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A sharpened version of Hardy's inequality for parameter $p = 5/4$

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Abstract

The purpose of this paper is to investigate a sharpened version of Hardy's inequality for parameter $p = 5/4$. By evaluating the weight coefficient $W(k, 5/4)$, sharpened Hardy's inequality that contains the best coefficient $\eta_{5/4} = 0.46\ldots$ is established.

MSC: 26D15; 26D20; 26D07

Keywords: Hardy's inequality; weight coefficient; sharpened inequality; exponential parameter; best coefficient

1 Introduction

Let $p > 1$, $1/p + 1/q = 1$, $a_n \geq 0$ $(n = 1, 2, \ldots)$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$. Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p < q^p \sum_{n=1}^{\infty} a_n^p, \tag{1}$$

where $q^p = (\frac{p}{p-1})^p$ is the best coefficient. Inequality (1) is called Hardy's inequality which is of great use in the field of modern mathematics (see [1, 2]).

A special case of (1) yields the following inequalities:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 < 4 \sum_{n=1}^{\infty} a_n^2, \tag{2}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^3 < \frac{27}{8} \sum_{n=1}^{\infty} a_n^3. \tag{3}$$

In 1998, Yang and Zhu [3] evaluated the weight coefficient $W(k, p)$,

$$W(k,p) = k^{1-1/p} \sum_{n=k}^{\infty} \frac{1}{n^p}\left(\sum_{j=1}^{n} \frac{1}{j^{1/p}}\right)^{p-1}, \quad k = 1, 2, \ldots, \tag{4}$$

and established an improved version of inequality (2) as follows:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 < 4 \sum_{n=1}^{\infty}\left(1 - \frac{1}{3\sqrt{n} + 5}\right) a_n^2. \tag{5}$$

With the same approach, that is, evaluating the weight coefficient $W(k,p)$, Huang [4–7] gave some improvements on Hardy's inequality for $p = 3$ and $p = 3/2$, *i.e.*,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^3 < \frac{27}{8}\sum_{n=1}^{\infty}\left(1-\frac{3}{19n^{2/3}}\right)a_n^3, \tag{6}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{3/2} \leq 3\sqrt{3}\sum_{n=1}^{\infty}\left(1-\frac{1}{5}\cdot\frac{1}{\sqrt[3]{n}+3}\right)a_n^{3/2}. \tag{7}$$

Some further extensions of Hardy's inequality related to the range of parameter p were given in Huang [7, 8].

In 2005, Yang [9] proved an inequality for the weight coefficient $W(k,2)$

$$W(k,2) = \sqrt{k}\sum_{n=k}^{\infty}\frac{1}{n^2}\left(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\right) \leq 4\left[1-\frac{1}{\sqrt{k}}\left(1-\frac{1}{4}W(1,2)\right)\right]$$

and established the following inequality:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^2 < 4\sum_{n=1}^{\infty}\left(1-\frac{\theta_2}{\sqrt{n}}\right)a_n^2, \tag{8}$$

where $\theta_2 = 1-\frac{1}{4}W(1,2) = 0.13788928\ldots$ is the best coefficient under the weight coefficient $W(k,2)$.

In 2009, Zhang and Xu made use of the monotonicity theorem [10–13] and obtained an improvement of inequality (1):

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}\left(1-\frac{c_p}{2(n-1/2)^{1-1/p}}\right)a_n^p, \tag{9}$$

where

$$c_p = \begin{cases} (p-1)[1-2^{1/p}(1-1/p)], & 1 < p \leq 2, \\ 1-2^{1-1/p}(1-1/p)^{p-1}, & p > 2. \end{cases}$$

By evaluating the weight coefficient $W(k,p)$, and with the help of an inequality-proving package called BOTTEMA [14, 15], He [16] investigated a sharpened version of Hardy's inequality for $p \in N$ and obtained the following improved version of inequality (3):

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^3 \leq \frac{27}{8}\sum_{n=1}^{\infty}\left(1-\frac{\theta_3}{n^{2/3}}\right)a_n^3, \tag{10}$$

where $\theta_3 = 1-\frac{8}{27}W(1,3) = 0.1673\ldots$ is the best coefficient under the weight coefficient $W(k,3)$.

In addition, in [16] the author wrote the computer program HDISCOVER to accomplish the automated verification of the following inequality for $p \in N$ (N is the set of natural numbers):

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \le \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}\left(1-\frac{\theta_p(1)}{n^{1-1/p}}\right)a_n^p, \tag{11}$$

where $\theta_p(1) = 1 - (\frac{p-1}{p})^p W(1,p)$ is the best coefficient of (11) under the weight coefficient $W(k,p)$.

