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An eigenvalue localization set for tensors and its applications

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Abstract

A new eigenvalue localization set for tensors is given and proved to be tighter than those presented by Li *et al.* (Linear Algebra Appl. 481:36-53, 2015) and Huang *et al.* (J. Inequal. Appl. 2016:254, 2016). As an application of this set, new bounds for the minimum eigenvalue of $\mathcal{M}$-tensors are established and proved to be sharper than some known results. Compared with the results obtained by Huang *et al.*, the advantage of our results is that, without considering the selection of nonempty proper subsets S of $N = \{1, 2, \ldots, n\}$, we can obtain a tighter eigenvalue localization set for tensors and sharper bounds for the minimum eigenvalue of $\mathcal{M}$-tensors. Finally, numerical examples are given to verify the theoretical results.

MSC: 15A18; 15A69; 65F10; 65F15

Keywords: $\mathcal{M}$-tensors; nonnegative tensors; minimum eigenvalue; localization set

1 Introduction

For a positive integer n, $n \geq 2$, N denotes the set $\{1, 2, \ldots, n\}$. $\mathbb{C}$ (respectively, $\mathbb{R}$) denotes the set of all complex (respectively, real) numbers. We call $\mathcal{A} = (a_{i_1 \cdots i_m})$ a complex (real) tensor of order m dimension n, denoted by $\mathbb{C}^{[m,n]}(\mathbb{R}^{[m,n]})$, if

$$a_{i_1 \cdots i_m} \in \mathbb{C}(\mathbb{R}),$$

where $i_j \in N$ for $j = 1, 2, \ldots, m$. $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $\mathbb{J} \subset N$ such that

$$a_{i_1 i_2 \cdots i_m} = 0, \quad \forall i_1 \in \mathbb{J}, \forall i_2, \ldots, i_m \notin \mathbb{J}.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible [3].

Given a tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, if there are $\lambda \in \mathbb{C}$ and $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C} \backslash \{0\}$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$ and x an eigenvector of $\mathcal{A}$ associated with λ, where $\mathcal{A}x^{m-1}$ is an n dimension vector whose ith component is

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,\dots,i_m \in N} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m}$$

and

$$x^{[m-1]} = \left(x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1}\right)^T.$$

If λ and x are all real, then λ is called an H-eigenvalue of $\mathcal{A}$ and x an H-eigenvector of $\mathcal{A}$ associated with λ; see [4, 5]. Moreover, the spectral radius $\rho(\mathcal{A})$ of $\mathcal{A}$ is defined as

$$\rho(\mathcal{A}) = \max\left\{|\lambda| : \lambda \in \sigma(\mathcal{A})\right\},$$

where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$, that is, $\sigma(\mathcal{A}) = \{\lambda : \lambda \text{ is an eigenvalue of } \mathcal{A}\}$; see [3, 6].

A real tensor $\mathcal{A}$ is called an $\mathcal{M}$-tensor if there exist a nonnegative tensor $\mathcal{B}$ and a positive number $\alpha > \rho(\mathcal{B})$ such that $\mathcal{A} = \alpha\mathcal{I} - \mathcal{B}$, where $\mathcal{I}$ is called the unit tensor with its entries

$$\delta_{i_1\cdots i_m} = \begin{cases} 1 & \text{if } i_1 = \cdots = i_m, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\tau(\mathcal{A})$ the minimal value of the real part of all eigenvalues of an $\mathcal{M}$-tensor $\mathcal{A}$. Then $\tau(\mathcal{A}) > 0$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector. If $\mathcal{A}$ is irreducible, then $\tau(\mathcal{A})$ is the unique eigenvalue with a positive eigenvector [7–9].

Recently, many people have focused on locating eigenvalues of tensors and using obtained eigenvalue inclusion theorems to determine the positive definiteness of an even-order real symmetric tensor or to give the lower and upper bounds for the spectral radius of nonnegative tensors and the minimum eigenvalue of $\mathcal{M}$-tensors. For details, see [1, 2, 10–14].

