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New real-variable characterizations of Hardy spaces associated with twisted convolution

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Abstract

In this paper, we give some new real-variables characterizations of the Hardy space associated with twisted convolution, including Poisson maximal function, area integral, and Littlewood-Paley g-function.

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Keywords: twisted convolution; Hardy space; Poisson maximal function; area integral; Littlewood-Paley g-function

1 Introduction

In this paper, we consider the $2n$ linear differential operators

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4}\bar{z}_j, \qquad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4}z_j \quad \text{on } \mathbb{C}^n, j = 1, 2, \ldots, n. \tag{1}$$

Together with the identity they generate a Lie algebra h^n which is isomorphic to the $2n+1$ dimensional Heisenberg algebra. The only nontrivial commutation relations are

$$[Z_j, \bar{Z}_j] = -\frac{1}{2}I, \quad j = 1, 2, \ldots, n. \tag{2}$$

The operator L defined by

$$L = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j)$$

is nonnegative, self-adjoint, and elliptic. Therefore it generates a diffusion semigroup $\{T_t^L\}_{t>0} = \{e^{-tL}\}_{t>0}$. The operators in (1) generate a family of 'twisted translations' τ_w on $\mathbb{C}^n$ defined on measurable functions by

$$\begin{aligned}(\tau_w f)(z) &= \exp\left(\frac{1}{2}\sum_{j=1}^{n}(w_j z_j + \bar{w}_j\bar{z}_j)\right)f(z) \\ &= f(z+w)\exp\left(\frac{i}{2}\operatorname{Im}(z\cdot\bar{w})\right).\end{aligned}$$

The 'twisted convolution' of two functions f and g on $\mathbb{C}^n$ can now be defined as

$$\begin{aligned}(f \times g)(z) &= \int_{\mathbb{C}^n} f(w)\tau_{-w}g(z)\,dw \\ &= \int_{\mathbb{C}^n} f(z-w)g(w)\bar{\omega}(z,w)\,dw,\end{aligned}$$

where $\omega(z,w) = \exp(\frac{i}{2}\operatorname{Im}(z\cdot\bar{w}))$. More about twisted convolution can be found in [1–3].

In [4], the authors defined the Hardy space $H_L^1(\mathbb{C}^n)$ associated with a twisted convolution. They gave several characterizations of $H_L^1(\mathbb{C}^n)$ via maximal functions, the atomic decomposition, and the behavior of the local Riesz transform. As applications, the boundedness of Hömander multipliers on Hardy spaces is considered in [5]. The 'twisted cancelation' and Weyl multipliers were introduced for the first time in [6]. Recently, Huang and Wang [7] defined the Hardy space $H_L^p(\mathbb{C}^n)$ associated with a twisted convolution for $\frac{2n}{2n+1} < p \le 1$. Huang gave the characterizations of the Hardy space associated with twisted convolution by the Lusin area integral function and the Littlewood-Paley function defined by the heat kernel in [8] and established the boundedness of the Weyl multiplier by these characterizations in [9]. Recently, Huang and Liu gave the molecular characterization of Hardy space associated with twisted convolution in [10]. The purpose of this paper is to give some new real-variable characterizations for $H_L^p(\mathbb{C}^n)$, including the Poisson maximal function, the Lusin area integral, and the Littlewood-Paley g-function defined by the Poisson kernel.

We first give some basic notations concerning $H_L^p(\mathbb{C}^n)$. Let $\mathcal{B}$ denote the class of C^∞-functions φ on $\mathbb{C}^n$, supported on the ball $B(0,1)$ such that $\|\varphi\|_\infty \le 1$ and $\|\nabla\varphi\|_\infty \le 2$. For $t>0$, let $\varphi_t(z) = t^{-2n}\varphi(z/t)$. Given $\sigma > 0$, $0 < \sigma \le +\infty$, and a tempered distribution f, define the grand maximal function

$$M_\sigma f(z) = \sup_{\varphi\in\mathcal{B}} \sup_{0<t<\sigma} \left|\varphi_t \times f(z)\right|.$$

Then the Hardy space $H_L^p(\mathbb{C}^n)$ can be defined by

$$H_L^p\left(\mathbb{C}^n\right) = \left\{f \in \mathcal{S}'\left(\mathbb{C}^n\right) : M_\infty f \in L^p\left(\mathbb{C}^n\right)\right\}.$$

For any $f \in H_L^p(\mathbb{C}^n)$, define $\|f\|_{H_L^p(\mathbb{C}^n)} = \|M_\infty f\|_{L^p}$.