Recently, based on the program HDISCOVER 2012 written by Deng, He and Wu [17], an automated verification of inequality (11) is achieved for $p \in Q$ ($Q$is the set of rational numbers).

For more detailed information of Hardy's inequality, we refer the interested readers to relevant research papers [10, 12, 18–23].

In this paper, by evaluating the weight coefficient $W(k,5/4)$, we establish an improvement of Hardy's inequality for parameter $p = 5/4$ as follows:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{5/4} \le 5^{5/4}\sum_{n=1}^{\infty}\left(1-\frac{1}{10}\cdot\frac{1}{n^{1/5}+\eta_{5/4}}\right)a_n^{5/4}, \tag{12}$$

where $\eta_{5/4} = \frac{5^{5/4}}{10[5^{5/4}-W(1,5/4)]} - 1 = 0.46\ldots$ is the best coefficient under the weight coefficient $W(k,5/4)$.

2 Lemmas

To prove the main results in Section 3, we will use the following lemmas.

Lemma 1 (see[22]) *If $p > 1$, then for all integers $n \ge 1$, it holds that*

$$\begin{aligned}&\frac{p}{p-1}n^{1-1/p} - \frac{p}{p-1} + \frac{1}{2} + \frac{1}{2n^{1/p}}\\ &\quad\le \sum_{j=1}^{n}\frac{1}{j^{1/p}} \le \frac{p}{p-1}n^{1-1/p} - \frac{p}{p-1} + \frac{1}{2} + \frac{1}{2n^{1/p}} + \frac{1}{12p} - \frac{1}{12pn^{1+1/p}}.\end{aligned} \tag{13}$$

Lemma 2 (see[3]) *If $p > 1$, then for all integers $n \ge k \ge 1$, it holds that*

$$\frac{1}{(p-1)k^{p-1}} + \frac{1}{2k^p} < \sum_{n=k}^{\infty}\frac{1}{n^p} < \frac{1}{(p-1)k^{p-1}} + \frac{1}{2k^p} + \frac{p}{12k^{p+1}}.$$

Lemma 3 *Let $p > 1$, $1/p + 1/q = 1$, and let g_r, g_l be the functions defined by*

$$g_r(x) = -\frac{6p+12q-1}{12pqx^{1/q}} + \frac{1}{2qx} - \frac{1}{12pqx^2}, \qquad g_l(x) = -\frac{p+2q}{2pqx^{1/q}} + \frac{1}{2qx}, \quad x \in [1,+\infty).$$

Then $-1 < g_r(x) < 0$, $-1 < g_l(x) < 0$.

Proof Since $p > 1$, $1/p + 1/q = 1$, hence $1/x^{1+1/p} \ge 1/x^2$ for $x \in [1,+\infty)$.

Further, we have

$$\begin{aligned} g_r'(x) &= \frac{6p+12q-1}{12pq^2x^{1+1/q}} - \frac{1}{2qx^2} + \frac{1}{6pqx^3} \\ &\geq \frac{6p+12q-1}{12pq^2x^2} - \frac{1}{2qx^2} + \frac{1}{6pqx^3} \\ &= \frac{(5px+x+2p)(p-1)}{12p^3x^3} > 0, \end{aligned}$$

and consequently, g_r is strictly increasing on $[1,+\infty)$.

Now, from $g_r(1) = -1/p > -1$ and $\lim_{x\to+\infty} g_r(x) = 0$, it follows that $g_r(x) \geq g_r(1) = -1/p > -1$ and $g_r(x) < 0$.

Similarly, from

$$g_l'(x) = \frac{p+2q}{2pq^2x^{1+1/q}} - \frac{1}{2qx^2} \geq \frac{p+2q}{2pq^2x^2} - \frac{1}{2qx^2} = \frac{p-1}{2p^2x^2} > 0,$$

$$g_l(1) = -1/p > -1 \quad \text{and} \quad \lim_{x\to+\infty} g_l(x) = 0,$$

we deduce that $-1 < g_l(x) < 0$.