In 2015, Li *et al.* [1] proposed the following Brauer-type eigenvalue localization set for tensors.

Theorem 1 ([1], Theorem 6) *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Delta(\mathcal{A}) = \bigcup_{i,j \in N, j \neq i} \Delta_i^j(\mathcal{A}),$$

where

$$\Delta_i^j(\mathcal{A}) = \left\{z \in \mathbb{C} : \left|(z - a_{i\cdots i})(z - a_{j\cdots j}) - a_{ij\cdots j}a_{ji\cdots i}\right| \leq |z - a_{j\cdots j}| r_i^j(\mathcal{A}) + |a_{ij\cdots j}| r_j^i(\mathcal{A})\right\},$$

$$r_i(\mathcal{A}) = \sum_{\delta_{ii_2\cdots i_m}=0} |a_{ii_2\cdots i_m}|, \qquad r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\cdots i_m}=0, \\ \delta_{ji_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| = r_i(\mathcal{A}) - |a_{ij\cdots j}|.$$

To reduce computations, Huang *et al.* [2] presented an S-type eigenvalue localization set by breaking N into disjoint subsets S and $\bar{S}$, where $\bar{S}$ is the complement of S in N.

Theorem 2 ([2], Theorem 3.1) *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\sigma(\mathcal{A}) \subseteq \Delta^S(\mathcal{A}) = \left(\bigcup_{i\in S, j\in\bar{S}} \Delta_i^j(\mathcal{A})\right) \cup \left(\bigcup_{i\in\bar{S}, j\in S} \Delta_i^j(\mathcal{A})\right).$$

Based on Theorem 2, Huang *et al.* [2] obtained the following lower and upper bounds for the minimum eigenvalue of $\mathcal{M}$-tensors.

Theorem 3 ([2], Theorem 3.6) *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an $\mathcal{M}$-tensor, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\min\left\{\min_{i\in S}\max_{j\in\bar{S}} L_{ij}(\mathcal{A}), \min_{i\in\bar{S}}\max_{j\in S} L_{ij}(\mathcal{A})\right\} \leq \tau(\mathcal{A}) \leq \max\left\{\max_{i\in S}\min_{j\in\bar{S}} L_{ij}(\mathcal{A}), \max_{i\in\bar{S}}\min_{j\in S} L_{ij}(\mathcal{A})\right\},$$

where

$$L_{ij}(\mathcal{A}) = \frac{1}{2}\left\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \left[\left(a_{i\cdots i} - a_{j\cdots j} - r_i^j(\mathcal{A})\right)^2 - 4a_{ij\cdots j} r_j(\mathcal{A})\right]^{\frac{1}{2}}\right\}.$$

The main aim of this paper is to give a new eigenvalue inclusion set for tensors and prove that this set is tighter than those in Theorems 1 and 2 without considering the selection of S. And then we use this set to obtain new lower and upper bounds for the minimum eigenvalue of $\mathcal{M}$-tensors and prove that new bounds are sharper than those in Theorem 3.

2 Main results

Now, we give a new eigenvalue inclusion set for tensors and establish the comparison between this set with those in Theorems 1 and 2.

Theorem 4 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$. Then*

$$\sigma(\mathcal{A}) \subseteq \Delta^{\cap}(\mathcal{A}) = \bigcup_{i\in N}\bigcap_{j\in N, j\neq i} \Delta_i^j(\mathcal{A}).$$

Proof For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n\backslash\{0\}$ be an associated eigenvector, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{1}$$