Definition 1 Let $\frac{2n}{2n+1} < p \le 1 \le q \le \infty$ and $p \ne q$. A function $a(z)$ is a $H_L^{p,q}$-atom for the Hardy space $H_L^p(\mathbb{C}^n)$ associated to a ball $B(z_0,r)$ if

(1) $\operatorname{supp} a \subset B(z_0,r)$;

(2) $\|a\|_q \le \left|B(z_0,r)\right|^{1/q-1/p}$;

(3) $\displaystyle\int_{\mathbb{C}^n} a(w)\bar{\omega}(z_0,w)\,dw = 0.$

We define the atomic Hardy space $H_L^{p,q}(\mathbb{C}^n)$ to be the set of all tempered distributions of the form $\sum_j \lambda_j a_j$ (the sum converges in the topology of $\mathcal{S}'(\mathbb{C}^n)$), where a_j are $H_L^{p,q}$-atoms and $\sum_j |\lambda_j|^p < +\infty$.

The atomic quasi-norm in $H_L^{p,q}(\mathbb{C}^n)$ is defined by

$$\|f\|_{L\text{-atom}} = \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$ and a_j are $H_L^{p,q}$-atoms.

The following result has been proved in [4] and [7].

Proposition 1 *Let* $\frac{2n}{2n+1} < p \leq 1$. *Then for a tempered distribution* f *on* $\mathbb{C}^n$, *the following are equivalent*:

(i) $M_\infty f \in L^p(\mathbb{C}^n)$.

(ii) *For some* σ, $0 < \sigma < +\infty$, $M_\sigma f \in L^p(\mathbb{C}^n)$.

(iii) *For some radial function* $\varphi \in \mathcal{S}$, *such that* $\int_{\mathbb{C}^n} \varphi(z)\,dz \neq 0$, *we have*

$$\sup_{0<t<1} \left|\varphi_t \times f(z)\right| \in L^p\left(\mathbb{C}^n\right).$$

(iv) f *can be decomposed as* $f = \sum_j \lambda_j a_j$, *where* a_j *are* $H_L^{p,q}$-*atoms and* $\sum_j |\lambda_j|^p < +\infty$.

Corollary 1 *Let* $\frac{2n}{2n+1} < p \leq 1$ *and* $1 < q \leq \infty$. *Then* $H_L^{p,q}(\mathbb{C}^n) = H_L^p(\mathbb{C}^n)$ *with equivalent norms*.

Let $\{P_t^L\}_{t>0}$ be the Poisson semigroup generated by the operator L. Then, for $f \in L^2(\mathbb{C}^n)$, the function $e^{-t\sqrt{L}}f$ has the special Hermite expansion (*cf.* [11])

$$e^{-t\sqrt{L}}f(z) = (2\pi)^{-n}\sum_{k=0}^{\infty} e^{-\sqrt{2k+n}t} f \times \varphi_k(z),$$

where φ_k are Laguerre functions. Therefore $e^{-t\sqrt{L}}f$ is given by the twisted convolution with the kernel

$$P_t(z) = (2\pi)^{-n}\sum_{k=0}^{\infty} e^{-\sqrt{2k+n}t} \varphi_k(z). \tag{3}$$

The Poisson maximal function is defined by

$$M_P(f)(z) = \sup_{t>0}\left|P_t \times f(z)\right|.$$

We can characterize the Hardy space $H_L^1(\mathbb{C}^n)$ as follows.

Theorem 1 $f \in H_L^1(\mathbb{C}^n)$ *if and only if* $f \in L^1(\mathbb{C}^n)$ *and* $M_P(f) \in L^1(\mathbb{C}^n)$. *Moreover, we have*

$$\|f\|_{H_L^1} \sim \left\|M_P(f)\right\|_{L^1}.$$

We define the area integral associated to $\{P_t^L\}_{t>0}$ by

$$\left(S_L^k f\right)(z) = \left(\int_0^{+\infty}\int_{|z-w|<t} \left|D_t^k f(w)\right|^2 \frac{dw\,dt}{t^{2n+1}}\right)^{1/2},$$

the Littlewood-Paley g-function by

$$\mathcal{G}_L^k(f)(z) = \left(\int_0^\infty \left|D_t^k f(z)\right|^2 \frac{dt}{t}\right)^{1/2},$$

and we consider the g_λ^*-function associated with L defined by

$$g_{\lambda,k}^* f(z) = \left(\int_0^\infty \int_{\mathbb{C}^n} \left(\frac{t}{t+|z-w|}\right)^{2\lambda n} \left|D_t^k f(w)\right|^2 \frac{dw\,dt}{t^{2n+1}}\right)^{1/2},$$

where $D_t^k f(z) = t^k(\partial_t^k P_t^L f)(z)$.

Now we can prove the main result of this paper.

