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On the quermassintegrals of convex bodies

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Abstract

The well-known question for quermassintegrals is the following: For which values of $i \in \mathbb{N}$ and every pair of convex bodies K and L, is it true that

$$\frac{W_i(K+L)}{W_{i+1}(K+L)} \geq \frac{W_i(K)}{\widetilde{W}_{i+1}(K)} + \frac{W_i(L)}{W_{i+1}(L)}?$$

In 2003, the inequality was proved if and only if $i = n-1$ or $i = n-2$. Following the problem, in the paper, we prove some interrelated results for the quermassintegrals of a convex body.

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1 Introduction

The origin of this work is an interesting inequality of Marcus and Lopes [1]. We write $E_i(x)$, $0 \leq i \leq n$, for the ith elementary symmetric function of an n-tuple $x = (x_1, \ldots, x_n)$ of positive real numbers. This is defined by $E_0(x) = 1$ and

$$E_i(x) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} x_{j_1} x_{j_2} \cdots x_{j_i}, \quad 1 \leq i \leq n.$$

In particular, $E_1(x) = x_1 + \cdots + x_n$, $E_2(x) = \sum_{i \neq j} x_i x_j, \ldots, E_n(x) = x_1 x_2 \cdots x_n$.

The Marcus-Lopes inequality (see also [2, p.33]) states that

$$\frac{E_i(x+y)}{E_{i-1}(x+y)} \geq \frac{E_i(x)}{E_{i-1}(x)} + \frac{E_i(y)}{E_{i-1}(y)} \tag{1.1}$$

for every pair of positive n-tuples x and y. This is a refinement of a further result concerning the symmetric functions, namely

$$\left[E_i(x+y)\right]^{1/i} \geq \left[E_i(x)\right]^{1/i} + \left[E_i(y)\right]^{1/i}. \tag{1.2}$$

A discussion of the cases of equality is contained in the reference [1].

A matrix analogue of (1.1) is the following result of Bergstrom [3] (see also the article [4] and [5, p.67] for an interesting proof): If K and L are positive definite matrices, and if K_i and L_i denote the submatrices obtained by deleting their ith row and column, then

$$\frac{\det(K+L)}{\det(K_i+L_i)} \geq \frac{\det(K)}{\det(K_i)} + \frac{\det(L)}{\det(L_i)}. \tag{1.3}$$

The following generalization of (1.3) was established by Ky Fan [5]:

$$\left(\frac{\det(K+L)}{\det(K_i+L_i)}\right)^{1/k} \geq \left(\frac{\det(K)}{\det(K_i)}\right)^{1/k} + \left(\frac{\det(L)}{\det(L_i)}\right)^{1/k}. \tag{1.4}$$

The proof is based on a minimum principle; see also Ky Fan [6] and Mirsky [7].

There is a remarkable similarity between inequalities about symmetric functions (or determinants of symmetric matrices) and inequalities about the mixed volumes of convex bodies. For example, the analogue of (1.2) in the Brunn-Minkowski theory is as follows.

If K and L are convex bodies in $\mathbb{R}^n$ and if $0 \leq i \leq n-1$, then

$$W_i(K+L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \tag{1.5}$$

with equality if and only if K and L are homothetic, where $W_i(K)$ is the ith quermassintegral of K (see Section 2).

In view of this analogue, Milman asked if there exists a version of (1.1) or (1.3) in the theory of mixed volumes (see [8, 9]).

Question For which values of $0 \leq i \leq n-1$, $i \in \mathbb{N}$, is it true that, for every pair of convex bodies K and L in $\mathbb{R}^n$, one has

$$\frac{W_i(K+L)}{W_{i+1}(K+L)} \geq \frac{W_i(K)}{W_{i+1}(K)} + \frac{W_i(L)}{W_{i+1}(L)}? \tag{1.6}$$

In 1991, the special case $i = 0$ was stated also in [10] as an open question. In the same paper it was also mentioned that (1.6) follows directly from the Aleksandrov-Fenchel inequality when $i = 0$ and L is a ball.

In 2002, it was proved in [9] that (1.6) is true for all $i = 1, \dots, n-1$ in the case where L is a ball.