Lemma 3 is proved. □

Lemma 4 *Let* $-1 < g(x) < 0$. *If* $\alpha \in (0,1]$, *then*

$$\begin{aligned} &\bigl(1+g(x)\bigr)\left(1+(\alpha-1)g(x)+\frac{(\alpha-1)(\alpha-2)}{2}g^2(x)\right) \\ &\quad \leq \bigl(1+g(x)\bigr)^\alpha \leq 1+\alpha g(x)+\frac{\alpha(\alpha-1)}{2}g^2(x). \end{aligned}$$

If $\alpha \in [1,2]$, *then*

$$\bigl(1+g(x)\bigr)^\alpha \geq 1+\alpha g(x)+\frac{\alpha(\alpha-1)}{2}g^2(x).$$

Proof When $\alpha \in (0,1]$. By using the Maclaurin formula

$$\begin{aligned} \bigl(1+g(x)\bigr)^\alpha &= 1+\alpha g(x)+\frac{\alpha(\alpha-1)}{2}g^2(x) \\ &\quad + \frac{\alpha(\alpha-1)(\alpha-2)(1+\theta g(x))^{\alpha-3}}{6}g^3(x), \quad \theta \in (0,1), \end{aligned}$$

and noticing $-1 < g(x) < 0$, we find

$$\begin{aligned} &1+\theta g(x) > 1+g(x) > 0, \\ &\frac{\alpha(\alpha-1)(\alpha-2)(1+\theta g(x))^{\alpha-3}}{6}g^3(x) \leq 0, \\ &\frac{(\alpha-1)(\alpha-2)(\alpha-3)(1+\theta g(x))^{\alpha-4}}{6}g^3(x) \geq 0. \end{aligned}$$

Thus

$$\begin{aligned}
&\big(1+g(x)\big)^{\alpha} \le 1+\alpha g(x)+\frac{\alpha(\alpha-1)}{2}g^2(x),\\
&\big(1+g(x)\big)^{\alpha} = \big(1+g(x)\big)\big(1+g(x)\big)^{\alpha-1}\\
&\qquad \ge \big(1+g(x)\big)\left(1+(\alpha-1)g(x)+\frac{(\alpha-1)(\alpha-2)}{2}g^2(x)\right).
\end{aligned}$$

When $\alpha \in [1,2]$. We have

$$\frac{\alpha(\alpha-1)(\alpha-2)(1+\theta g(x))^{\alpha-3}}{6}g^3(x) \ge 0.$$

Thus

$$\big(1+g(x)\big)^{\alpha} \ge 1+\alpha g(x)+\frac{\alpha(\alpha-1)}{2}g^2(x).$$

The proof of Lemma 4 is complete. □

Lemma 5 *Let $p>1$, $1/p+1/q=1$, $n\ge k\ge 1$, and let $[x]$ denote the greatest integer less than or equal to the real number x. Then we have*

$$\begin{aligned}
W(k,p) \le{}& q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\left[\frac{1}{n^{1+1/q}}\big(1+g_r(n)\big)^{[p]-1}\right.\\
&\left.\times\left(1+\big(p-[p]\big)g_r(n)+\frac{(p-[p])(p-[p]-1)}{2}g_r^2(n)\right)\right].
\end{aligned}$$

Proof By Lemma 1 and the identity $pq=p+q$, $q(p-1)=p$, it follows that

$$\begin{aligned}
W(k,p) &= k^{1-1/p}\sum_{n=k}^{\infty}\frac{1}{n^p}\left(\sum_{j=1}^{n}\frac{1}{j^{1/p}}\right)^{p-1}\\
&\le k^{1-1/p}\sum_{n=k}^{\infty}\frac{1}{n^p}\left(\frac{p}{p-1}n^{1-1/p}-\frac{p}{p-1}+\frac{1}{2}+\frac{1}{2n^{1/p}}+\frac{1}{12p}-\frac{1}{12pn^{1+1/p}}\right)^{p-1}\\
&= k^{1/q}\sum_{n=k}^{\infty}\frac{1}{n^p}\left(qn^{1/q}-\frac{6p+12q-1}{12p}+\frac{1}{2n^{1/p}}-\frac{1}{12pn^{1+1/p}}\right)^{p-1}\\
&= k^{1/q}\sum_{n=k}^{\infty}\frac{1}{n^p}q^{p-1}n^{(p-1)/q}\left(1-\frac{6p+12q-1}{12pqn^{1/q}}+\frac{1}{2qn}-\frac{1}{12pqn^2}\right)^{p-1}\\
&= q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\frac{1}{n^{1+1/q}}\big(1+g_r(n)\big)^{p-1}.
\end{aligned}$$