Theorem 2

(a) *A function $f \in H_L^1(\mathbb{C}^n)$ if and only if its Lusin area integral $S_L^k f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$. Moreover, we have*

$$\|f\|_{H_L^1} \sim \left\|S_L^k f\right\|_{L^1}.$$

(b) *A function $f \in H_L^1(\mathbb{C}^n)$ if and only if its Littlewood-Paley g-function $\mathcal{G}_L^k f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$. Moreover, we have*

$$\|f\|_{H_L^1} \sim \left\|\mathcal{G}_L^k f\right\|_{L^1}.$$

(c) *A function $f \in H_L^1(\mathbb{C}^n)$ if and only if its g_λ^*-function $g_{\lambda,k}^* f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$, where $\lambda > 3$. Moreover, we have*

$$\|f\|_{H_L^1} \sim \left\|g_{\lambda,k}^* f\right\|_{L^1}.$$

Remark 1 In this paper, we just give the proofs of our results for $p = 1$. In fact, we can prove the case $\frac{2n}{2n+1} < p < 1$ under more conditions (such as that f vanishes weakly at infinity). The proofs of the case $\frac{2n}{2n+1} < p < 1$ are quite similar to the case $p = 1$, so we omit them.

Throughout the article, we will use C to denote a positive constant, which is independent of the main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{B_1}{B_2} \leq C$.

2 Preliminaries

In this section, we give some preliminaries that we will use in the sequel.

Let $K_t(z)$ be the heat kernel of $\{T_t^L\}_{t>0}$. Then we can get (*cf.* [11])

$$K_t(z) = (4\pi)^{-n}(\sinh t)^{-n} e^{-\frac{1}{4}|z|^2(\coth t)}. \tag{4}$$

It is easy to prove that the heat kernel $K_t(z)$ has the following estimates (*cf.* [8]).

Lemma 1 *There exists a positive constant $C > 0$ such that*

(i) $|K_t(z)| \leq Ct^{-n}e^{-C\frac{|z|^2}{t}}$;

(ii) $|\nabla K_t(z)| \leq Ct^{-n-\frac{1}{2}}e^{-C\frac{|z|^2}{t}}$.

Let $Q_t^k(z)$ be the twisted convolution kernel of $Q_t^k = t^{2k}\partial_s^k T_s^L|_{s=t^2}$. Then

$$Q_t^k(z) = t^{2k}\partial_s^k K_s(z)|_{s=t^2}.$$

We have the following estimates [8].

Lemma 2 *There exist constants $C, C_k > 0$ such that*
(i) $|Q_t^k(z)| \leq C_k t^{-2n} e^{-Ct^{-2}|z|^2}$;
(ii) $|\nabla Q_t^k(z)| \leq C_k t^{-2n-1} e^{-Ct^{-2}|z|^2}$.

By the subordination formula, we can give the following estimates as regards the Poisson kernel.

Lemma 3 *There exist constants $C_k > 0$, $A > 0$ such that*
(a)

$$0 < P_t(z) \leq C_k \frac{t}{(t^2 + A|z|^2)^{(2n+1)/2}}; \tag{5}$$

(b)

$$\left|\nabla P_t(z)\right| \leq C_k \frac{\sqrt{t}}{(t^2 + A|z|^2)^{(2n+1)/2}}. \tag{6}$$

Lemma 4 *Let $D_t^k(z)$ be the integral kernel of the operator D_t^k. Then there exist constants $C_k > 0$, $A > 0$, such that*
(a)

$$\left|D_t^k(z)\right| \leq C_k \frac{t}{(t^2 + A|z|^2)^{(2n+1)/2}};$$

(b)

$$\left|\nabla D_t^k(z)\right| \leq C_k \frac{\sqrt{t}}{(t^2 + A|z|^2)^{(2n+1)/2}}.$$

We also need some basic properties about the tent space (*cf.* [12]).

Let $0 < p < \infty$, and $1 \leq q \leq \infty$. Then the tent space T_q^p is defined as the space of functions f on $\mathbb{C}^n \times \mathbb{R}^+$, so that

$$\left(\int_{\Gamma(z)} \left|f(w,t)\right|^q \frac{dw\,dt}{t^{2n+1}}\right)^{1/q} \in L^p\left(\mathbb{C}^n\right), \quad \text{when } 1 \leq q < \infty$$

and

$$\sup_{(w,t)\in\Gamma(z)} \left|f(w,t)\right| \in L^p\left(\mathbb{C}^n\right), \quad \text{when } q = \infty,$$

where $\Gamma(z)$ is the standard cone whose vertex is $z \in \mathbb{C}^n$, *i.e.*,

$$\Gamma(z) = \left\{(w,t) : |w - z| < t\right\}.$$

Assume $B(z_0, r)$ is a ball in $\mathbb{C}^n$, its tent $\hat{B}$ is defined by $\hat{B} = \{(w,t) : |w - z_0| \le r - t\}$. A function $a(z,t)$ supported in a tent $\hat{B}$, B a ball in $\mathbb{C}^n$, is said to be an atom in the tent space T_q^p if and only if it satisfies

$$\left(\int_{\hat{B}} \left|a(z,t)\right|^2 \frac{dz\,dt}{t}\right)^{1/2} \le |B|^{1/2-1/p}.$$

The atomic decomposition of T_q^p is stated as follows.