Let $|x_p| = \max\{|x_i| : i \in N\}$. Then $|x_p| > 0$. For any $j \in N, j \neq p$, then from (1) we have

$$\lambda x_p^{m-1} = \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{pi_2\cdots i_m} x_{i_2}\cdots x_{i_m} + a_{p\cdots p} x_p^{m-1} + a_{pj\cdots j} x_j^{m-1}$$

and

$$\lambda x_j^{m-1} = \sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} a_{ji_2\cdots i_m} x_{i_2}\cdots x_{i_m} + a_{j\cdots j} x_j^{m-1} + a_{jp\cdots p} x_p^{m-1},$$

equivalently,

$$(\lambda - a_{p\cdots p})x_p^{m-1} - a_{pj\cdots j}x_j^{m-1} = \sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{pi_2\cdots i_m}x_{i_2}\cdots x_{i_m} \tag{2}$$

and

$$(\lambda - a_{j\cdots j})x_j^{m-1} - a_{jp\cdots p}x_p^{m-1} = \sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m}. \tag{3}$$

Solving x_p^{m-1} from (2) and (3), we get

$$\begin{aligned}&\bigl((\lambda - a_{p\cdots p})(\lambda - a_{j\cdots j}) - a_{pj\cdots j}a_{jp\cdots p}\bigr)x_p^{m-1}\\ &\quad = (\lambda - a_{j\cdots j})\sum_{\substack{\delta_{pi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{pi_2\cdots i_m}x_{i_2}\cdots x_{i_m} + a_{pj\cdots j}\sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{pi_2\cdots i_m}=0}} a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m}.\end{aligned}$$

Taking absolute values and using the triangle inequality yields

$$\begin{aligned}&\bigl|(\lambda - a_{p\cdots p})(\lambda - a_{j\cdots j}) - a_{pj\cdots j}a_{jp\cdots p}\bigr||x_p|^{m-1}\\ &\quad \le |\lambda - a_{j\cdots j}|r_p^j(\mathcal{A})|x_p|^{m-1} + |a_{pj\cdots j}|r_j^p(\mathcal{A})|x_p|^{m-1}.\end{aligned}$$

Furthermore, by $|x_p| > 0$, we have

$$\bigl|(\lambda - a_{p\cdots p})(\lambda - a_{j\cdots j}) - a_{pj\cdots j}a_{jp\cdots p}\bigr| \le |\lambda - a_{j\cdots j}|r_p^j(\mathcal{A}) + |a_{pj\cdots j}|r_j^p(\mathcal{A}),$$

which implies that $\lambda \in \Delta_p^j(\mathcal{A})$. From the arbitrariness of j, we have $\lambda \in \bigcap_{j\in N, j\neq p}\Delta_p^j(\mathcal{A})$. Furthermore, we have $\lambda \in \bigcup_{i\in N}\bigcap_{j\in N, j\neq i}\Delta_i^j(\mathcal{A})$. The conclusion follows. □

Next, a comparison theorem is given for Theorems 1, 2 and 4.

Theorem 5 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$, S be a nonempty proper subset of N. Then*

$$\Delta^{\cap}(\mathcal{A}) \subseteq \Delta^S(\mathcal{A}) \subseteq \Delta(\mathcal{A}).$$

Proof By Theorem 3.2 in [2], $\Delta^S(\mathcal{A}) \subseteq \Delta(\mathcal{A})$. Here, only $\Delta^{\cap}(\mathcal{A}) \subseteq \Delta^S(\mathcal{A})$ is proved. Let $z \in \Delta^{\cap}(\mathcal{A})$, then there exists some $i_0 \in N$ such that $z \in \Delta_{i_0}^j(\mathcal{A}), \forall j \in N, j \neq i_0$. Let $\bar{S}$ be the complement of S in N. If $i_0 \in S$, then taking $j \in \bar{S}$, obviously, $z \in \bigcup_{i_0\in S, j\in\bar{S}}\Delta_{i_0}^j(\mathcal{A}) \subseteq \Delta^S(\mathcal{A})$. If $i_0 \in \bar{S}$, then taking $j \in S$, obviously, $z \in \bigcup_{i_0\in\bar{S}, j\in S}\Delta_{i_0}^j(\mathcal{A}) \subseteq \Delta^S(\mathcal{A})$. The conclusion follows. □

Remark 1 Theorem 5 shows that the set $\Delta^{\cap}(\mathcal{A})$ in Theorem 4 is tighter than those in Theorems 1 and 2, that is, $\Delta^{\cap}(\mathcal{A})$ can capture all eigenvalues of $\mathcal{A}$ more precisely than $\Delta(\mathcal{A})$ and $\Delta^S(\mathcal{A})$.

