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# New iterative criteria for strong $\mathcal{H}$ -tensors and an application

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## Abstract

Strong  $\mathcal{H}$ -tensors play an important role in identifying the positive definiteness of even-order real symmetric tensors. In this paper, some new iterative criteria for identifying strong  $\mathcal{H}$ -tensors are obtained. These criteria only depend on the elements of the tensors, and it can be more effective to determine whether a given tensor is a strong  $\mathcal{H}$ -tensor or not by increasing the number of iterations. Some numerical results show the feasibility and effectiveness of the algorithm.

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**Keywords:** strong  $\mathcal{H}$ -tensors; positive definiteness; irreducible; non-zero elements chain

## 1 Introduction

A tensor can be regarded as a higher-order generalization of a matrix. Let  $\mathbb{C}(\mathbb{R})$  denote the set of all complex (real) numbers and  $N = \{1, 2, \dots, n\}$ . We call  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  an  $m$ th-order  $n$ -dimensional complex (real) tensor, if

$$a_{i_1 i_2 \dots i_m} \in \mathbb{C}(\mathbb{R}),$$

where  $i_j = 1, 2, \dots, n$  for  $j = 1, 2, \dots, m$  [1, 2]. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2. A tensor  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  is called symmetric [3], if

$$a_{i_1 i_2 \dots i_m} = a_{\pi(i_1 i_2 \dots i_m)}, \quad \forall \pi \in \Pi_m,$$

where  $\Pi_m$  is the permutation group of  $m$  indices. Furthermore, an  $m$ th-order  $n$ -dimensional tensor  $\mathcal{I} = (\delta_{i_1 i_2 \dots i_m})$  is called the unit tensor [4], if its entries

$$\delta_{i_1 i_2 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor. If there exist a number  $\lambda \in \mathbb{C}$  and a non-zero vector  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$  that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then we call  $\lambda$  an eigenvalue of  $\mathcal{A}$  and  $x$  the eigenvector of  $\mathcal{A}$  associated with  $\lambda$  [1, 5–7], where  $\mathcal{A}x^{m-1}$  and  $\lambda x^{[m-1]}$  are vectors, whose  $i$ th components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, i_3, \dots, i_m \in N} a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m}$$

and

$$(x^{[m-1]})_i = x_i^{m-1},$$

respectively. In particular, if  $\lambda$  and  $x$  are restricted in the real field, then we call  $\lambda$  an  $H$ -eigenvalue of  $\mathcal{A}$  and  $x$  an  $H$ -eigenvector of  $\mathcal{A}$  associated with  $\lambda$  [1].

In addition, the spectral radius of a tensor  $\mathcal{A}$  is defined as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

Analogous with that of  $M$ -matrices, comparison matrices and  $H$ -matrices, the definitions of  $\mathcal{M}$ -tensors, comparison tensors and strong  $\mathcal{H}$ -tensors are given by the following.

**Definition 1.1** ([8]) Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor.  $\mathcal{A}$  is called an  $\mathcal{M}$ -tensor if there exist a non-negative tensor  $\mathcal{B}$  and a positive real number  $\eta \geq \rho(\mathcal{B})$  such that  $\mathcal{A} = \eta \mathcal{I} - \mathcal{B}$ . If  $\eta > \rho(\mathcal{B})$ , then  $\mathcal{A}$  is called a strong  $\mathcal{M}$ -tensor.

**Definition 1.2** ([9]) Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor. We call another tensor  $\mathcal{M}(\mathcal{A}) = (m_{i_1 i_2 \dots i_m})$  as the comparison tensor of  $\mathcal{A}$  if

$$m_{i_1 i_2 \dots i_m} = \begin{cases} |a_{i_1 i_2 \dots i_m}|, & \text{if } (i_2, i_3, \dots, i_m) = (i_1, i_1, \dots, i_1); \\ -|a_{i_1 i_2 \dots i_m}|, & \text{if } (i_2, i_3, \dots, i_m) \neq (i_1, i_1, \dots, i_1). \end{cases}$$

**Definition 1.3** ([10]) Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor.  $\mathcal{A}$  is called a strong  $\mathcal{H}$ -tensor if there is a positive vector  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$  such that

$$|a_{ii \dots i}| x_i^{m-1} > \sum_{\substack{i_2, i_3, \dots, i_m \in N, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| x_{i_2} \cdots x_{i_m}, \quad \forall i \in N. \quad (1.1)$$

**Definition 1.4** ([10]) Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor.  $\mathcal{A}$  is called a diagonally dominant tensor if

$$|a_{ii \dots i}| \geq \sum_{\substack{i_2, i_3, \dots, i_m \in N, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|, \quad \forall i \in N. \quad (1.2)$$

We call  $\mathcal{A}$  a strictly diagonally dominant tensor if all strict inequalities in (1.2) hold.

**Definition 1.5** ([4]) An  $m$ th-order  $n$ -dimensional complex tensor  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  is called reducible, if there exists a nonempty proper index subset  $I \subset N$  such that

$$a_{i_1 i_2 \dots i_m} = 0, \quad \forall i_1 \in I, \forall i_2, \dots, i_m \notin I.$$

We call  $\mathcal{A}$  irreducible if  $\mathcal{A}$  is not reducible.