Proposition 2 *When $0 < p \le 1$, then for any $f \in T_2^p$ can be written as $f = \sum \lambda_k a_k$, where a_k are atoms and $\sum |\lambda_k|^p \le C\|f\|_{T_2^p}^p$.*

3 The proofs of the main results

Let

$$M_H f(z) = \sup_{t>0} \left|K_t \times f(z)\right|, \quad f \in L^1\left(\mathbb{C}^n\right)$$

be the heat maximal function. Then we can characterize $H_L^1(\mathbb{C}^n)$ by the maximal function $M_H f$ as follows (*cf.* [4] or [8]).

Lemma 5 *$f \in H_L^1(\mathbb{C}^n)$ if and only if $M_H f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$.*

Now, we give the proof of Theorem 1.

Proof of Theorem 1 If $f \in H_L^1(\mathbb{C}^n)$, then, by Lemma 5, we get $M_H f \in L^1(\mathbb{C}^n)$. Since

$$P_t(z) = \frac{1}{\sqrt{\pi}} \int_0^\infty K_{t^2/4\mu}(z) e^{-\mu} \mu^{-1/2}\,d\mu,$$

we have $\|M_P(f)\|_{L^1} \le C\|M_H(f)\|_{L^1}$, *i.e.*, $M_P f \in L^1(\mathbb{C}^n)$.

For the reverse, there exists a function η defined on $(1,\infty)$ that is rapidly decreasing at ∞ and satisfies the moment conditions (*cf.* [13])

$$\int_1^\infty \eta(t)\,dt = 1, \qquad \int_1^\infty t^k \eta(t)\,dt = 0, \quad k = 1, 2, \ldots.$$

Let

$$\Phi(z) = \int_1^\infty \eta(t) P_t(z)\,dt. \tag{7}$$

Since

$$\left(1 + s^2\right)^{-(2n+1)/2} = \sum_{k<R} a_k s^k + O\left(s^R\right), \quad 0 \le s < \infty$$

for appropriate binomial coefficients a_k, we have

$$\frac{t}{(t^2 + A|z|^2)^{(2n+1)/2}} = \sum_{k<R} a_k t |z|^{-1-2n} \left(\frac{t}{|z|}\right)^k + O\left(t^{R+1} |z|^{-2n-1-R}\right). \tag{8}$$

By (8) and Lemma 3, we know that Φ and any derivative of Φ are rapidly decreasing. Thus $\Phi \in \mathcal{S}$ and

$$\int_{\mathbb{C}^n} \Phi(z)\,dz = \int_1^\infty \eta(t)\,dt = 1.$$

Therefore,

$$M_\Phi(f)(z) \le M_P(f)(z) \int_1^\infty |\eta(t)|\,dt \le CM_P(f)(z).$$

This proves that $M_P(f) \in L^1(\mathbb{C}^n)$ implies $f \in H_L^1(\mathbb{C}^n)$ and the proof of Theorem 1 is complete. □

In order to get our results, we need the following lemma (*cf.* Lemma 5 in [8]).

Lemma 6

(i) *The operators S_L^k and $\mathcal{G}_L^k$ are isometries on $L^2(\mathbb{C}^n)$ up to constant factors. Exactly,*

$$\|\mathcal{G}_L^k f\|_{L^2} \sim \|f\|_{L^2}, \qquad \|S_L^k f\|_{L^2} \sim \|f\|_{L^2}.$$

(ii) *When $\lambda > 1$, there exists a constant $C > 0$, such that*

$$C^{-1}\|f\|_{L^2} \le \|g_{\lambda,k}^* f\|_{L^2} \le C\|f\|_{L^2}.$$

We define the new Lusin type area integral operator by

$$\left(S_{L,\alpha}^k f\right)(z) = \left(\int_0^{+\infty} \int_{|z-w|<\alpha t} \left|D_t^k f(w)\right|^2 \frac{dw\,dt}{t^{2n+1}}\right)^{1/2},$$

where $\alpha > 0$.

Lemma 7 *It is easy to see that the above definition of the area integral operator is independent of α in the sense of $\|(S_L^\alpha f)\|_{L^p} \sim \|(S_L^\beta f)\|_{L^p}$, for $0 < \alpha < \beta < \infty$ and $0 < p < \infty$ (cf. [12]). In the following, we use S_L^k to denote $S_{L,1}^k$.*

Proof of Theorem 2 (a) By Lemma 4, we can prove that there exists a constant $C > 0$ such that for any atom $a(z)$ of $H_L^1(\mathbb{C}^n)$, we have

$$\|S_L^k a\|_{L^1} \le C. \tag{9}$$

In the following, we will show that $f \in H_L^1(\mathbb{C}^n)$ when $S_L^k f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$.

