n |a_{pj}| + \sigma \sum_{j=1, j \neq p}^n |a_{jp}| \right) \omega_q. \tag{10}$$

Multiplying inequalities (9) with (10), we have

$$|\sigma^2 - |a_{pp}|^2| |\sigma^2 - |a_{qq}|^2| \leq (|a_{pp}|r_p(A) + \sigma c_p(A))(|a_{qq}|r_q(A) + \sigma c_q(A)).$$

Case II: We suppose $\omega_q = |y_q|, \omega_p = |y_p|$, similar to the proof of Theorem 2, the q th equations in (8) imply

$$\begin{aligned} |\sigma^2 - |a_{qq}|^2| \omega_q &\leq |a_{qq}| \sum_{j=1, j \neq q}^n |a_{jq}| |y_j| + \sigma \sum_{j=1, j \neq q}^n |a_{qj}| |x_j| \\ &\leq \left(|a_{qq}| \sum_{j=1, j \neq q}^n |a_{jq}| + \sigma \sum_{j=1, j \neq q}^n |a_{qj}| \right) \omega_p. \end{aligned} \tag{11}$$

Similarly, the p th equations in (8) imply

$$|\sigma^2 - |a_{pp}|^2| \omega_p \leq \left(|a_{pp}| \sum_{j=1, j \neq p}^n |a_{jp}| + \sigma \sum_{j=1, j \neq p}^n |a_{pj}| \right) \omega_q. \tag{12}$$

Multiplying inequalities (11) with (12), we have

$$|\sigma^2 - |a_{pp}|^2| |\sigma^2 - |a_{qq}|^2| \leq (|a_{pp}|c_p(A) + \sigma r_p(A))(|a_{qq}|c_q(A) + \sigma r_q(A)).$$

Case III: We suppose $\omega_q = |y_q|$, $\omega_p = |x_p|$, similar to the proof of Theorem 2, the q th equations in (8) imply

$$\begin{aligned}
 |\sigma^2 - |a_{qq}|^2| \omega_q &\leq |a_{qq}| \sum_{j=1, j \neq q}^n |a_{jq}| |y_j| + \sigma \sum_{j=1, j \neq q}^n |a_{qj}| |x_j| \\
 &\leq \left(|a_{qq}| \sum_{j=1, j \neq q}^n |a_{jq}| + \sigma \sum_{j=1, j \neq q}^n |a_{qj}| \right) \omega_p.
 \end{aligned}
 \tag{13}$$

Similarly, the p th equations in (8) imply

$$|\sigma^2 - |a_{pp}|^2| \omega_p \leq \left(|a_{pp}| \sum_{j=1, j \neq p}^n |a_{pj}| + \sigma \sum_{j=1, j \neq p}^n |a_{jp}| \right) \omega_q.
 \tag{14}$$

Multiplying inequalities (13) with (14), we have

$$|\sigma^2 - |a_{pp}|^2| |\sigma^2 - |a_{qq}|^2| \leq (|a_{pp}| r_p(A) + \sigma c_p(A)) (|a_{qq}| c_q(A) + \sigma r_q(A)).$$

Case IV: We suppose $\omega_q = |x_q|$, $\omega_p = |y_p|$, similar to the proof of Cases I, II, III, we can get

$$|\sigma^2 - |a_{pp}|^2| |\sigma^2 - |a_{qq}|^2| \leq (|a_{pp}| c_p(A) + \sigma r_p(A)) (|a_{qq}| c_q(A) + \sigma r_q(A)).$$