**Definition 1.6** ([2]) Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional tensor and a  $n$ -by- $n$  matrix  $X = (x_{ij})$  on mode- $k$  is defined

$$(\mathcal{A} \times_k X)_{i_1 \dots j_k \dots i_m} = \sum_{i_k=1}^n a_{i_1 \dots i_k \dots i_m} x_{i_k j_k}.$$

According to Definition 1.6, we denote

$$(\mathcal{A} X^{m-1}) := \mathcal{A} \times_2 X \times_3 \dots \times_m X.$$

Particularly, for  $X = \text{diag}(x_1, x_2, \dots, x_n)$ , the product of the tensor  $\mathcal{A}$  and the matrix  $X$  is given by

$$\mathcal{B} = (b_{i_1 i_2 \dots i_m}) = \mathcal{A} X^{m-1}, \quad b_{i_1 i_2 \dots i_m} = a_{i_1 i_2 \dots i_m} x_{i_2} x_{i_3} \dots x_{i_m}, \quad i_j \in N, j \in \{1, 2, \dots, m\}.$$

**Definition 1.7** ([2]) Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor. For some  $i, j \in N$  ( $i \neq j$ ), if there exist indices  $k_1, k_2, \dots, k_r$  with

$$\sum_{\substack{i_2, i_3, \dots, i_m \in N, \\ \delta_{k_s i_2 \dots i_m} = 0, \\ k_{s+1} \in \{i_2, i_3, \dots, i_m\}}} |a_{k_s i_2 \dots i_m}| \neq 0, \quad s = 0, 1, \dots, r,$$

where  $k_0 = i, k_{r+1} = j$ , we call there is a non-zero elements chain from  $i$  to  $j$ .

For an  $m$ th degree homogeneous polynomial of  $n$  variables  $f(x)$  is denoted as

$$f(x) = \sum_{i_1, i_2, \dots, i_m \in N} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m}, \quad (1.3)$$

where  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ . When  $m$  is even,  $f(x)$  is called positive definite if

$$f(x) > 0, \quad \text{for any } x \in \mathbb{R}^n, x \neq 0.$$

The homogeneous polynomial  $f(x)$  in (1.3) is equivalent to the tensor product of an  $m$ th-order  $n$ -dimensional symmetric tensor  $\mathcal{A}$  and  $x^m$  defined by [11]

$$f(x) = \mathcal{A} x^m = \sum_{i_1, i_2, \dots, i_m \in N} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m}, \quad (1.4)$$

where  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ . It is well known that the positive definiteness of multivariate polynomial  $f(x)$  plays an important role in the stability study of nonlinear autonomous systems [8, 12]. For  $n \leq 3$ , the positive definiteness of the multivariate polynomial form can be checked by a method based on the Sturm theorem [13]. However, for  $n > 3$  and  $m \geq 4$ , it is difficult to determine a given even-order multivariate polynomial  $f(x)$  is positive semi-definite or not because the problem is NP-hard. For solving this problem, Qi [1] pointed out that  $f(x)$  defined by (1.4) is positive definite if and only if the real symmetric tensor  $\mathcal{A}$  is positive definite, and provided an eigenvalue method to verify the positive definiteness of  $\mathcal{A}$  when  $m$  is even (see Lemma 1.1).

**Lemma 1.1** ([1]) *Let  $\mathcal{A}$  be an even-order real symmetric tensor, then  $\mathcal{A}$  is positive definite if and only if all of its  $H$ -eigenvalues are positive.*

Although from Lemma 1.1 we can verify the positive definiteness of an even-order symmetric tensor  $\mathcal{A}$  (the positive definiteness of the  $m$ th-degree homogeneous polynomial  $f(x)$ ) by computing the  $H$ -eigenvalues of  $\mathcal{A}$ . In [14–16], for a non-negative tensor, some algorithms are provided to compute its largest eigenvalue. And in [17, 18], based on semi-definite programming approximation schemes, some algorithms are also given to compute eigenvalues for general tensors with moderate sizes. However, it is difficult to compute all these  $H$ -eigenvalues when  $m$  and  $n$  are large. Recently, by introducing the definition of strong  $\mathcal{H}$ -tensor [9, 10], Li *et al.* [10] provided a practical sufficient condition for identifying the positive definiteness of an even-order symmetric tensor (see Lemma 1.2).

**Lemma 1.2** ([10]) *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an even-order real symmetric tensor with  $a_{k \dots k} > 0$  for all  $k \in N$ . If  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor, then  $\mathcal{A}$  is positive definite.*

As mentioned in [19], it is still difficult to determine a strong  $\mathcal{H}$ -tensor in practice by using the definition of strong  $\mathcal{H}$ -tensor because the conditions ‘there is a positive vector  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$  such that, for all  $i \in N$ , the Inequality (1.1) holds’ in Definition 1.3 is unverifiable for there are an infinite number of positive vector in  $\mathbb{R}^n$ . Therefore, much literature has focused on researching how to determine that a given tensor is a strong  $\mathcal{H}$ -tensor by using the elements of the tensors without Definition 1.3 recently, consequently, the corresponding even-order real symmetric tensor is positive definite. Therefore, the main aim of this paper is to study some new iterative criteria for identifying strong  $\mathcal{H}$ -tensors only depending on the elements of the tensors.

Before presenting our results, we review the existing ones that relate to the criteria for strong  $\mathcal{H}$ -tensors. Let  $S$  be an arbitrary nonempty subset of  $N$  and let  $N \setminus S$  be the complement of  $S$  in  $N$ . Given an  $m$ th-order  $n$ -dimensional complex tensor  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ , we denote