Combining Lemmas 3 and 4, we obtain

$$\begin{aligned}
&W(k,p)\\
&\quad\le q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\left[\frac{1}{n^{1+1/q}}\big(1+g_r(n)\big)^{[p]-1}\big(1+g_r(n)\big)^{p-[p]}\right]
\end{aligned}$$

$$\le q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\biggl[\frac{1}{n^{1+1/q}}\bigl(1+g_r(n)\bigr)^{[p]-1} \times \biggl(1+\bigl(p-[p]\bigr)g_r(n)+\frac{(p-[p])(p-[p]-1)}{2}g_r^2(n)\biggr)\biggr].$$

This completes the proof of Lemma 5. □

Lemma 6 *Let* $1/p+1/q=1$, $n\ge k\ge 1$. *If* $p\in(1,2)$, *then*

$$W(k,p)\ge q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\biggl[\frac{1}{n^{1+1/q}}\bigl(1+g_l(n)\bigr)^{[p]} \times \biggl(1+\bigl(p-[p]-1\bigr)g_l(n)+\frac{(p-[p]-1)(p-[p]-2)}{2}g_l^2(n)\biggr)\biggr].$$

If $p\in[2,+\infty)$, *then*

$$W(k,p)\ge q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\biggl[\frac{1}{n^{1+1/q}}\bigl(1+g_l(n)\bigr)^{[p]-2} \times \biggl(1+\bigl(p-[p]+1\bigr)g_l(n)+\frac{(p-[p]+1)(p-[p])}{2}g_l^2(n)\biggr)\biggr].$$

Proof Since $pq=p+q$, $q(p-1)=p$, using Lemma 1 gives

$$\begin{aligned}
W(k,p)&=k^{1-1/p}\sum_{n=k}^{\infty}\frac{1}{n^p}\Biggl(\sum_{j=1}^{n}\frac{1}{j^{1/p}}\Biggr)^{p-1}\\
&\ge k^{1-1/p}\sum_{n=k}^{\infty}\frac{1}{n^p}\biggl(\frac{p}{p-1}n^{1-1/p}-\frac{p}{p-1}+\frac{1}{2}+\frac{1}{2n^{1/p}}\biggr)^{p-1}\\
&=k^{1/q}\sum_{n=k}^{\infty}\frac{1}{n^p}\biggl(qn^{1/q}-\frac{p+2q}{2p}+\frac{1}{2n^{1/p}}\biggr)^{p-1}\\
&=k^{1/q}\sum_{n=k}^{\infty}\frac{1}{n^p}q^{p-1}n^{(p-1)/q}\biggl(1-\frac{p+2q}{2pqn^{1/q}}+\frac{1}{2qn}\biggr)^{p-1}\\
&=q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\frac{1}{n^{1+1/q}}\bigl(1+g_l(n)\bigr)^{p-1}.
\end{aligned}$$

When $p\in(1,2)$. From Lemmas 3 and 4, we have

$$\begin{aligned}
W(k,p)&\ge q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\frac{1}{n^{1+1/q}}\bigl(1+g_l(n)\bigr)^{[p]-1}\bigl(1+g_l(n)\bigr)^{p-[p]}\\
&\ge q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\biggl[\frac{1}{n^{1+1/q}}\bigl(1+g_l(n)\bigr)^{[p]}\\
&\quad\times\biggl(1+\bigl(p-[p]-1\bigr)g_l(n)+\frac{(p-[p]-1)(p-[p]-2)}{2}g_l^2(n)\biggr)\biggr].
\end{aligned}$$

When $p \in [2, +\infty)$. Using Lemmas 3 and 4, we obtain

$$\begin{aligned} W(k,p) &\geq q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\frac{1}{n^{1+1/q}}\bigl(1+g_l(n)\bigr)^{[p]-2}\bigl(1+g_l(n)\bigr)^{p-[p]+1} \\ &\geq q^{p-1}k^{1/q}\sum_{n=k}^{\infty}\biggl[\frac{1}{n^{1+1/q}}\bigl(1+g_l(n)\bigr)^{[p]-2} \\ &\quad \times\biggl(1+\bigl(p-[p]+1\bigr)g_l(n)+\frac{(p-[p]+1)(p-[p])}{2}g_l^2(n)\biggr)\biggr]. \end{aligned}$$

Lemma 6 is proved. □

Lemma 7 (see[3]) *Let $p > 1$, $a_n \geq 0$ $(n = 1, 2, \ldots)$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$. Then*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leq \sum_{k=1}^{\infty}\left[k^{1-1/p}\sum_{n=k}^{\infty}\frac{1}{n^p}\left(\sum_{j=1}^{n}\frac{1}{j^{1/p}}\right)^{p-1}a_k^p\right] = \sum_{k=1}^{\infty}W(k,p)a_k^p.$$