Thus, we complete the proof. □

Remark 2 Since

$$\begin{aligned}
 (|a_{ii}| r_i(A) + \sigma c_i(A)) (|a_{jj}| r_j(A) + \sigma c_j(A)) &\leq (|a_{ii}| + \sigma) (|a_{jj}| + \sigma) s_i s_j, \\
 (|a_{ii}| c_i(A) + \sigma r_i(A)) (|a_{jj}| c_j(A) + \sigma r_j(A)) &\leq (|a_{ii}| + \sigma) (|a_{jj}| + \sigma) s_i s_j, \\
 (|a_{ii}| r_i(A) + \sigma c_i(A)) (|a_{jj}| c_j(A) + \sigma r_j(A)) &\leq (|a_{ii}| + \sigma) (|a_{jj}| + \sigma) s_i s_j
 \end{aligned}$$

and

$$(|a_{ii}| r_i(A) + \sigma c_i(A)) (|a_{jj}| c_j(A) + \sigma r_j(A)) \leq (|a_{ii}| + \sigma) (|a_{jj}| + \sigma) s_i s_j,$$

the results in Theorem 3 are always better than the results in Theorem 1(ii).

We now establish comparison results between $\Gamma(A)$ and $\Omega(A)$.

Theorem 4 *If a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, then*

$$\sigma(A) \in \Omega(A) \subseteq \Gamma(A).$$

Proof Let z be any point of $\Omega_3(A)$. Then there are $i, j \in N$, $i \neq j$, such that $z \in \Omega_3(A)$, i.e.,

$$|z^2 - |a_{ii}|^2| |z^2 - |a_{jj}|^2| \leq (|a_{ii}| r_i(A) + z c_i(A)) (|a_{jj}| c_j(A) + z r_j(A)).
 \tag{15}$$

If $(|a_{ii}|r_i(A) + zc_i(A))(|a_{jj}|c_j(A) + zr_j(A)) = 0$, then

$$|z^2 - |a_{ii}|^2| = 0$$

or

$$|z^2 - |a_{jj}|^2| = 0.$$

Therefore, $z \in \Gamma_1(A) \cup \Gamma_2(A)$. Moreover, if $(|a_{ii}|r_i(A) + zc_i(A))(|a_{jj}|c_j(A) + zr_j(A)) > 0$, then from inequality (15), we have

$$\frac{|z^2 - |a_{ii}|^2|}{|a_{ii}|r_i(A) + zc_i(A)} \frac{|z^2 - |a_{jj}|^2|}{|a_{jj}|c_j(A) + zr_j(A)} \leq 1. \tag{16}$$

Hence, from inequality (16), we have that

$$\frac{|z^2 - |a_{ii}|^2|}{|a_{ii}|r_i(A) + zc_i(A)} \leq 1$$

or

$$\frac{|z^2 - |a_{jj}|^2|}{|a_{jj}|c_j(A) + zr_j(A)} \leq 1.$$

That is, $z \in \Gamma_1(A)$ or $z \in \Gamma_2(A)$, i.e., $z \in \Gamma(A)$.