Theorem A *If K is a convex body and B is a ball in $\mathbb{R}^n$, then for $0 \leq i \leq n-1$, $i \in \mathbb{N}$,*

$$\frac{W_i(K+B)}{W_{i+1}(K+B)} \geq \frac{W_i(K)}{W_{i+1}(K)} + \frac{W_i(B)}{W_{i+1}(B)}. \tag{1.7}$$

In 2003, it was proved in [8] that (1.6) holds true for every pair of convex bodies K and L in $\mathbb{R}^n$ if and only if $i = n-2$ or $i = n-1$.

Theorem B *Let $0 \leq i \leq n-1$, then*

$$\frac{W_i(K+L)}{W_{i+1}(K+L)} \geq \frac{W_i(K)}{W_{i+1}(K)} + \frac{W_i(L)}{W_{i+1}(L)} \tag{1.8}$$

is true for every pair of convex bodies K and L in $\mathbb{R}^n$ if and only if $i = n-1$ or $i = n-2$.

In this paper, following the above results, we prove the following interest results.

Theorem 1.1 *Let $0 \leq i \leq n-1$ and for every convex body K and L in $\mathbb{R}^n$. Then the function*

$$g(t) = \frac{W_i(K+tL)}{W_{i+1}(K+tL)} \tag{1.9}$$

is a convex function on $t \in [0, +\infty)$ if and only if $i = n-1$ or $i = n-2$.

Theorem 1.2 *Let $0 \le i \le n-1$ and for every convex body K and L in $\mathbb{R}^n$. Then*

$$\begin{aligned}&(n-i)W_{i+2}(K)\big(W_{i+1}(K)^2 - W_i(K)W_{i+2}(K)\big)\\&\quad\ge (n-i-2)W_i(K)\big(W_{i+2}^2(K) - W_{i+1}(K)W_{i+3}(K)\big)\end{aligned} \tag{1.10}$$

if and only if $i = n-1$ or $i = n-2$.

2 Notations and preliminaries

The setting for this paper is an n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n.

We use $V(K)$ for the n-dimensional volume of a convex body K. Let $h(K,\cdot): S^{n-1} \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$; *i.e.*, for $u \in S^{n-1}$,

$$h(K,u) = \mathrm{Max}\{u\cdot x : x \in K\},$$

where $u \cdot x$ denotes the usual inner product u and x in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathcal{K}^n$, *i.e.*, for $K, L \in \mathcal{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot): S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$ by

$$\rho(K,u) = \mathrm{Max}\{\lambda \ge 0 : \lambda u \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K, L \in \mathcal{S}^n$, then $\tilde{\delta}(K,L) = |\rho_K - \rho_L|_\infty$.

If $K_i \in \mathcal{K}^n$ $(i = 1,2,\ldots,r)$ and λ_i $(i = 1,2,\ldots,r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in the λ_i given by (see, *e.g.*, [11] or [12])

$$V(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1,\ldots,i_n} \lambda_{i_1}\cdots\lambda_{i_n} V_{i_1,\ldots,i_n}, \tag{2.1}$$

where the sum is taken over all n-tuples $(i_1,\ldots,i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1,\ldots,i_n}$ depends only on the bodies $K_{i_1},\ldots,K_{i_n}$ and is uniquely determined by (2.1). It is called the mixed volume of $K_{i_1},\ldots,K_{i_n}$, and is written as $V(K_{i_1},\ldots,K_{i_n})$. Let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1,\ldots,K_n)$ is written as $V_i(K,L)$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$, then the mixed volume $V_i(K,B)$ is written as $W_i(K)$ and is called the quermassintegral of a convex body K.

It is convenient to write relation (2.1) in the form (see [12, p.137])

$$\begin{aligned}&V(\lambda_1 K_1 + \cdots + \lambda_s K_s)\\&\quad= \sum_{p_1+\cdots+p_r=n}\ \sum_{1\le i_1<\cdots<i_r\le s} \frac{n!}{p_1!\cdots p_r!}\lambda_{i_1}^{p_1}\cdots\lambda_{i_r}^{p_r} V(\underbrace{K_{i_1},\ldots,K_{i_1}}_{p_1},\ldots,\underbrace{K_{i_r},\ldots,K_{i_r}}_{p_r}).\end{aligned} \tag{2.2}$$

Let $s = 2$, $\lambda_1 = 1$, $K_1 = K$, $K_2 = B$, we have

$$V(K + \lambda B) = \sum_{i=0}^{n} \binom{n}{i} \lambda^i W_i(K),$$

known as formula 'Steiner decomposition'.