$$\begin{aligned} N^{m-1} &= \{i_2 i_3 \dots i_m : i_j \in N, j = 2, 3, \dots, m\}; \\ S^{m-1} &= \{i_2 i_3 \dots i_m : i_j \in S, j = 2, 3, \dots, m\}; \\ N^{m-1} \setminus S^{m-1} &= \{i_2 i_3 \dots i_m : i_2 i_3 \dots i_m \in N^{m-1} \text{ and } i_2 i_3 \dots i_m \notin S^{m-1}\}; \\ r_i(\mathcal{A}) &= \sum_{\substack{i_2, i_3, \dots, i_m \in N, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| = \sum_{i_2, i_3, \dots, i_m \in N} |a_{ii_2 \dots i_m}| - |a_{ii \dots i}|; \\ r_i^j(\mathcal{A}) &= \sum_{\substack{\delta_{ii_2 \dots i_m} = 0, \\ \delta_{ji_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| = r_i(\mathcal{A}) - |a_{ij \dots j}|; \\ N_1 = N_1(\mathcal{A}) &= \{i \in N : |a_{ii \dots i}| > r_i(\mathcal{A})\}; \\ N_2 = N_2(\mathcal{A}) &= \{i \in N : 0 < |a_{ii \dots i}| \leq r_i(\mathcal{A})\}; \\ s_i &= \frac{|a_{ii \dots i}|}{r_i(\mathcal{A})}, \quad t_i = \frac{r_i(\mathcal{A})}{|a_{ii \dots i}|}, \quad \bar{r} = \max \left\{ \max_{i \in N_2} s_i, \max_{i \in N_1} t_i \right\}; \\ r &= \max_{i \in N_1} \left\{ \frac{\sum_{i_2, i_3, \dots, i_m \in N^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}|}{|a_{ii \dots i}| - \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1}, \delta_{ii_2 \dots i_m} = 0} |a_{ii_2 \dots i_m}|} \right\}; \end{aligned}$$

$$R_i^{(1)}(\mathcal{A}) = \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}| + r \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|, \quad \forall i \in N_1.$$

In [10], Li *et al.* obtained the following result.

**Lemma 1.3** Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be a complex tensor of order  $m$  dimension  $n$ . If there is an index  $i \in N$  such that for all  $j \in N, j \neq i$ ,

$$|a_{ii \dots i}| (|a_{jj \dots j}| - r_j^i(\mathcal{A})) > r_i(\mathcal{A}) |a_{ji \dots i}|,$$

then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.

In [20], Wang and Sun derived the following result.

**Lemma 1.4** Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an order  $m$  dimension  $n$  complex tensor. If

$$|a_{ii \dots i}| s_i > \bar{r} \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| + \sum_{i_2, i_3, \dots, i_m \in N_1} \max_{j \in \{i_2, i_3, \dots, i_m\}} \{t_j\} |a_{ii_2 \dots i_m}|, \quad \forall i \in N_2,$$

then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.

Recently, Li *et al.* in [19] showed the following.

**Lemma 1.5** Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an order  $m$  dimension  $n$  complex tensor. If

$$|a_{ii \dots i}| > \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| + \sum_{i_2, i_3, \dots, i_m \in N_1} \max_{j \in \{i_2, i_3, \dots, i_m\}} \frac{r_j(\mathcal{A})}{|a_{jj \dots j}|} |a_{ii_2 \dots i_m}|, \quad \forall i \in N_2,$$

then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.

In the sequel, Wang *et al.* in [21] proved the following result.

**Lemma 1.6** Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be a complex tensor with order  $m$  and dimension  $n$ . If for all  $i \in N_2, j \in N_1$ ,

$$\begin{aligned} & \left( R_j^{(1)}(\mathcal{A}) - \sum_{\substack{j_2, j_3, \dots, j_m \in N_1^{m-1}, \\ \delta_{jj_2 \dots j_m} = 0}} \max_{k \in \{j_2, j_3, \dots, j_m\}} \frac{R_k^{(1)}(\mathcal{A})}{|a_{kk \dots k}|} |a_{jj_2 \dots j_m}| \right) \\ & \times \left( |a_{ii \dots i}| - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| \right) \\ & > \sum_{t_2, t_3, \dots, t_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{jt_2 \dots t_m}| \sum_{l_2, l_3, \dots, l_m \in N_1^{m-1}} \max_{l \in \{l_2, l_3, \dots, l_m\}} \frac{R_l^{(1)}(\mathcal{A})}{|a_{ll \dots l}|} |a_{il_2 \dots l_m}|, \end{aligned}$$

then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.

In this paper, we continue this research on criteria for strong  $\mathcal{H}$ -tensors; inspired by the ideas of [21], we obtain some new iterative criteria for strong  $\mathcal{H}$ -tensors, which improve the aforementioned Lemmas 1.3-1.6. As applications of the new iterative criteria for strong  $\mathcal{H}$ -tensors, we establish some sufficient conditions of the positive definiteness for an even-order real symmetric tensor. Numerical examples are implemented to illustrate these facts.

Now, some notations are given, which will be used throughout this paper. Let

$$\begin{aligned} Z &= \{0, 1, 2, \dots\}, \quad h^{(0)} = r, \quad \delta_i^{(0)} = 1, \quad \delta_i^{(1)} = \frac{R_i^{(1)}(\mathcal{A})}{|a_{ii\dots i}|}, \quad \forall i \in N_1; \\ h^{(1)} &= \max_{i \in N_1} \left\{ \frac{\sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}|}{R_i^{(1)}(\mathcal{A}) - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(1)} |a_{ii_2 \dots i_m}|} \right\}; \\ R_i^{(l+1)}(\mathcal{A}) &= \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}| \\ &\quad + h^{(l)} \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l)} |a_{ii_2 \dots i_m}|, \quad \forall i \in N_1, l \in Z; \\ \delta_i^{(l+1)} &= \frac{R_i^{(l+1)}(\mathcal{A})}{|a_{ii\dots i}|}, \quad \forall i \in N_1, l \in Z; \\ h^{(l+1)} &= \max_{i \in N_1} \left\{ \frac{\sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}|}{R_i^{(l+1)}(\mathcal{A}) - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l+1)} |a_{ii_2 \dots i_m}|} \right\}, \quad l \in Z. \end{aligned}$$

The remainder of the paper is organized as follows. In Section 2.1, some criteria for identifying strong  $\mathcal{H}$ -tensors are obtained; as an interesting application of these criteria, some sufficient conditions of the positive definiteness for an even-order real symmetric tensor are presented in Section 2.2. Numerical examples are given to verify the corresponding results. Finally, some conclusions are given to end this paper in Section 3.