In the following, we give new lower and upper bounds for the minimum eigenvalue of $\mathcal{M}$-tensors.

Theorem 6 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible $\mathcal{M}$-tensor. Then*

$$\min_{i\in N}\max_{j\neq i} L_{ij}(\mathcal{A}) \leq \tau(\mathcal{A}) \leq \max_{i\in N}\min_{j\neq i} L_{ij}(\mathcal{A}).$$

Proof Let $x = (x_1, x_2, \dots, x_n)^T$ be an associated positive eigenvector of $\mathcal{A}$ corresponding to $\tau(\mathcal{A})$, i.e.,

$$\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}. \tag{4}$$

(I) Let $x_q = \min\{x_i : i \in N\}$. For any $j \in N, j \neq q$, we have by (4) that

$$\tau(\mathcal{A})x_q^{m-1} = \sum_{\substack{\delta_{qi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{qi_2\cdots i_m}x_{i_2}\cdots x_{i_m} + a_{q\cdots q}x_q^{m-1} + a_{qj\cdots j}x_j^{m-1}$$

and

$$\tau(\mathcal{A})x_j^{m-1} = \sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m} + a_{j\cdots j}x_j^{m-1} + a_{jq\cdots q}x_q^{m-1},$$

equivalently,

$$\bigl(\tau(\mathcal{A}) - a_{q\cdots q}\bigr)x_q^{m-1} - a_{qj\cdots j}x_j^{m-1} = \sum_{\substack{\delta_{qi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{qi_2\cdots i_m}x_{i_2}\cdots x_{i_m} \tag{5}$$

and

$$\bigl(\tau(\mathcal{A}) - a_{j\cdots j}\bigr)x_j^{m-1} - a_{jq\cdots q}x_q^{m-1} = \sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m}. \tag{6}$$

Solving x_q^{m-1} by (5) and (6), we get

$$\begin{aligned}
&\bigl(\bigl(\tau(\mathcal{A}) - a_{q\cdots q}\bigr)\bigl(\tau(\mathcal{A}) - a_{j\cdots j}\bigr) - a_{qj\cdots j}a_{jq\cdots q}\bigr)x_q^{m-1} \\
&\quad = \bigl(\tau(\mathcal{A}) - a_{j\cdots j}\bigr) \sum_{\substack{\delta_{qi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} a_{qi_2\cdots i_m}x_{i_2}\cdots x_{i_m} + a_{qj\cdots j} \sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m}.
\end{aligned}$$

From Theorem 2.1 in [9], we have $\tau(\mathcal{A}) \leq \min_{i\in N} a_{i\cdots i}$ and

$$\begin{aligned}
&\bigl(\bigl(a_{q\cdots q} - \tau(\mathcal{A})\bigr)\bigl(a_{j\cdots j} - \tau(\mathcal{A})\bigr) - a_{qj\cdots j}a_{jq\cdots q}\bigr)x_q^{m-1} \\
&\quad = \bigl(a_{j\cdots j} - \tau(\mathcal{A})\bigr) \sum_{\substack{\delta_{qi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} |a_{qi_2\cdots i_m}|x_{i_2}\cdots x_{i_m} + |a_{qj\cdots j}| \sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}} |a_{ji_2\cdots i_m}|x_{i_2}\cdots x_{i_m}.
\end{aligned}$$