3 Main results

Theorem 1 *For an arbitrary natural number k, the following inequality holds true*:

$$W(k, 5/4) < R_{5/4}(k),$$

where

$$\begin{aligned} R_{5/4}(k) = 5^{1/4}\biggl(5 &- \frac{133}{240k^{1/5}} - \frac{17{,}689}{144{,}000k^{2/5}} + \frac{25}{48k} - \frac{19}{192k^{6/5}} - \frac{17{,}689}{480{,}000k^{7/5}} + \frac{467}{4{,}224k^2} \\ &+ \frac{133}{18{,}000k^{11/5}} + \frac{97}{38{,}400k^3} + \frac{133}{60{,}000k^{16/5}} + \frac{61}{504{,}000k^4} + \frac{19}{240{,}000k^5}\biggr). \end{aligned}$$

Proof Using Lemma 5 gives

$$W(k,5/4) \leq 5^{1/4}k^{1/5}\sum_{n=k}^{\infty}\left[\frac{1}{n^{6/5}}\left(1+\frac{1}{4}g_r(n)-\frac{3}{32}g_r^2(n)\right)\right] = 5^{1/4}k^{1/5}\sum_{n=k}^{\infty}r_{5/4}(n),$$

where

$$\begin{aligned} r_{5/4}(n) = \frac{1}{n^{6/5}} &- \frac{133}{600n^{7/5}} - \frac{17{,}689}{240{,}000n^{8/5}} + \frac{1}{40n^{11/5}} + \frac{133}{8{,}000n^{12/5}} - \frac{41}{9{,}600n^{16/5}} \\ &- \frac{133}{60{,}000n^{17/5}} + \frac{1}{4{,}000n^{21/5}} - \frac{1}{60{,}000n^{26/5}}. \end{aligned}$$

Hence

$$\begin{aligned} W(k,5/4) \leq 5^{1/4}k^{1/5}\sum_{n=k}^{\infty}\biggl(&\frac{1}{n^{6/5}} - \frac{133}{600n^{7/5}} - \frac{17{,}689}{240{,}000n^{8/5}} + \frac{1}{40n^{11/5}} + \frac{133}{8{,}000n^{12/5}} \\ &- \frac{41}{9{,}600n^{16/5}} - \frac{133}{60{,}000n^{17/5}} + \frac{1}{4{,}000n^{21/5}} - \frac{1}{60{,}000n^{26/5}}\biggr). \end{aligned}$$

Using Lemma 2 and taking $p = 6/5, 7/5, 8/5, 11/5, 12/5, 16/5, 17/5, 21/5, 26/5$ in the right-hand side of inequality (13), respectively, we get

$$\sum_{n=k}^{\infty} \frac{1}{n^{6/5}} < \frac{5}{k^{1/5}} + \frac{1}{2k^{6/5}} + \frac{1}{10k^{11/5}},$$

$$-\sum_{n=k}^{\infty} \frac{133}{600n^{7/5}} < -\frac{133}{240k^{2/5}} - \frac{133}{1{,}200k^{7/5}},$$

$$\ldots,$$

$$\sum_{n=k}^{\infty} \frac{1}{4{,}000n^{21/5}} < \frac{1}{12{,}800k^{16/5}} + \frac{1}{8{,}000k^{21/5}} + \frac{7}{80{,}000k^{26/5}},$$

$$-\sum_{n=k}^{\infty} \frac{1}{60{,}000n^{26/5}} < -\frac{1}{252{,}000k^{21/5}} - \frac{1}{120{,}000k^{26/5}}.$$

Adding up the above inequalities, we obtain

$$W(k, 5/4) < R_{5/4}(k).$$

Theorem 1 is proved. □

Theorem 2 *For an arbitrary natural number k, the following inequality holds true*:

$$W(k, 5/4) > L_{5/4}(k),$$

where

$$\begin{aligned} L_{5/4}(k) = 5^{1/4}\biggl(5 &- \frac{9}{16k^{1/5}} - \frac{81}{640k^{2/5}} - \frac{15{,}309}{25{,}600k^{3/5}} + \frac{25}{48k} - \frac{45}{448k^{6/5}} + \frac{3{,}159}{51{,}200k^{7/5}} \\ &- \frac{15{,}309}{64{,}000k^{8/5}} + \frac{17}{1{,}408k^{2}} - \frac{129}{5{,}120k^{11/5}} + \frac{891}{12{,}800k^{12/5}} - \frac{45{,}927}{640{,}000k^{13/5}} \\ &- \frac{27}{102{,}400k^{3}} - \frac{567}{64{,}000k^{16/5}} + \frac{1}{12{,}800k^{4}} - \frac{3{,}213}{640{,}000k^{21/5}}\biggr). \end{aligned}$$