We adopt the following notations throughout this paper. The calligraphy letters  $\mathcal{A}, \mathcal{B}, \mathcal{H}, \dots$  denote tensors; the capital letters  $A, B, D, \dots$  represent matrices; the lowercase letters  $x, y, \dots$  refer to vectors.

## 2 Main results

### 2.1 Criteria for identifying strong $\mathcal{H}$ -tensors

In this subsection, we give some new criteria for identifying strong  $\mathcal{H}$ -tensors by making use of elements of tensors only. For the convenience of our discussion, we start with the following lemmas, which will be useful in the next proofs.

**Lemma 2.1** *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor, then, for all  $i \in N_1, l = 1, 2, \dots$ ,*

- $1 \geq h^{(l)} \geq 0$ ;
- $1 > \delta_i^{(1)} \geq h^{(1)} \delta_i^{(1)} \geq \delta_i^{(2)} \geq \dots \geq \delta_i^{(l)} \geq h^{(l)} \delta_i^{(l)} \geq \delta_i^{(l+1)} \geq \dots \geq 0$ .

*Proof* Since  $i \in N_1$ , we have  $0 \leq r < 1$ . Moreover, for  $i \in N_1$ , we get

$$r \geq \frac{\sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}|}{|a_{ii \dots i}| - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|}, \quad |a_{ii \dots i}| - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| > 0,$$

which implies

$$r|a_{ii \dots i}| \geq \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}| + r \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| = R_i^{(1)}(\mathcal{A}).$$

From the above inequality,  $\forall i \in N_1$ , we obtain

$$0 \leq \delta_i^{(1)} = \frac{R_i^{(1)}(\mathcal{A})}{|a_{ii \dots i}|} \leq r < 1.$$

Together with the expression of  $R_i^{(1)}(\mathcal{A})$ , for  $\forall i \in N_1$ , we deduce that

$$\begin{aligned} & \frac{\sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}|}{R_i^{(1)}(\mathcal{A}) - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(1)} |a_{ii_2 \dots i_m}|} \\ &= \frac{R_i^{(1)}(\mathcal{A}) - r \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|}{R_i^{(1)}(\mathcal{A}) - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(1)} |a_{ii_2 \dots i_m}|} \leq 1. \end{aligned}$$

Combining the expression of  $h^{(1)}$  and the above inequality results in

$$0 \leq h^{(1)} \leq 1. \quad (2.1)$$

Besides, for  $\forall i \in N_1$ ,

$$R_i^{(1)}(\mathcal{A}) = \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}| + r \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| \leq r_i(\mathcal{A}) < |a_{ii \dots i}|,$$

that is,

$$\delta_i^{(1)} = \frac{R_i^{(1)}(\mathcal{A})}{|a_{ii \dots i}|} \leq \frac{r_i(\mathcal{A})}{|a_{ii \dots i}|} < 1. \quad (2.2)$$

Since

$$h^{(1)} = \max_{i \in N_1} \left\{ \frac{\sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}|}{R_i^{(1)}(\mathcal{A}) - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(1)} |a_{ii_2 \dots i_m}|} \right\},$$

for  $\forall i \in N_1$ , we have

$$h^{(1)} \geq \frac{\sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}|}{R_i^{(1)}(\mathcal{A}) - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}}, \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(1)} |a_{ii_2 \dots i_m}|},$$

which entails

$$\begin{aligned} h^{(1)} R_i^{(1)}(\mathcal{A}) &\geq \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}| + h^{(1)} \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}}, \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(1)} |a_{ii_2 \dots i_m}| \\ &= R_i^{(2)}(\mathcal{A}). \end{aligned}$$

Dividing by  $|a_{ii \dots i}|$  on both sides of the above inequality yields

$$h^{(1)} \delta_i^{(1)} = h^{(1)} \frac{R_i^{(1)}(\mathcal{A})}{|a_{ii \dots i}|} \geq \frac{R_i^{(2)}(\mathcal{A})}{|a_{ii \dots i}|} = \delta_i^{(2)}. \quad (2.3)$$

For  $i \in N_1$ , it follows from (2.1)-(2.3) that

$$1 > \delta_i^{(1)} \geq h^{(1)} \delta_i^{(1)} \geq \delta_i^{(2)} \geq 0.$$

Furthermore, by the expression of  $R_i^{(2)}(\mathcal{A})$  and the above inequality, for  $i \in N_1$ , we have

$$\begin{aligned} &\frac{\sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}|}{R_i^{(2)}(\mathcal{A}) - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}}, \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(2)} |a_{ii_2 \dots i_m}|} \\ &= \frac{R_i^{(2)}(\mathcal{A}) - h^{(1)} \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}}, \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(1)} |a_{ii_2 \dots i_m}|}{R_i^{(2)}(\mathcal{A}) - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}}, \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(2)} |a_{ii_2 \dots i_m}|} \leq 1. \end{aligned}$$

Combining the expression of  $h^{(2)}$  and the above inequality results in

$$0 \leq h^{(2)} \leq 1. \quad (2.4)$$

In the same manner as applied in the proof of (2.3), for  $i \in N_1$ , we obtain

$$h^{(2)} \delta_i^{(2)} \geq \delta_i^{(3)}. \quad (2.5)$$

For  $i \in N_1$ , it follows from inequalities (2.4) and (2.5) that

$$\delta_i^{(2)} \geq h^{(2)} \delta_i^{(2)} \geq \delta_i^{(3)} \geq 0.$$

By an analogical proof as above, we can derive that, for  $i \in N_1, l = 3, 4, \dots$ ,

$$1 \geq h^{(l)} \geq 0;$$



$$\delta_i^{(2)} \geq h^{(2)} \delta_i^{(2)} \geq \delta_i^{(3)} \geq h^{(3)} \delta_i^{(3)} \geq \dots \geq \delta_i^{(l+1)} \geq h^{(l+1)} \delta_i^{(l+1)} \geq \delta_i^{(l+2)} \geq \dots \geq 0.$$