On the other hand, for convex bodies K and L, (2.2) can show the following special case:

$$W_i(K + \lambda L) = \sum_{j=0}^{n-i} \binom{n-i}{j} \lambda^j V(\underbrace{K,\ldots,K}_{n-i-j}, \underbrace{B,\ldots,B}_{i}, \underbrace{L,\ldots,L}_{j}). \tag{2.3}$$

3 Proof of main results

Proof of Theorem 1.1 If $s, t \in [0, \infty)$, from (1.8), if and only if $i = n-1$ or $i = n-2$, we have

$$\begin{aligned}
g\left(\frac{t+s}{2}\right) &= \frac{W_i(K + \frac{t+s}{2}L)}{W_{i+1}(K + \frac{t+s}{2}L)} \\
&= \frac{W_i(\frac{K+tL}{2} + \frac{K+sL}{2})}{W_{i+1}(\frac{K+tL}{2} + \frac{K+sL}{2})} \\
&\geq \frac{W_i(\frac{K+tL}{2})}{W_{i+1}(\frac{K+tL}{2})} + \frac{W_i(\frac{K+sL}{2})}{W_{i+1}(\frac{K+sL}{2})} \\
&= \frac{1}{2}\frac{W_i(K + tL)}{W_{i+1}(K + tL)} + \frac{1}{2}\frac{W_i(K + sL)}{W_{i+1}(K + sL)} \\
&= \frac{1}{2}\big(g(t) + g(s)\big).
\end{aligned} \tag{3.1}$$

Hence the function $g(t)$ is a convex function on $[0, +\infty)$ for every star body K and L if and only if $i = n-1$ or $i = n-2$. □

Proof of Theorem 1.2 Let K be a convex body in $\mathbb{R}^n$. For every $i \geq 0$, we set

$$f_i(t) = W_i(K + tB),$$

then from (2.3)

$$\begin{aligned}
f_i(t + \varepsilon) &= W_i\big((K + tB) + \varepsilon B\big) \\
&= \sum_{j=0}^{n-i} \binom{n-i}{j} \varepsilon^j W_{i+j}(K + tB) \\
&= f_i(t) + \varepsilon(n-i) f_{i+1}(t) + O\big(\varepsilon^2\big).
\end{aligned}$$

Therefore

$$f_i'(t) = (n-i) f_{i+1}(t).$$

The derivative of the function

$$g_i(t) = \frac{f_i(t)}{f_{i+1}(t)} = \frac{W_i(K + tB)}{W_{i+1}(K + tB)}$$

is thus given by

$$g_i'(t) = (n-i) - (n-i-1)\frac{f_i(t)f_{i+2}(t)}{f_{i+1}^2(t)}. \tag{3.2}$$

Since $g_i(x)$ is a convex function if and only if $i = n-1$ or $i = n-2$, hence by differentiating the both sides of (3.2), we obtain for $t \in (0, +\infty)$

$$(n-i)f_{i+2}(t)f_{i+1}^2(t) + (n-i-2)f_i(t)f_{i+1}(t)f_{i+3}(t) - 2(n-i-1)f_i(t)f_{i+2}^2(t) \geq 0$$

if and only if $i = n-1$ or $i = n-2$.

This can be equivalently written in the form

$$(n-i)f_{i+2}(t)\big(f_{i+1}^2(t) - f_i(t)f_{i+2}(t)\big) \geq (n-i-2)f_i(t)\big(f_{i+2}^2(t) - f_{i+1}(t)f_{i+3}(t)\big)$$

if and only if $i = n-1$ or $i = n-2$.

Letting $t \to 0^+$, we conclude Theorem 1.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CJZ and WSC jointly contributed to the main results Theorems 1.1-1.2. All authors read and approved the final manuscript.

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