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On the Hermite-Hadamard type inequalities

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Abstract

In the present paper, we establish some new Hermite-Hadamard type inequalities involving two functions. Our results in a special case yield recent results on Hermite-Hadamard type inequalities.

MSC: 26D15

Keywords: Hermite-Hadamard inequality; Barnes-Godunova-Levin inequality; Minkowski integral inequality; Hölder inequality

1 Introduction

The following inequality is well known in the literature as Hermite-Hadamard's inequality [1].

Theorem 1.1 *Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on an interval of real numbers. Then the following Hermite-Hadamard inequality for convex functions holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

If the function f is concave, the inequality (1.1) can be written as follows:

$$f\left(\frac{a+b}{2}\right) \geq \frac{1}{b-a} \int_a^b f(x) dx \geq \frac{f(a)+f(b)}{2}. \quad (1.2)$$

Recently, many generalizations, extensions and variants of this inequality have appeared in the literature (see, e.g., [2–10]) and the references given therein. In particular, in 2010, Özdemir and Dragomir [11] established some new Hermite-Hadamard inequalities and other integral inequalities involving two functions in \mathbb{R} . Following this work, the main purpose of the present paper is to establish some dual Hermite-Hadamard type inequalities involving two functions in \mathbb{R}^2 . Our results provide some new estimates on such type of inequalities.

2 Preliminaries

A region $D \subset \mathbb{R}^2$ is called convex if it contains the close line segment joining any two of its points, or equivalently, if $\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2 \in D$ whenever $x(x_1, y_1), y(x_2, y_2) \in D$ and $0 \leq \lambda \leq 1$.

Let $z = f(x, y)$ be a duality function on the convex region $D \subset \mathbb{R}^2$. $z = f(x, y)$ is called a duality convex function on the convex region D if

$$f[\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2] \leq \lambda f(x_1, y_1) + (1 - \lambda)f(x_2, y_2), \tag{2.1}$$

whenever $(x_1, y_1), (x_2, y_2) \in D$ and $0 \leq \lambda \leq 1$.

If the function $f(x, y)$ is concave, the inequality (2.1) can be written as follows:

$$f[\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2] \geq \lambda f(x_1, y_1) + (1 - \lambda)f(x_2, y_2). \tag{2.2}$$

Let $x = (x_{11}, \dots, x_{1n}, \dots, x_{m1}, \dots, x_{mn})$ and $p = (p_{11}, \dots, p_{1n}, \dots, p_{m1}, \dots, p_{mn})$ be two positive nm -tuples, and let $r \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then, on putting $P_{mn} = \sum_{k_2=1}^n \sum_{k_1=1}^m p_{k_1 k_2}$, it easy follows that if $-\infty \leq r < s \leq +\infty$, then

$$M_{mn}^{[r]} \leq M_{mn}^{[s]} \tag{2.3}$$

(also see, e.g., [1, p.15]). Here, the r th power mean of x with weights p is the following:
 $M_{mn}^{[r]} = (\frac{1}{P_{mn}} \sum_{k_2=1}^n \sum_{k_1=1}^m p_{k_1 k_2} x_{k_1 k_2}^r)^{1/r}$ if $r \neq +\infty, 0, -\infty$; $M_{mn}^{[r]} = (\prod_{k_2=1}^n \prod_{k_1=1}^m x_{k_1 k_2}^{p_{k_1 k_2}})^{1/P_{mn}}$ if $r = 0$; $M_{mn}^{[r]} = \min(x_{11}, \dots, x_{1n}, \dots, x_{m1}, \dots, x_{mn})$ if $r = -\infty$ and $M_{mn}^{[r]} = \max(x_{11}, \dots, x_{1n}, \dots, x_{m1}, \dots, x_{mn})$ if $r = +\infty$.

Let $f(x, y) : [a, b] \times [c, d] \rightarrow \mathbb{R}$, and $p \geq 1$. Now, we define the p -norm of the function $f(x, y)$ on $[a, b] \times [c, d]$ as follows:

$$\|f(x, y)\|_p = \left(\int_a^b \int_c^d |f(x, y)|^p dx dy \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|f(x, y)\|_p = \sup|f(x, y)|, \quad p = \infty,$$

and $L^p([a, b] \times [c, d])$ is the set of all functions $f(x, y) : [a, b] \times [c, d] \rightarrow \mathbb{R}$ such that $\|f(x, y)\|_p < \infty$.

Lemma 2.1 (see [12]) (Barnes-Godunova-Levin inequality) *Let $f(x, y), g(x, y)$ be nonnegative concave functions on $[a, b] \times [c, d]$, then for $p, q > 1$ we have*

$$\|f(x, y)\|_p \|g(x, y)\|_q \leq B(p, q) \int_a^b \int_c^d f(x, y)g(x, y) dx dy, \tag{2.4}$$

where

$$B(p, q) = \frac{6[(b - a)(d - c)]^{1/p+1/q-1}}{(p + 1)^{1/p}(q + 1)^{1/q}}.$$

Lemma 2.2 (see [1]) (Hermite-Hadamard inequality) *Let $f(x, y) : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a convex function. Then the following dual Hermite-Hadamard inequality for convex functions holds:*

$$f\left(\frac{a + c}{2}, \frac{b + d}{2}\right) \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dx dy \leq \frac{f(a, b) + f(c, d)}{2}. \tag{2.5}$$

The inequality is reversed if the function $f(x, y)$ is concave.