Proof Utilizing Lemma 6 gives

$$\begin{aligned} W(k, 5/4) &\geq 5^{1/4}k^{1/5}\sum_{n=k}^{\infty}\biggl[\frac{1}{n^{6/5}}\bigl(1 + g_l(n)\bigr)\biggl(1 - \frac{3}{4}g_l(n) + \frac{21}{32}g_l^2(n)\biggr)\biggr] \\ &= 5^{1/4}k^{1/5}\sum_{n=k}^{\infty} l_{5/4}(n), \end{aligned}$$

where

$$\begin{aligned} l_{5/4}(n) = \frac{1}{n^{6/5}} &- \frac{9}{40n^{7/5}} - \frac{243}{3{,}200n^{8/5}} - \frac{15{,}309}{32{,}000n^{9/5}} + \frac{1}{40n^{11/5}} + \frac{27}{1{,}600n^{12/5}} \\ &+ \frac{5{,}103}{32{,}000n^{13/5}} - \frac{3}{3{,}200n^{16/5}} - \frac{567}{32{,}000n^{17/5}} + \frac{21}{32{,}000n^{21/5}}. \end{aligned}$$

Hence

$$W(k,5/4) \geq 5^{1/4}k^{1/5}\sum_{n=k}^{\infty}\left(\frac{1}{n^{6/5}} - \frac{9}{40n^{7/5}} - \frac{243}{3{,}200n^{8/5}} - \frac{15{,}309}{32{,}000n^{9/5}} + \frac{1}{40n^{11/5}} + \frac{27}{1{,}600n^{12/5}} + \frac{5{,}103}{32{,}000n^{13/5}} - \frac{3}{3{,}200n^{16/5}} - \frac{567}{32{,}000n^{17/5}} + \frac{21}{32{,}000n^{21/5}}\right).$$

Using Lemma 2 and taking $p = 6/5, 7/5, 8/5, 9/5, 11/5, 12/5, 13/5, 16/5, 17/5, 21/5$ in the left-hand side of inequality (13), respectively, we get

$$\sum_{n=k}^{\infty}\frac{1}{n^{6/5}} > \frac{5}{k^{1/5}} + \frac{1}{2k^{6/5}},$$

$$-\sum_{n=k}^{\infty}\frac{9}{40n^{7/5}} > -\frac{9}{16k^{2/5}} - \frac{9}{80k^{7/5}} - \frac{21}{800k^{12/5}},$$

$$\ldots,$$

$$-\sum_{n=k}^{\infty}\frac{567}{32{,}000n^{17/5}} > -\frac{189}{25{,}600k^{12/5}} - \frac{567}{64{,}000k^{17/5}} - \frac{3{,}213}{640{,}000k^{22/5}},$$

$$\sum_{n=k}^{\infty}\frac{21}{32{,}000n^{21/5}} > \frac{21}{102{,}400k^{16/5}} + \frac{21}{64{,}000k^{21/5}}.$$

Adding up the above inequalities, we obtain

$$W(k,5/4) > L_{5/4}(k).$$

Theorem 2 is proved. □

Theorem 3 *Let $a_n \geq 0$ $(n = 1, 2, \ldots)$, $0 < \sum_{n=1}^{\infty} a_n^{5/4} < \infty$. Then*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{5/4} \leq 5^{5/4}\sum_{n=1}^{\infty}\left(1 - \frac{1}{10}\cdot\frac{1}{n^{1/5} + \eta_{5/4}}\right)a_n^{5/4}, \tag{14}$$

where $\eta_{5/4} = \frac{5^{5/4}}{10[5^{5/4} - W(1,5/4)]} - 1 = 0.46\ldots$ is the best possible under the weight coefficient $W(k,5/4)$.

Proof By Lemma 7, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{5/4} \leq \sum_{k=1}^{\infty} W(k,5/4)a_k^{5/4}.$$

Therefore, to prove inequality (14), it suffices to show that

$$W(k,5/4) \leq 5^{5/4}\left(1 - \frac{1}{10}\cdot\frac{1}{k^{1/5} + \eta_{5/4}}\right). \tag{15}$$

Obviously, inequality (15) becomes an equality for $k = 1$. In what follows, we will assume that $k \geq 2$.