We first assume that $f \in L^1(\mathbb{C}^n) \cap L^2(\mathbb{C}^n)$. When $S_L^k f \in L^1(\mathbb{C}^n)$, we know $D_t^k f \in T_2^1$. By Proposition 2, we get

$$D_t^k f(z) = \sum_j \lambda_j a_j(z,t), \tag{10}$$

where $a_j(z,t)$ are atoms of T_2^1 and $\sum_j |\lambda_j| < \infty$. By the spectrum theorem (*cf.* [14]), we can prove

$$f(z) = 4\int_0^\infty D_t^k\bigl(D_t^k f(z)\bigr)\frac{dt}{t}. \tag{11}$$

By (10) and (11), we get

$$f(z) = 4\int_0^{+\infty} D_t^k\biggl(\sum_j \lambda_j a_j(z,t)\biggr)\frac{dt}{t} = C\sum_j \lambda_j \int_0^{+\infty} D_t^k a_j(z,t)\frac{dt}{t}.$$

Therefore, it is sufficient to prove $\alpha_j = \int_0^{+\infty} D_t^k a_j(z,t)\frac{dt}{t}$, $i = 1,2,\ldots$, are bounded in $H_L^1(\mathbb{C}^n)$ uniformly, *i.e.*, there exists a constant $C > 0$ such that for any atom $a(z,t)$ in T_2^1,

$$\|\alpha\|_{H_L^1} = \biggl\|\int_0^{+\infty} D_t^k a(z,t)\frac{dt}{t}\biggr\|_{H_L^1} \le C.$$

We assume that $a(z,t)$ is supported in $\hat{B}(z_0,r)$, where $\hat{B}(z_0,r)$ denotes the tent of the ball $B(z_0,r)$, then

$$\Bigl\|\sup_{t>0}\bigl|e^{-t\sqrt{L}}\alpha(z)\bigr|\Bigr\|_{L^1} \le \Bigl\|\Bigl(\sup_{t>0}\bigl|e^{-t\sqrt{L}}\alpha(z)\bigr|\Bigr)\chi_{B^*}\Bigr\|_{L^1} + \Bigl\|\Bigl(\sup_{t>0}\bigl|e^{-t\sqrt{L}}\alpha(z)\bigr|\Bigr)\chi_{(B^*)^c}\Bigr\|_{L^1} = I_1 + I_2,$$

where $B^* = B(z_0, 2r)$.

By the Hölder inequality, we get

$$I_1 \le \bigl|B^*\bigr|^{1/2}\biggl(\int_{\mathbb{C}^n}\Bigl(\sup_{t>0}\bigl|e^{-t\sqrt{L}}\alpha(z)\bigr|\Bigr)^2 dz\biggr)^{1/2} \le \bigl|B^*\bigr|^{1/2}\|\alpha\|_{L^2}.$$

By the self-adjointness of D_t^k and Lemma 5, we can get

$$\begin{aligned}
\|\alpha\|_{L^2} &= \sup_{\|\beta\|_{L^2}\le 1}\int_{\mathbb{C}^n}\alpha(z)\bar{\beta}(z)\,dz \\
&= \sup_{\|\beta\|_{L^2}\le 1}\int_{\mathbb{C}^n}\biggl(\int_0^{+\infty} D_t^k a(z,t)\frac{dt}{t}\biggr)\bar{\beta}(z)\,dz \\
&= \sup_{\|\beta\|_{L^2}\le 1}\int_0^{+\infty}\int_{\mathbb{C}^n} D_t^k a(z,t)\bar{\beta}(z)\,dz\frac{dt}{t} \\
&= \sup_{\|\beta\|_{L^2}\le 1}\int_0^{+\infty}\int_{\mathbb{C}^n} a(z,t)D_t^k\bar{\beta}(z)\,dz\frac{dt}{t} \\
&\le \sup_{\|\beta\|_{L^2}\le 1}\biggl(\int_{\mathbb{C}^n}\int_0^{+\infty}\bigl|a(z,t)\bigr|^2\frac{dz\,dt}{t}\biggr)^{1/2} \\
&\quad\times\biggl(\int_{\mathbb{C}^n}\int_0^{+\infty}\bigl|D_t^k\bar{\beta}(z)\bigr|^2\frac{dz\,dt}{t}\biggr)^{1/2} \\
&\le |B|^{-1/2}\|\beta\|_{L^2} \le |B|^{-1/2}.
\end{aligned}$$

This gives the proof of $I_1 \le C$.