Hence,

$$\begin{aligned}&\big(\big(a_{q\cdots q}-\tau(\mathcal{A})\big)\big(a_{j\cdots j}-\tau(\mathcal{A})\big)-|a_{qj\cdots j}||a_{jq\cdots q}|\big)x_q^{m-1}\\&\quad\geq\big(a_{j\cdots j}-\tau(\mathcal{A})\big)\sum_{\substack{\delta_{qi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}}|a_{qi_2\cdots i_m}|x_q^{m-1}+|a_{qj\cdots j}|\sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}}|a_{ji_2\cdots i_m}|x_q^{m-1}.\end{aligned}$$

From $x_q > 0$, we have

$$\begin{aligned}&\big(a_{q\cdots q}-\tau(\mathcal{A})\big)\big(a_{j\cdots j}-\tau(\mathcal{A})\big)-|a_{qj\cdots j}||a_{jq\cdots q}|\\&\quad\geq\big(a_{j\cdots j}-\tau(\mathcal{A})\big)\sum_{\substack{\delta_{qi_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}}|a_{qi_2\cdots i_m}|+|a_{qj\cdots j}|\sum_{\substack{\delta_{ji_2\cdots i_m}=0,\\ \delta_{qi_2\cdots i_m}=0}}|a_{ji_2\cdots i_m}|\\&\quad=\big(a_{j\cdots j}-\tau(\mathcal{A})\big)r_q^j(\mathcal{A})+|a_{qj\cdots j}|r_j^q(\mathcal{A}),\end{aligned}$$

equivalently,

$$\big(a_{q\cdots q}-\tau(\mathcal{A})\big)\big(a_{j\cdots j}-\tau(\mathcal{A})\big)-\big(a_{j\cdots j}-\tau(\mathcal{A})\big)r_q^j(\mathcal{A})-|a_{qj\cdots j}|r_j(\mathcal{A})\geq 0,$$

that is,

$$\tau(\mathcal{A})^2-\big(a_{q\cdots q}+a_{j\cdots j}-r_q^j(\mathcal{A})\big)\tau(\mathcal{A})+a_{q\cdots q}a_{j\cdots j}-a_{j\cdots j}r_q^j(\mathcal{A})+a_{qj\cdots j}r_j(\mathcal{A})\geq 0.$$

Solving for $\tau(\mathcal{A})$ gives

$$\tau(\mathcal{A})\leq\frac{1}{2}\Big\{a_{q\cdots q}+a_{j\cdots j}-r_q^j(\mathcal{A})-\big[\big(a_{q\cdots q}-a_{j\cdots j}-r_q^j(\mathcal{A})\big)^2-4a_{qj\cdots j}r_j(\mathcal{A})\big]^{\frac{1}{2}}\Big\}=L_{qj}(\mathcal{A}).$$

For the arbitrariness of j, we have $\tau(\mathcal{A})\leq\min_{j\neq q}L_{qj}(\mathcal{A})$. Furthermore, we have

$$\tau(\mathcal{A})\leq\max_{i\in N}\min_{j\neq i}L_{ij}(\mathcal{A}).$$

(II) Let $x_p=\max\{x_i : i\in N\}$. Similar to (I), we have

$$\tau(\mathcal{A})\geq\min_{i\in N}\max_{j\neq i}L_{ij}(\mathcal{A}).$$

The conclusion follows from (I) and (II). □

Similar to the proof of Theorem 3.6 in [2], we can extend the results of Theorem 6 to a more general case.

Theorem 7 *Let $\mathcal{A}=(a_{i_1\cdots i_m})\in\mathbb{R}^{[m,n]}$ be an $\mathcal{M}$-tensor. Then*

$$\min_{i\in N}\max_{j\neq i}L_{ij}(\mathcal{A})\leq\tau(\mathcal{A})\leq\max_{i\in N}\min_{j\neq i}L_{ij}(\mathcal{A}).$$

By Theorems 3, 6 and 7 in [13], the following comparison theorem is obtained easily.