By Theorem 1 $W(k, 5/4) < R_{5/4}(k)$, we need only to prove that

$$R_{5/4}(k) \leq 5^{5/4}\left(1 - \frac{1}{10} \cdot \frac{1}{k^{1/5} + \eta_{5/4}}\right).$$

Note that

$$\eta_{5/4} = \frac{5^{5/4}}{10[5^{5/4} - W(1, 5/4)]} - 1 = 0.44\ldots > \frac{11}{25},$$

it suffices to show

$$R_{5/4}(k) \leq 5^{5/4}\left(1 - \frac{1}{10} \cdot \frac{1}{k^{1/5} + 11/25}\right). \tag{16}$$

Substituting $k = x^5$ in (16), inequality (16) becomes

$$R_{5/4}\left(x^5\right) \leq 5^{5/4}\left(1 - \frac{1}{10} \cdot \frac{1}{x + 11/25}\right), \quad \text{where } x \geq \sqrt[5]{2},$$

which is equivalent to the following inequality:

$$\begin{aligned}
&5^{1/4}\left(5 - \frac{133}{240x} - \frac{17{,}689}{144{,}000x^2} + \frac{25}{48x^5} - \frac{19}{192x^6} - \frac{17{,}689}{480{,}000x^7} + \frac{467}{4{,}224x^{10}} + \frac{133}{18{,}000x^{11}}\right. \\
&\qquad \left. + \frac{97}{38{,}400x^{15}} + \frac{133}{60{,}000x^{16}} + \frac{61}{504{,}000x^{20}} + \frac{19}{240{,}000x^{25}}\right) \\
&\quad \leq 5^{5/4}\left(1 - \frac{1}{10} \cdot \frac{1}{x + 11/25}\right) \qquad (17) \\
&\quad \Leftrightarrow 5 - \frac{133}{240x} - \frac{17{,}689}{144{,}000x^2} + \frac{25}{48x^5} - \frac{19}{192x^6} - \frac{17{,}689}{480{,}000x^7} + \frac{467}{4{,}224x^{10}} + \frac{133}{18{,}000x^{11}} \\
&\qquad + \frac{97}{38{,}400x^{15}} + \frac{133}{60{,}000x^{16}} + \frac{61}{504{,}000x^{20}} + \frac{19}{240{,}000x^{25}} \leq 5 - \frac{1}{2} \cdot \frac{1}{x + 11/25} \\
&\quad \Leftrightarrow -\frac{133}{240x} - \frac{17{,}689}{144{,}000x^2} + \frac{25}{48x^5} - \frac{19}{192x^6} - \frac{17{,}689}{480{,}000x^7} + \frac{467}{4{,}224x^{10}} + \frac{133}{18{,}000x^{11}} \\
&\qquad + \frac{97}{38{,}400x^{15}} + \frac{133}{60{,}000x^{16}} + \frac{61}{504{,}000x^{20}} + \frac{19}{240{,}000x^{25}} + \frac{1}{2} \cdot \frac{1}{x + 11/25} \leq 0 \\
&\quad \Leftrightarrow -\frac{f(x)}{221{,}760{,}000x^{25}(25x + 11)} \leq 0,
\end{aligned}$$

where

$$\begin{aligned}
f(x) &= 300{,}300{,}000x^{25} + 2{,}032{,}838{,}500x^{24} + 299{,}651{,}660x^{23} - 2{,}887{,}500{,}000x^{21} \\
&\quad - 721{,}875{,}000x^{20} + 445{,}702{,}950x^{19} + 89{,}895{,}498x^{18} - 612{,}937{,}500x^{16} \\
&\quad - 310{,}656{,}500x^{15} - 18{,}024{,}160x^{14} - 14{,}004{,}375x^{11} - 18{,}451{,}125x^{10} \\
&\quad - 5{,}407{,}248x^9 - 671{,}000x^6 - 295{,}240x^5 - 438{,}900x - 193{,}116.
\end{aligned}$$