By Lemma 2, we can prove

$$\begin{aligned}
&\sup_{s>0}\left|e^{-s\sqrt{L}}\int_0^{+\infty}D_t^k a(z,t)\frac{dt}{t}\right| \\
&\quad=\sup_{s>0}\left|e^{-s\sqrt{L}}\int_0^{+\infty}(-t\sqrt{L})^k e^{-t\sqrt{L}}a(z,t)\frac{dt}{t}\right| \\
&\quad=\sup_{s>0}\left|\int_0^{+\infty}(-t\sqrt{L})^k e^{-(s+t)\sqrt{L}}a(z,t)\frac{dt}{t}\right| \\
&\quad=\sup_{s>0}\left|\int_0^{+\infty}\left(\frac{t}{s+t}\right)^k\left(-(s+t)\sqrt{L}\right)^k e^{-(s+t)\sqrt{L}}a(z,t)\frac{dt}{t}\right| \\
&\quad=\sup_{s>0}\left|\int_0^{+\infty}\left(\frac{t}{s+t}\right)^k\int_{\mathbb{C}^n}D_{s+t}^k(z-w)a(w,t)\frac{dw\,dt}{t}\right| \\
&\quad\le\sup_{s>0}\int_0^{+\infty}\frac{t}{s+t}\int_{\mathbb{C}^n}\frac{s+t}{((s+t)^2+A|z-w|^2)^{(2n+1)/2}}\left|a(w,t)\right|\frac{dw\,dt}{t} \\
&\quad\le\sup_{s>0}\left(\int_0^r\int_B(s+t)^{-4n}\left(1+A\frac{|z-w|^2}{(s+t)^2}\right)^{-(2n+1)}\left(\frac{t}{s+t}\right)^2\frac{dw\,dt}{t}\right)^{1/2} \\
&\qquad\times\left(\int_0^r\int_B\left|a(w,t)\right|^2\frac{dw\,dt}{t}\right)^{1/2} \\
&\quad\le|B|^{-1/2}|z-z_0|^{-(2n+1)}\left(\int_0^r\int_B t\,dw\,dt\right)^{1/2} \\
&\quad\le Cr|z-z_0|^{-(2n+1)}.
\end{aligned}$$

Then we get

$$I_2\le Cr\int_{(B^*)^c}|z-z_0|^{-(2n+1)}\,dz\le C.$$

When $f\in L^1(\mathbb{C}^n)$, we can proceed similarly to Proposition 14 in [15]. In fact, we let $f_s=T_{2^{-s}}^L f$, $s\ge 0$. Then, by $f\in L^1(\mathbb{C}^n)$ and Lemma 3, we know $f_s\in L^2(\mathbb{C}^n)$ and $\|S_L^k f_s\|_1\le\|S_L^k f\|_1$. By the above proof, we get

$$\|f_s\|_{H_L^1(\mathbb{C}^n)}\lesssim\left\|S_L^k f_s\right\|_{L^1}\le\left\|S_L^k f\right\|_{L^1}.$$

By the monotone convergence theorem, we have

$$\|f_s-f_n\|_{H_L^1}\le\left\|S_L^k(f_s-f_n)\right\|_{L^1}\to 0,\quad\text{when } s,n\to+\infty.$$

Therefore, $\{f_s\}$ is a Cauchy sequence in $H_L^1(\mathbb{C}^n)$ and there exists $g\in H_L^1(\mathbb{C}^n)$ such that

$$\lim_{s\to+\infty}f_s=g\quad\text{in } H_L^1\left(\mathbb{C}^n\right).$$

As

$$\lim_{s\to+\infty}f_s=f\quad\text{in } (BMO_L)^*,$$

we know $f=g\in H_L^1(\mathbb{C}^n)$ and $\|f\|_{H_L^1(\mathbb{C}^n)}\lesssim\|S_L^k f\|_{L^1}$.

This gives the proof of Theorem 2(a).

(b) Firstly, by Lemma 4, we can prove that there exists a positive constant C such that for any atom $a(z)$ of $H^1_L(\mathbb{C}^n)$, we have

$$\|\mathcal{G}^k_L a\|_{L^1} \le C.$$

For the reverse, by (a), it is sufficient to prove

$$\|S^{k+1}_L f\|_{L^1} \le C\|\mathcal{G}^k_L f\|_{L^1}. \tag{12}$$

Our proof is motivated by [16]. Let

$$F(z)(t) = \left(\partial^k_t e^{-t\sqrt{L}} f\right)(z), \qquad V(z,s) = e^{-s\sqrt{L}} F(z).$$

Then

$$V(z,s)(t) = e^{-s\sqrt{L}}\left(\partial^k_t e^{-t\sqrt{L}} f\right)(z) = \left(\partial^k_t e^{-(s+t)\sqrt{L}} f\right)(z).$$

Therefore

$$\begin{aligned}\int_0^{+\infty} \left|V(z,s)(t)\right|^2 t^{2k-1}\,dt &= \int_0^{+\infty} \left|\left(\partial^k_t e^{-(s+t)\sqrt{L}} f\right)(z)\right|^2 t^{2k-1}\,dt \\ &= \int_s^{+\infty} \left|\left(\partial^k_t e^{-t\sqrt{L}} f\right)(z)\right|^2 (t-s)^{2k-1}\,dt.\end{aligned}$$