Theorem 8 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an $\mathcal{M}$-tensor, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\min_{i\in N} R_i(\mathcal{A}) \le \min_{j\ne i} L_{ij}(\mathcal{A}) \le \min\Big\{\min_{i\in S}\max_{j\in \bar{S}} L_{ij}(\mathcal{A}), \min_{i\in \bar{S}}\max_{j\in S} L_{ij}(\mathcal{A})\Big\} \le \min_{i\in N}\max_{j\ne i} L_{ij}(\mathcal{A})$$
$$\le \max_{i\in N}\min_{j\ne i} L_{ij}(\mathcal{A}) \le \max\Big\{\max_{i\in S}\min_{j\in \bar{S}} L_{ij}(\mathcal{A}), \max_{i\in \bar{S}}\min_{j\in S} L_{ij}(\mathcal{A})\Big\},$$

where $R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m\in N} a_{ii_2\cdots i_m}$.

Remark 2 Theorem 8 shows that the bounds in Theorem 7 are shaper than those in Theorem 3, Theorem 2.1 of [9] and Theorem 4 of [13] without considering the selection of S, which is also the advantage of our results.

3 Numerical examples

In this section, two numerical examples are given to verify the theoretical results.

Example 1 Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,4]}$ be an irreducible $\mathcal{M}$-tensor with elements defined as follows:

$$\mathcal{A}(:,:,1) = \begin{pmatrix} 62 & -3 & -4 & -2 \\ -4 & -2 & -2 & -1 \\ -3 & -1 & -3 & -3 \\ -3 & -3 & -2 & -2 \end{pmatrix}, \qquad \mathcal{A}(:,:,2) = \begin{pmatrix} 0 & -4 & -3 & -3 \\ -1 & 28 & -2 & -2 \\ -1 & -2 & -2 & -4 \\ -2 & -2 & -3 & -1 \end{pmatrix},$$

$$\mathcal{A}(:,:,3) = \begin{pmatrix} -2 & -1 & -2 & -1 \\ -1 & -1 & -1 & -2 \\ -2 & -4 & 63 & -4 \\ -4 & -4 & -2 & -2 \end{pmatrix}, \qquad \mathcal{A}(:,:,4) = \begin{pmatrix} -4 & -2 & -2 & -1 \\ -1 & -2 & -3 & -1 \\ -2 & -3 & -3 & -2 \\ -2 & -2 & -4 & 61 \end{pmatrix}.$$

By Theorem 2.1 in [9], we have

$$2 = \min_{i\in N} R_i(\mathcal{A}) \le \tau(\mathcal{A}) \le \min\Big\{\max_{i\in N} R_i(\mathcal{A}), \min_{i\in N} a_{i\cdots i}\Big\} = 28.$$

By Theorem 4 in [13], we have

$$\tau(\mathcal{A}) \ge \min_{j\ne i} L_{ij}(\mathcal{A}) = 2.3521.$$

By Theorem 3, we have

$$\text{if } S=\{1\}, \bar{S}=\{2,3,4\}, \quad 3.6685 \le \tau(\mathcal{A}) \le 24.2948;$$
$$\text{if } S=\{2\}, \bar{S}=\{1,3,4\}, \quad 3.6685 \le \tau(\mathcal{A}) \le 19.7199;$$
$$\text{if } S=\{3\}, \bar{S}=\{1,2,4\}, \quad 2.3569 \le \tau(\mathcal{A}) \le 27.7850;$$
$$\text{if } S=\{4\}, \bar{S}=\{1,2,3\}, \quad 2.3521 \le \tau(\mathcal{A}) \le 27.8536;$$
$$\text{if } S=\{1,2\}, \bar{S}=\{3,4\}, \quad 2.3569 \le \tau(\mathcal{A}) \le 27.7850;$$
$$\text{if } S=\{1,3\}, \bar{S}=\{2,4\}, \quad 3.6685 \le \tau(\mathcal{A}) \le 23.0477;$$

$$\text{if } S = \{1,4\}, \bar{S} = \{2,3\}, \quad 3.6685 \le \tau(\mathcal{A}) \le 23.9488.$$