From the hypothesis $x \geq \sqrt[5]{2} > 1.14$, we have

$$\begin{aligned}
&300{,}300{,}000x^{25} + 2{,}032{,}838{,}500x^{24} + 299{,}651{,}660x^{23} - 2{,}887{,}500{,}000x^{21} \\
&\qquad - 721{,}875{,}000x^{20} + 445{,}702{,}950x^{19} + 89{,}895{,}498x^{18} \\
&\quad > \bigl(300{,}300{,}000 \times 1.14^4 + 2{,}032{,}838{,}500 \times 1.14^3 + 299{,}651{,}660 \times 1.14^2 \\
&\qquad - 2{,}887{,}500{,}000\bigr)x^{21} - 721{,}875{,}000x^{20} + 445{,}702{,}950x^{19} + 89{,}895{,}498x^{18} \\
&\quad = 1{,}020{,}861{,}716x^{21} - 721{,}875{,}000x^{20} + 445{,}702{,}950x^{19} + 89{,}895{,}498x^{18} \\
&\quad = \bigl[(1{,}020{,}861{,}716x - 721{,}875{,}000)x^2 + 445{,}702{,}950x + 89{,}895{,}498\bigr]x^{18} \\
&\quad > \bigl[(1{,}020{,}861{,}716 \times 1.14 - 721{,}875{,}000) \times 1.14^2 \\
&\qquad + 445{,}702{,}950 \times 1.14 + 89{,}895{,}498\bigr]x^{18} \\
&\quad = 1{,}172{,}299{,}661x^{18}.
\end{aligned}$$

Further, we have

$$\begin{aligned}
f(x) &> 1{,}172{,}299{,}661x^{18} - 612{,}937{,}500x^{16} - 310{,}656{,}500x^{15} \\
&\quad - 18{,}024{,}160x^{14} - 14{,}004{,}375x^{11} - 18{,}451{,}125x^{10} \\
&\quad - 5{,}407{,}248x^9 - 671{,}000x^6 - 295{,}240x^5 - 438{,}900x - 193{,}116 \\
&> 1{,}172{,}299{,}661x^{18} - 612{,}937{,}500x^{18} - 310{,}656{,}500x^{18} \\
&\quad - 18{,}024{,}160x^{18} - 14{,}004{,}375x^{18} \\
&\quad - 18{,}451{,}125x^{18} - 5{,}407{,}248x^{18} - 671{,}000x^{18} - 295{,}240x^{18} \\
&\quad - 438{,}900x^{18} - 193{,}116x^{18} \\
&= 191{,}220{,}497x^{18} > 0.
\end{aligned}$$

Consequently, inequality (17) holds true, and inequality (14) is proved.

Let us now show that $\eta_{5/4} = \frac{5^{5/4}}{10[5^{5/4} - W(1,5/4)]} - 1 = 0.46\ldots$ is the best possible under the weight coefficient $W(k, 5/4)$.

Consider inequality (14) in a general form as

$$W(k, 5/4) \leq 5^{5/4}\left(1 - \frac{1}{10} \cdot \frac{1}{k^{1/5} + \eta_{5/4}}\right). \tag{18}$$

Putting $k = 1$ in (18) yields

$$\eta_{5/4} \geq \frac{5^{5/4}}{10[5^{5/4} - W(1, 5/4)]}.$$

Thus the best possible value for $\eta_{5/4}$ in (18) should be $\eta_{\min} = \frac{5^{5/4}}{10[5^{5/4} - W(1,5/4)]}$.

This completes the proof of Theorem 3. □

Remark 1 From the definition of $W(k,p)$ and in the same way as in [17], we can establish the following accurate estimates of $W(1,5/4)$:

$$6.965042829 < W(1,5/4) < 6.967740323. \tag{19}$$

Further, the approximation of $\eta_{5/4}$ can be derived as follows:

$$\eta_{5/4} = \frac{5^{5/4}}{10[5^{5/4} - W(1,5/4)]} - 1 = 0.46\ldots.$$

Remark 2 For $p = 5/4$, inequality (11) can be written as

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{5/4} \leq 5^{5/4}\sum_{n=1}^{\infty}\left(1 - \frac{1-(\frac{1}{5})^{5/4}W(1,5/4)}{n^{1/5}}\right)a_n^{5/4}. \tag{20}$$

It is easy to observe that

$$\frac{1}{10}\cdot\frac{1}{n^{1/5}+\eta_{5/4}} > \frac{1}{10}\cdot\frac{1}{n^{1/5}+4{,}711/10{,}000} > \frac{7}{100n^{1/5}}$$

and

$$1 - \left(\frac{1}{5}\right)^{5/4} W(1,5/4) < 1 - \left(\frac{1}{5}\right)^{5/4} \times 6.967740323 = 0.06808\ldots < \frac{7}{100},$$

hence

$$\frac{1-(\frac{1}{5})^{5/4}W(1,5/4)}{n^{1/5}} < \frac{1}{10}\cdot\frac{1}{n^{1/5}+\eta_{5/4}}.$$

This implies that inequality (14) is stronger than inequality (11).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YD finished the proof and the writing work. SW gave YD some advice on the proof and writing. DH gave YD lots of help in revising the paper. All authors read and approved the final manuscript.

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