Hence

$$\sup_{s>0} \int_0^{+\infty} \left|V(z,s)(t)\right|^2 t^{2k-1}\,dt \le \int_0^{+\infty} \left|\left(t^k \partial^k_t e^{-t\sqrt{L}} f\right)(z)\right|^2 \frac{dt}{t} = \left(\mathcal{G}^k_L f(z)\right)^2.$$

Let $\mathbf{X} = L^2((0,\infty), t^{2k-1}\,dt)$. Then

$$\sup_{s>0} \left\|e^{-s\sqrt{L}} F(z)\right\|_{\mathbf{X}} = \mathcal{G}^k_L f(z) \in L^1\left(\mathbb{C}^n\right).$$

Therefore $F \in H^1_{\mathbf{X}}(\mathbb{C}^n)$, here $H^1_{\mathbf{X}}(\mathbb{C}^n)$ can be seen as a vector-valued Hardy space. This shows that $\widetilde{S^1_L} F(z) \in L^1(\mathbb{C}^n)$, where

$$\widetilde{S^1_L} F(z) = \left(\int_0^{+\infty} \int_{|z-w|<2t} \left\|D^1_t F(w)\right\|^2_{\mathbf{X}} \frac{dw\,dt}{t^{2n+1}}\right)^{1/2}.$$

By

$$\begin{aligned}\left(S^1_L F(z)\right)^2 &= \int_0^{+\infty} \int_{|z-w|<2t} \left\|D^1_t(z)\right\|^2_{\mathbf{X}} \frac{dw\,dt}{t^{2n+1}} \\ &= \int_0^{+\infty} \int_{|z-w|<2t} \int_0^{+\infty} \left|(-t\sqrt{L}) e^{-t\sqrt{L}} F(w)(s)\right|^2 s^{2k-1}\,ds \frac{dw\,dt}{t^{2n+1}} \\ &= \int_0^{+\infty} \int_0^{+\infty} \int_{|z-w|<2t} \left|(-\sqrt{L})^{k+1} e^{-(s+t)\sqrt{L}} f(w)\right|^2 t^{1-2n} s^{2k-1}\,dw\,dt\,ds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty}\int_s^{+\infty}\int_{|z-w|<2(t-s)} \bigl|(-\sqrt{L})^{k+1}e^{-t\sqrt{L}}f(w)\bigr|^2(t-s)^{1-2n}s^{2k-1}\,dw\,dt\,ds \\
&= \int_0^{+\infty}\int_0^{t}\int_{|z-w|<2(t-s)} \bigl|(-\sqrt{L})^{k+1}e^{-t\sqrt{L}}f(w)\bigr|^2(t-s)^{1-2n}s^{2k-1}\,dw\,ds\,dt \\
&\geq \int_0^{+\infty}\int_0^{t/2}\int_{|z-w|<2(t-s)} \bigl|(-\sqrt{L})^{k+1}e^{-t\sqrt{L}}f(w)\bigr|^2(t-s)^{1-2n}s^{2k-1}\,dw\,ds\,dt \\
&\geq \int_0^{+\infty}\int_0^{t/2}\int_{|z-w|<t} \bigl|(-\sqrt{L})^{k+1}e^{-t\sqrt{L}}f(w)\bigr|^2t^{1-2n}s^{2k-1}\,dw\,ds\,dt \\
&= \frac{1}{2k2^{2k}}\int_0^{+\infty}\int_{|z-w|<t} \bigl|(-t\sqrt{L})^{k+1}e^{-t\sqrt{L}}f(w)\bigr|^2t^{-1-2n}\,dw\,dt \\
&= \frac{1}{2k2^{2k}}\int_0^{+\infty}\int_{|z-w|<t} \bigl|D_t^{k+1}f(w)\bigr|^2\frac{dw\,dt}{t^{2n+1}} = \frac{1}{2k2^{2k}}\bigl(S_L^{k+1}f(z)\bigr)^2,
\end{aligned}
$$

we get $S_L^{k+1}f \in L^1(\mathbb{C}^n)$. Then $f \in H_L^1(\mathbb{C}^n)$ follows from (a).

This completes the proof of Theorem 2(b).