By Theorem 7, we have

$$3.6685 \le \tau(\mathcal{A}) \le 19.7199.$$

In fact, $\tau(\mathcal{A}) = 14.4049$. Hence, this example verifies Theorem 8 and Remark 2, that is, the bounds in Theorem 7 are sharper than those in Theorem 3, Theorem 2.1 of [9] and Theorem 4 of [13] without considering the selection of S.

Example 2 Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be an $\mathcal{M}$-tensor with elements defined as follows:

$$a_{1111} = 6, \qquad a_{1222} = -1, \qquad a_{2111} = -2, \qquad a_{2222} = 5,$$

other $a_{ijkl} = 0$. By Theorem 7, we have

$$4 \le \tau(\mathcal{A}) \le 4.$$

In fact, $\tau(\mathcal{A}) = 4$.

4 Conclusions

In this paper, we give a new eigenvalue inclusion set for tensors and prove that this set is tighter than those in [1, 2]. As an application, we obtain new lower and upper bounds for the minimum eigenvalue of $\mathcal{M}$-tensors and prove that the new bounds are sharper than those in [2, 9, 13]. Compared with the results in [2], the advantage of our results is that, without considering the selection of S, we can obtain a tighter eigenvalue localization set for tensors and sharper bounds for the minimum eigenvalue of $\mathcal{M}$-tensors.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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References

1. Li, CQ, Chen, Z, Li, YT: A new eigenvalue inclusion set for tensors and its applications. Linear Algebra Appl. **481**, 36-53 (2015)
2. Huang, ZG, Wang, LG, Xu, Z, Cui, JJ: A new *S*-type eigenvalue inclusion set for tensors and its applications. J. Inequal. Appl. **2016**, 254 (2016)
3. Chang, KQ, Zhang, T, Pearson, K: Perron-Frobenius theorem for nonnegative tensors. Commun. Math. Sci. **6**, 507-520 (2008)
4. Qi, LQ: Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. **40**, 1302-1324 (2005)
5. Lim, LH: Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing. CAMSAP, vol. 05, pp. 129-132 (2005)
6. Yang, YN, Yang, QZ: Further results for Perron-Frobenius theorem for nonnegative tensors. SIAM J. Matrix Anal. Appl. **31**, 2517-2530 (2010)
7. Ding, WY, Qi, LQ, Wei, YM: $\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors. Linear Algebra Appl. **439**, 3264-3278 (2013)
8. Zhang, LP, Qi, LQ, Zhou, GL: $\mathcal{M}$-tensors and some applications. SIAM J. Matrix Anal. Appl. **35**, 437-452 (2014)
9. He, J, Huang, TZ: Inequalities for $\mathcal{M}$-tensors. J. Inequal. Appl. **2014**, 114 (2014)

10. Li, CQ, Li, YT, Kong, X: New eigenvalue inclusion sets for tensors. Numer. Linear Algebra Appl. **21**, 39-50 (2014)
11. Li, CQ, Li, YT: An eigenvalue localization set for tensor with applications to determine the positive (semi-)definiteness of tensors. Linear Multilinear Algebra **64**(4), 587-601 (2016)
12. Li, CQ, Jiao, AQ, Li, YT: An *S*-type eigenvalue location set for tensors. Linear Algebra Appl. **493**, 469-483 (2016)
13. Zhao, JX, Sang, CL: Two new lower bounds for the minimum eigenvalue of $\mathcal{M}$-tensors. J. Inequal. Appl. **2016**, 268 (2016)
14. He, J: Bounds for the largest eigenvalue of nonnegative tensors. J. Comput. Anal. Appl. **20**(7), 1290-1301 (2016)