The proof is completed.  $\square$

**Lemma 2.2** ([10]) *If  $\mathcal{A}$  is a strictly diagonally dominant tensor, then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.*

**Lemma 2.3** ([10]) *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor. If  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor, then  $N_1 \neq \emptyset$ .*

By Lemma 2.2, if  $N_2 = \emptyset$  ( $\mathcal{A}$  is a strictly diagonally dominant tensor), then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor; by Lemma 2.3, if  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor, then  $N_1 \neq \emptyset$ . Hence, we always assume that  $N_1 \neq \emptyset, N_2 \neq \emptyset$ . In addition, we also assume that  $\mathcal{A}$  satisfies  $a_{ii \dots i} \neq 0, r_i(\mathcal{A}) \neq 0, \forall i \in N$ .

**Lemma 2.4** ([10]) *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor. If  $\mathcal{A}$  is irreducible,*

$$|a_{ii \dots i}| \geq r_i(\mathcal{A}), \quad \forall i \in N,$$

*and strictly inequality holds for at least one  $i$ , then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.*

**Lemma 2.5** ([10]) *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional tensor. If there exists a positive diagonal matrix  $X$  such that  $AX^{m-1}$  is a strong  $\mathcal{H}$ -tensor, then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.*

**Lemma 2.6** ([22]) *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor. If*

- (i)  $|a_{ii \dots i}| \geq r_i(\mathcal{A}), \forall i \in N$ ,
  - (ii)  $N_1 = \{i \in N : |a_{ii \dots i}| > r_i(\mathcal{A})\} \neq \emptyset$ ,
  - (iii) *for any  $i \notin N_1$ , there exists a non-zero elements chain from  $i$  to  $j$  such that  $j \in N_1$ ,*
- then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.*

**Theorem 2.1** *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor. If there exists  $l \in \mathbb{Z}$  such that*

$$\begin{aligned} |a_{ii \dots i}| &> h^{(l+1)} \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l+1)} |a_{ii_2 \dots i_m}| \\ &+ \sum_{\substack{i_2, i_3, \dots, i_m \in N^{m-1} \setminus N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|, \quad \forall i \in N_2, \end{aligned} \quad (2.6)$$

*then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.*

*Proof* By the expression of  $h^{(l+1)}$ , it follows that

$$h^{(l+1)} \geq \frac{\sum_{i_2, i_3, \dots, i_m \in N^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}|}{R_i^{(l+1)}(\mathcal{A}) - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l+1)} |a_{ii_2 \dots i_m}|}, \quad \forall i \in N_1,$$

equivalently,

$$\begin{aligned} h^{(l+1)} R_i^{(l+1)}(\mathcal{A}) &\geq \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}| \\ &\quad + h^{(l+1)} \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l+1)} |a_{ii_2 \dots i_m}|. \end{aligned} \quad (2.7)$$

From Lemma 2.1, we have

$$0 \leq h^{(l+1)} \delta_i^{(l+1)} < 1, \quad \forall i \in N_1.$$

Together with Inequality (2.6), there exists a  $\varepsilon > 0$ , sufficiently small such that for all  $i \in N_1$ ,

$$0 < h^{(l+1)} \delta_i^{(l+1)} + \varepsilon < 1, \quad (2.8)$$

and for all  $i \in N_2$ ,

$$\begin{aligned} |a_{ii \dots i}| - h^{(l+1)} \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l+1)} |a_{ii_2 \dots i_m}| - \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| \\ > \varepsilon \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1}} |a_{ii_2 \dots i_m}|. \end{aligned} \quad (2.9)$$

Let the matrix  $X = \text{diag}(x_1, x_2, \dots, x_n)$ , where

$$x_i = \begin{cases} (h^{(l+1)} \delta_i^{(l+1)} + \varepsilon)^{\frac{1}{m-1}}, & i \in N_1; \\ 1, & i \in N_2. \end{cases}$$

We see by Inequality (2.8) that  $(h^{(l+1)} \delta_i^{(l+1)} + \varepsilon)^{\frac{1}{m-1}} < 1$  ( $\forall i \in N_1$ ), as  $\varepsilon \neq \infty, x_i \neq \infty$ , which shows that  $X$  is a diagonal matrix with positive entries. Let  $\mathcal{B} = \mathcal{A}X^{m-1}$ . Next, we will prove that  $\mathcal{B}$  is strictly diagonally dominant.

For any  $i \in N_1$ , it follows from (2.7) that

$$\begin{aligned} r_i(\mathcal{B}) &\leq \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| (h^{(l+1)} \delta_{i_2}^{(l+1)} + \varepsilon)^{\frac{1}{m-1}} \dots (h^{(l+1)} \delta_{i_m}^{(l+1)} + \varepsilon)^{\frac{1}{m-1}} \\ &\quad + \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}| \\ &\leq \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| \left( h^{(l+1)} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l+1)} + \varepsilon \right) \\ &\quad + \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}} |a_{ii_2 \dots i_m}| \\ &\leq \varepsilon \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| + h^{(l+1)} R_i^{(l+1)}(\mathcal{A}) \end{aligned}$$

$$\begin{aligned}
&< \varepsilon |a_{ii\dots i}| + h^{(l+1)} R_i^{(l+1)}(\mathcal{A}) \\
&= |a_{ii\dots i}| (\varepsilon + h^{(l+1)} \delta_i^{(l+1)}) \\
&= |b_{ii\dots i}|.
\end{aligned}$$