(c) By $S_L^k f(z) \leq (\frac{1}{2})^{2\lambda n} g_{\lambda,k}^* f(z)$, we know $f \in H_L^1(\mathbb{C}^n)$ when $g_{\lambda,k}^* f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$. In the following, we show there exists a constant $C > 0$ such that for any atom $a(z)$ of $H_L^1(\mathbb{C}^n)$, we have

$$\bigl\|g_{\lambda,k}^* a\bigr\|_{L^1} \leq C.$$

Without loss of generality, we may assume $a(z)$ is supported in $B(0,r)$, then

$$
\begin{aligned}
g_{\lambda,k}^* a(z)^2 &= \int_0^\infty\int_{\mathbb{C}^n}\left(\frac{t}{t+|z-w|}\right)^{2\lambda n}\bigl|D_t^k a(w)\bigr|^2\frac{dw\,dt}{t^{2n+1}} \\
&= \int_0^\infty\int_{|z-w|<t}\left(\frac{t}{t+|z-w|}\right)^{2\lambda n}\bigl|D_t^k a(w)\bigr|^2\frac{dw\,dt}{t^{2n+1}} \\
&\quad + \sum_{i=1}^\infty\int_0^\infty\int_{2^{i-1}t\leq|z-w|<2^it}\left(\frac{t}{t+|z-w|}\right)^{2\lambda n}\bigl|D_t^k a(w)\bigr|^2\frac{dw\,dt}{t^{2n+1}} \\
&\leq CS_L^1 a(z)^2 + \sum_{i=1}^\infty 2^{-2i\lambda n}S_{L,2^i}^k a(z)^2.
\end{aligned}
$$

Therefore,

$$\bigl\|g_{\lambda,k}^* a\bigr\|_{L^1} \leq C\bigl\|S_L^1 a\bigr\|_{L^1} + \sum_{i=1}^\infty 2^{-i\lambda n}\bigl\|S_{L,2^i}^k a\bigr\|_{L^1}.$$

By part (a), we have $\|S_L^k a\|_{L^1} \leq C$. In the following, we will prove that

$$\bigl\|S_{L,2^i}^k a\bigr\|_{L^1} \leq C2^{3in}. \tag{13}$$

First, by Lemma 5, we can obtain

$$\bigl\|S_{L,2^i}^k a\bigr\|_{L^1(B(0,2^{i+2}r))} \leq \bigl|B\bigl(0,2^{i+2}r\bigr)\bigr|^{1/2}\bigl\|S_{L,2^i}^k a\bigr\|_{L^2} \leq C2^{2in}. \tag{14}$$

Let $z \notin B(0, 2^{i+2}r)$. We have

$$\begin{aligned} S^k_{L,2^i}a(z)^2 &\leq \int_0^\infty \int_{|z-w|<2^i t} \left(\int_{B(0,r)} \left| D^k_t(w-v) - D^k_t(w) \right| \left| a(v) \right| dv \right)^2 \frac{dw\,dt}{t^{2n+1}} \\ &\leq \int_0^{\frac{|z|}{2^{i+1}}} \int_{|z-w|<2^i t} (\cdots)^2 \frac{dw\,dt}{t^{2n+1}} + \int_{\frac{|z|}{2^{i+1}}}^\infty \int_{|z-w|<2^i t} (\cdots)^2 \frac{dw\,dt}{t^{2n+1}} \\ &= I_1 + I_2. \end{aligned}$$

For $z \notin B(0, 2^{i+2}r)$, when $|z-w| < 2^i t \leq \frac{|z|}{2}$, we have $|w| \sim |z|$. By Lemma 4, we get

$$\begin{aligned} I_1 &\leq C \int_0^{\frac{|z|}{2^{i+1}}} \int_{|z-w|<2^i t} \left(\int_{B(0,r)} \frac{\sqrt{t}}{(t^2 + A|w|^2)^{(2n+1)/2}} |v| \left| a(v) \right| dv \right)^2 \frac{dw\,dt}{t^{2n+1}} \\ &\leq C 2^{2in} \int_0^{\frac{|z|}{2^{i+1}}} t^{-4n} \left(\frac{|z|}{t} \right)^{-(4n+3)} \left(\frac{r}{t} \right)^2 \frac{dt}{t} \leq C 2^{2in-i} \frac{r^2}{|z|^{4n+2}}. \end{aligned}$$

By Lemma 4 again, we get

$$\begin{aligned} I_2 &\leq C \int_{\frac{|z|}{2^{i+1}}}^\infty \int_{|z-w|<2^i t} \left(\int_{B(0,r)} t^{-2n} \left(\frac{r}{t} \right) \left| a(v) \right| dv \right)^2 \frac{dw\,dt}{t^{2n+1}} \\ &\leq C 2^{2in} \int_{\frac{|z|}{2^{i+1}}}^\infty t^{-4n} \left(\frac{r}{t} \right)^2 \frac{dt}{t} \leq C 2^{i(6n+2)} \frac{r^2}{|z|^{2(2n+1)}}. \end{aligned}$$

Thus,

$$\int_{|z| \geq 2^{i+2}r} \left| S^k_{L,2^i}a(z) \right| dz \leq C 2^{3in+i} \int_{|z| \geq 2^{i+2}r} \frac{r}{|z|^{2n+1}} dz \leq C 2^{3in}. \tag{15}$$

Therefore, when $\lambda > 3$, we prove $\|g^*_{\lambda,k}a\|_{L^1} \leq C$. Then Theorem 2(c) is proved. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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