Similarly, if z is any point of $\Omega_1(A)$ or $\Omega_2(A)$, we can get

$$\sigma(A) \in \Omega_1(A) \subseteq \Gamma(A)$$

and

$$\sigma(A) \in \Omega_2(A) \subseteq \Gamma(A).$$

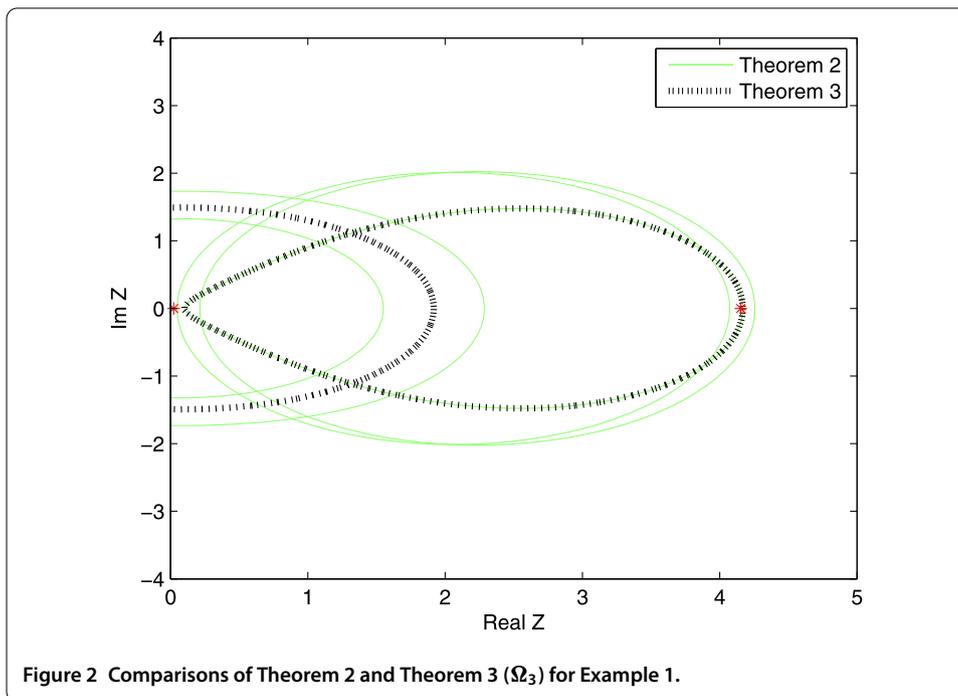
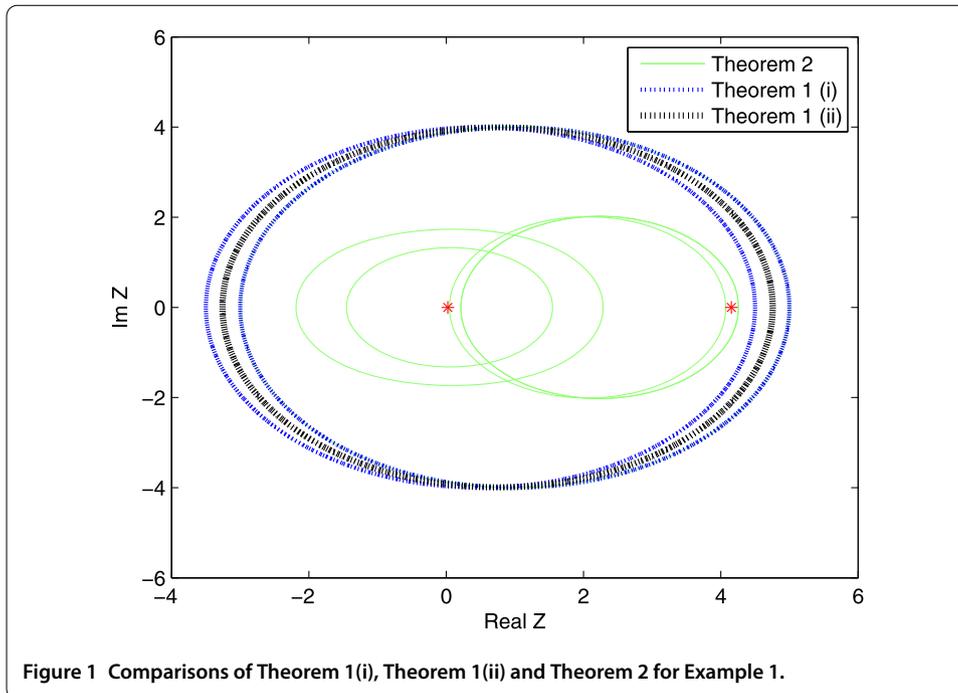
Thus, we complete the proof. □

3 Numerical example

Example 1 Let

$$A = \begin{bmatrix} 1 & 4 \\ 0.1 & 0.5 \end{bmatrix}.$$

The singular values of A are $\sigma_1 = 4.1544$ and $\sigma_2 = 0.0241$. From Figure 1, it is easy to see that Theorem 2 is better than Theorem 1 for certain examples. In Figure 2, we can see that the results in Theorem 3 are tighter than the results in Theorem 2, which is analyzed in Theorem 4.



4 Conclusion

In this paper, some new inclusion sets for singular values are given. Theoretical analysis and numerical example show that these estimates are more efficient than recent corresponding results in some cases.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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