For any  $i \in N_2$ , it follows from (2.9) that

$$\begin{aligned}
r_i(\mathcal{B}) &\leq \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1}} |a_{ii_2\dots i_m}| (h^{(l+1)} \delta_{i_2}^{(l+1)} + \varepsilon)^{\frac{1}{m-1}} \dots (h^{(l+1)} \delta_{i_m}^{(l+1)} + \varepsilon)^{\frac{1}{m-1}} \\
&\quad + \sum_{\substack{i_2, i_3, \dots, i_m \in N^{m-1} \setminus N_1^{m-1}, \\ \delta_{ii_2\dots i_m} = 0}} |a_{ii_2\dots i_m}| \\
&\leq \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1}} |a_{ii_2\dots i_m}| \left( h^{(l+1)} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l+1)} + \varepsilon \right) \\
&\quad + \sum_{\substack{i_2, i_3, \dots, i_m \in N^{m-1} \setminus N_1^{m-1}, \\ \delta_{ii_2\dots i_m} = 0}} |a_{ii_2\dots i_m}| \\
&< |a_{ii\dots i}| = |b_{ii\dots i}|.
\end{aligned}$$

Therefore, from the above inequalities, we conclude that  $|b_{ii\dots i}| > r_i(\mathcal{B})$  for all  $i \in N$ ,  $\mathcal{B}$  is strictly diagonally dominant, and by Lemma 2.2,  $\mathcal{B}$  is a strong  $\mathcal{H}$ -tensor. Further, by Lemma 2.5,  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.  $\square$

**Remark 2.1** If  $N_1$  contains only one element, then Theorem 2.1 reduces to Lemma 1.3, and if  $l = 0$ , then Theorem 2.1 reduces to Lemma 1.6.

**Theorem 2.2** Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional complex tensor. If  $\mathcal{A}$  is irreducible and there exists  $l \in \mathbb{Z}$  such that for all  $i \in N_2$

$$|a_{ii\dots i}| \geq h^{(l+1)} \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l+1)} |a_{ii_2\dots i_m}| + \sum_{\substack{i_2, i_3, \dots, i_m \in N^{m-1} \setminus N_1^{m-1}, \\ \delta_{ii_2\dots i_m} = 0}} |a_{ii_2\dots i_m}|, \quad (2.10)$$

in addition, the strict inequality holds for at least one  $i \in N_2$ , then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.

*Proof* Notice that  $\mathcal{A}$  is irreducible; this implies

$$\sum_{i_2, i_3, \dots, i_m \in N^{m-1} \setminus N_1^{m-1}} |a_{ii_2\dots i_m}| > 0, \quad i \in N_1.$$

Let the matrix  $X = \text{diag}(x_1, x_2, \dots, x_n)$ , where

$$x_i = \begin{cases} (h^{(l+1)} \delta_i^{(l+1)})^{\frac{1}{m-1}}, & i \in N_1; \\ 1, & i \in N_2. \end{cases}$$

Adopting the same procedure as in the proof of Theorem 2.1, we conclude that  $|b_{i\dots i}| \geq r_i(\mathcal{B})$  for all  $i \in N$ . Moreover, the strict inequality holds for at least one  $i \in N_2$ , thus, there exists at least an  $i \in N$  such that  $|b_{ii\dots i}| > r_i(\mathcal{B})$ .

On the other hand, since  $\mathcal{A}$  is irreducible, and so is  $\mathcal{B}$ . Then by Lemma 2.4, we see that  $\mathcal{B}$  is a strong  $\mathcal{H}$ -tensor. By Lemma 2.5,  $\mathcal{A}$  is also a strong  $\mathcal{H}$ -tensor.  $\square$

**Remark 2.2** If  $l = 0$ , then Theorem 2.2 reduces to Theorem 2.6 of [21].

Let

$$J = \left\{ i \in N_2 : |a_{ii \dots i}| > h^{(l+1)} \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l+1)} |a_{ii_2 \dots i_m}| \right. \\ \left. + \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| \right\}.$$

**Theorem 2.3** Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an  $m$ th-order  $n$ -dimensional tensor. If for all  $i \in N_2$

$$|a_{ii \dots i}| \geq h^{(l+1)} \sum_{i_2, i_3, \dots, i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \delta_j^{(l+1)} |a_{ii_2 \dots i_m}| + \sum_{\substack{i_2, i_3, \dots, i_m \in N_1^{m-1} \setminus N_1^{m-1}, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|,$$

and if  $\forall i \in N \setminus J \neq \emptyset$ , there exists a non-zero elements chain from  $i$  to  $j$  such that  $j \in J \neq \emptyset$ , then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.

*Proof* Let the matrix  $X = \text{diag}(x_1, x_2, \dots, x_n)$ , where

$$x_i = \begin{cases} (h^{(l+1)} \delta_i^{(l+1)})^{\frac{1}{m-1}}, & i \in N_1; \\ 1, & i \in N_2. \end{cases}$$

Similar to the proof of Theorem 2.1, we can obtain  $|b_{ii \dots i}| \geq r_i(\mathcal{B})$  for all  $i \in N$ , and there exists at least an  $i \in N_2$  such that  $|b_{ii \dots i}| > r_i(\mathcal{B})$ .

On the other hand, if  $|b_{ii \dots i}| = r_i(\mathcal{B})$ , then  $i \in N \setminus J$ ; by the assumption, we know that there exists a non-zero elements chain of  $\mathcal{A}$  from  $i$  to  $j$ , such that  $j \in J$ . Then there exists a non-zero elements chain of  $\mathcal{B}$  from  $i$  to  $j$ , such that  $j$  satisfies  $|b_{jj \dots j}| > r_j(\mathcal{B})$ .

Based on the above analysis, we conclude that the tensor  $\mathcal{B}$  satisfies the conditions of Lemma 2.6, so  $\mathcal{B}$  is a strong  $\mathcal{H}$ -tensor. By Lemma 2.5,  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor. The proof is completed.  $\square$

**Remark 2.3** If  $l = 0$ , then Theorem 2.3 reduces to Theorem 2.7 of [21].

**Remark 2.4** From Lemma 2.1, we can also obtain smaller iterative coefficients  $h^{(l+1)} \delta_i^{(l+1)}$  by increasing  $l$ . Therefore, Theorem 2.1 in this paper can be more effective to determine whether a given tensor is a strong  $\mathcal{H}$ -tensor or not by increasing the number of iterations.

**Example 2.1** Consider a tensor  $\mathcal{A} = (a_{ijk})$  with 3-order and 4-dimension defined as follows:

$$\mathcal{A} = [A(1, :, :), A(2, :, :), A(3, :, :), A(4, :, :)],$$

$$A(1, :, :) = \begin{pmatrix} 15.5 & 0.5 & 0.5 & 0 \\ 0.5 & 20 & 1 & 0 \\ 0 & \frac{5}{6} & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 & 0.5 \end{pmatrix}, \quad A(2, :, :) = \begin{pmatrix} 1 & 0 & 0.5 & 0.5 \\ 0.5 & 12 & 0 & 0.5 \\ 1 & 0.5 & 0.5 & 0.5 \\ 1 & 0.5 & 0.5 & 0.5 \end{pmatrix},$$

$$A(3, :, :) = \begin{pmatrix} 1 & 1 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 8 & 0 \\ 0.5 & 0.5 & 0 & 1 \end{pmatrix}, \quad A(4, :, :) = \begin{pmatrix} 0.5 & 0.5 & 1 & 0 \\ 0.5 & 1 & 0 & 0.5 \\ 0.5 & 0 & 1 & 0 \\ 1 & 1 & 0.5 & 10 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 15.5, \quad r_1(\mathcal{A}) = \frac{155}{6}, \quad |a_{222}| = 12, \quad r_2(\mathcal{A}) = 8,$$

$$|a_{333}| = 8, \quad r_3(\mathcal{A}) = 6, \quad |a_{444}| = 10, \quad r_4(\mathcal{A}) = 8,$$

so  $N_1(\mathcal{A}) = \{2, 3, 4\}$ ,  $N_2(\mathcal{A}) = \{1\}$ . First of all, it can be verified that Lemmas 1.3-1.6 cannot determine whether the tensor  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor or not. However, Theorem 2.1 in this paper can verify that the tensor  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor when  $l = 1$ .

In fact, by Lemma 1.3,

$$|a_{333}|(|a_{111}| - r_1^3) = -78.6667 < 3 = r_3|a_{133}|,$$

by Lemma 1.4,  $\bar{r} = \max\{s_1, \max_{i \in N_2} t_i\} = \max\{\frac{r_1(\mathcal{A})}{a_{111}}, \max_{i \in N_2} \frac{a_{iii}}{r_i(\mathcal{A})}\} = 0.8$ ,

$$|a_{111}|s_1 = 9.3 < 17.8833 = \bar{r} \sum_{\substack{i_2, i_3 \in N^2 \setminus N_1^2, \\ \delta_{1i_2i_3} = 0}} |a_{1i_2i_3}| + \sum_{i_2, i_3 \in N_1^2} \max_{j \in \{i_2, i_3\}} \{t_j\} |a_{1i_2i_3}|,$$

by Lemma 1.5,

$$|a_{111}| = 15.5 < 18.1833 = \sum_{\substack{i_2, i_3 \in N^2 \setminus N_1^2, \\ \delta_{1i_2i_3} = 0}} |a_{1i_2i_3}| + \sum_{i_2, i_3 \in N_1^2} \max_{j \in \{i_2, i_3\}} \frac{r_j(\mathcal{A})}{|a_{jjj}|} |a_{1i_2i_3}|,$$

and, by Lemma 1.6,

$$\left( R_2^{(1)}(\mathcal{A}) - \sum_{\substack{i_2, i_3 \in N_1^2, \\ \delta_{2i_2i_3} = 0}} \max_{j \in \{i_2, i_3\}} \frac{R_j^{(1)}(\mathcal{A})}{|a_{jjj}|} |a_{2i_2i_3}| \right) \left( |a_{111}| - \sum_{\substack{i_2, i_3 \in N^2 \setminus N_1^2, \\ \delta_{1i_2i_3} = 0}} |a_{1i_2i_3}| \right)$$

$$= 4.5417 \times 14 = 63.5838$$

$$< 63.8127 = \sum_{i_2, i_3 \in N^2 \setminus N_1^2} |a_{2i_2i_3}| \sum_{i_2, i_3 \in N_1^2} \max_{j \in \{i_2, i_3\}} \frac{R_j^{(1)}(\mathcal{A})}{|a_{jjj}|} |a_{1i_2i_3}|.$$

However, by calculation with Matlab 7.11.0,  $r = 0.667$  and the results of  $R_i^{(l+1)}(\mathcal{A}), \delta_i^{(l+1)}, h^{(l+1)}$  ( $i \in \{2, 3, 4\}$ ) of Theorem 2.1 in this paper are given in Table 1 for the total number of

**Table 1** The results of  $R_i^{(l+1)}(\mathcal{A})$  and  $\delta_i^{(l+1)}$  and  $h^{(l+1)}$  ( $i \in \{2, 3, 4\}$ )

$l$	$R_2^{(l+1)}(\mathcal{A})$	$R_3^{(l+1)}(\mathcal{A})$	$R_4^{(l+1)}(\mathcal{A})$	$\delta_2^{(l+1)}$	$\delta_3^{(l+1)}$	$\delta_4^{(l+1)}$	$h^{(l+1)}$
0	6.8333	5.000	6.6667	0.5694	0.6250	0.6667	0.9908
1	6.7706	4.9128	6.5046	0.5642	0.6141	0.6505	0.9937
2	6.7261	4.8782	6.4636	0.5605	0.6098	0.6464	0.9999
3	6.7255	4.8777	6.4628	0.5605	0.6097	0.6463	1.0000
4	6.7254	4.8776	6.4627	0.5604	0.6097	0.6463	1.0000

iterations  $l = 4$ . When  $l = 1$ , we can get

$$|a_{111}| = 15.5 > 15.4300 = h^{(2)} \sum_{i_2, i_3 \in N_1^2} \max_{j \in \{i_2, i_3\}} \delta_j^{(2)} |a_{1i_2i_3}| + \sum_{\substack{i_2, i_3 \in N^2 \setminus N_1^2, \\ \delta_{1i_2i_3} = 0}} |a_{1i_2i_3}|,$$

we see that  $\mathcal{A}$  satisfies the conditions of Theorem 2.1, then  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor. In fact, there exists a positive diagonal matrix  $X = \text{diag}(1, 0.7489, 0.7812, 0.7978)$  such that  $\mathcal{A}X^2$  is strictly diagonally dominant.

## 2.2 An application: the positive definiteness of an even-order real symmetric tensor

In this subsection, by making use of the results in Section 2.1, we present new criteria for identifying the positive definiteness of an even-order real symmetric tensor.

From Lemma 1.2 and Theorems 2.1-2.3, we easily obtain the following result.

**Theorem 2.4** Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  be an even-order real symmetric tensor with  $m$ th-order  $n$ -dimension, and  $a_{i \dots i} > 0$  for all  $i \in N$ . If  $\mathcal{A}$  satisfies one of the following conditions:

- (i) all the conditions of Theorem 2.1;
- (ii) all the conditions of Theorem 2.2;
- (iii) all the conditions of Theorem 2.3,

then  $\mathcal{A}$  is positive definite.

**Example 2.2** Let

$$\begin{aligned} f(x) = \mathcal{A}x^4 = & 16x_1^4 + 21x_2^4 + 23x_3^4 + 19x_4^4 - 8x_1^3x_4 + 12x_1^2x_2x_3 \\ & - 12x_2x_3^2x_4 + 4x_2x_4^3 + 4x_3x_4^3 - 24x_1x_2x_3x_4 \end{aligned}$$

be a 4th-degree homogeneous polynomial. We can get an 4th-order 4-dimensional real symmetric tensor  $\mathcal{A} = (a_{i_1 i_2 i_3 i_4})$ , where

$$a_{1111} = 16, \quad a_{2222} = 21, \quad a_{3333} = 23, \quad a_{4444} = 19,$$

$$a_{1114} = a_{1141} = a_{1411} = a_{4111} = -2,$$

$$a_{2444} = a_{4244} = a_{4424} = a_{4442} = 1,$$

$$a_{3444} = a_{4344} = a_{4434} = a_{4443} = 1,$$

$$a_{1123} = a_{1132} = a_{1213} = a_{1312} = a_{1231} = a_{1321} = 1,$$

$$a_{2113} = a_{2131} = a_{2311} = a_{3112} = a_{3121} = a_{3211} = 1,$$

$$a_{3234} = a_{3243} = a_{3324} = a_{3342} = a_{3423} = a_{3432} = -1,$$

$$a_{2334} = a_{2343} = a_{2433} = a_{4233} = a_{4323} = a_{4332} = -1,$$

$$a_{1234} = a_{1243} = a_{1324} = a_{1342} = a_{1423} = a_{1432} = -1,$$

$$a_{2134} = a_{2143} = a_{2314} = a_{2341} = a_{2413} = a_{2431} = -1,$$

$$a_{3124} = a_{3142} = a_{3214} = a_{3241} = a_{3412} = a_{3421} = -1,$$

$$a_{4123} = a_{4132} = a_{4213} = a_{4231} = a_{4312} = a_{4321} = -1,$$

and other  $a_{i_1 i_2 i_3 i_4} = 0$ . By calculation, we have

$$|a_{1111}| = 16 < 18 = r_1(\mathcal{A})$$

and

$$|a_{2222}|(a_{1111} - r_1(\mathcal{A}) + |a_{1222}|) = -42 < 0 = r_2(\mathcal{A})|a_{1222}|.$$

Hence,  $\mathcal{A}$  is not a strictly diagonally dominant tensor defined in [23], or a quasi-doubly strictly diagonally dominant tensor defined in [22], so we cannot use Theorem 3 in [23] and Theorem 4 in [22] to identify the positive definiteness of  $\mathcal{A}$ . Further, it can be verified that  $\mathcal{A}$  satisfies all the conditions of Theorem 2.1. Thus, from Theorem 2.4, we can see that  $\mathcal{A}$  is positive definite, that is,  $f(x)$  is positive definite. In fact, there exists a positive diagonal matrix  $X = \text{diag}(1, 0.8110, 0.8243, 0.8043)$  such that  $\mathcal{A}X^3$  is strictly diagonally dominant. Therefore,  $\mathcal{A}$  is a strong  $\mathcal{H}$ -tensor.

### 3 Conclusions

In this paper, we give some criteria for identifying a strong  $\mathcal{H}$ -tensor which only depend on the elements of tensors, and by increasing the number of iterations, we can determine whether a given tensor is a strong  $\mathcal{H}$ -tensor or not more effective. We also present new criteria for identifying the positive definiteness of an even-order real symmetric tensor based on these criteria.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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