Lemma 2.3 (see [13]) (A reversed Minkowski integral inequality) *Let $f(x, y)$ and $g(x, y)$ be positive functions satisfying*

$$0 < m \leq \frac{f(x, y)}{g(x, y)} \leq M, \quad (x, y) \in [a, b] \times [c, d]. \tag{2.6}$$

Then

$$\|f(x, y)\|_p + \|g(x, y)\|_p \leq c \|f(x, y) + g(x, y)\|_p, \tag{2.7}$$

where $c = [M(m + 1) + (M + 1)] / [(m + 1)(M + 1)]$.

3 Main results

Our main results are established in the following theorems.

Theorem 3.1 *Let $p, q > 1$ and let $f(x, y), g(x, y) : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be nonnegative functions such that $f(x, y)^p$ and $g(x, y)^q$ are concave on $[a, b] \times [c, d]$. Then*

$$\begin{aligned} & \frac{f(a, b) + f(c, d)}{2} \times \frac{g(a, b) + g(c, d)}{2} \\ & \leq \frac{1}{[(b - a)(d - c)]^{1/p+1/q}} B(p, q) \int_a^b \int_c^d f(x, y)g(x, y) \, dx \, dy, \end{aligned} \tag{3.1}$$

where $B(p, q)$ is the Barnes-Godunova-Levin constant given by (2.4).

Proof Observe that whenever $f^p(x, y)$ is concave on $[a, b] \times [c, d]$, the nonnegative function $f(x, y)$ is also concave on $[a, b] \times [c, d]$. Namely,

$$f[\lambda a + (1 - \lambda)c, \lambda b + (1 - \lambda)d]^p \geq \lambda f(a, b)^p + (1 - \lambda)f(c, d)^p,$$

that is,

$$f[\lambda a + (1 - \lambda)c, \lambda b + (1 - \lambda)d] \geq ((\lambda f(a, b)^p + (1 - \lambda)f(c, d)^p))^{1/p},$$

and $p > 1$, using the power-mean inequality (2.3), we obtain

$$f[\lambda a + (1 - \lambda)c, \lambda b + (1 - \lambda)d] \geq \lambda f(a, b) + (1 - \lambda)f(c, d).$$

For $q > 1$, similarly, if $g^q(x, y)$ is concave on $[a, b] \times [c, d]$, the nonnegative function $g(x, y)$ is concave on $[a, b] \times [c, d]$.

In view that $f^p(x, y)$ and $g^q(x, y)$ are concave functions on $[a, b] \times [c, d]$, from Lemma 2.2, we get

$$\begin{aligned} \left(\frac{f(a, b)^p + f(c, d)^p}{2} \right)^{1/p} & \leq \frac{1}{[(b - a)(d - c)]^{1/p}} \left(\int_a^b \int_c^d f(x, y)^p \, dx \, dy \right)^{1/p} \\ & \leq f\left(\frac{a + c}{2}, \frac{b + d}{2} \right), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \left(\frac{g(a,b)^p + g(c,d)^q}{2}\right)^{1/q} &\leq \frac{1}{[(b-a)(d-c)]^{1/q}} \left(\int_a^b \int_c^d g(x,y)^q dx dy\right)^{1/q} \\ &\leq g\left(\frac{a+c}{2}, \frac{b+d}{2}\right). \end{aligned} \tag{3.3}$$

By multiplying the above inequalities, we obtain

$$\begin{aligned} \left(\frac{f(a,b)^p + f(c,d)^p}{2}\right)^{1/p} \left(\frac{g(a,b)^p + g(c,d)^q}{2}\right)^{1/q} \\ \leq \frac{1}{[(b-a)(d-c)]^{1/p+1/q}} \left(\int_a^b \int_c^d f(x,y)^p dx dy\right)^{1/p} \left(\int_a^b \int_c^d g(x,y)^q dx dy\right)^{1/q}. \end{aligned} \tag{3.4}$$

If $p, q > 1$, then it is easy to show that

$$\left(\frac{f(a,b)^p + f(c,d)^p}{2}\right)^{1/p} \geq \frac{f(a,b) + f(c,d)}{2}, \tag{3.5}$$

and

$$\left(\frac{g(a,b)^q + g(c,d)^q}{2}\right)^{1/q} \geq \frac{g(a,b) + g(c,d)}{2}. \tag{3.6}$$

Thus, by applying Barnes-Godunova-Levin inequality to the right-hand side of (3.4) with (3.5), (3.6), we get (3.1).

The proof is complete. □

Remark 3.1 By multiplying inequalities (3.2), (3.3), we obtain

$$\begin{aligned} \frac{1}{[(b-a)(d-c)]^{1/p+1/q}} \left(\int_a^b \int_c^d f(x,y)^p dx dy\right)^{1/p} \left(\int_a^b \int_c^d g(x,y)^q dx dy\right)^{1/q} \\ \leq f\left(\frac{a+c}{2}, \frac{b+d}{2}\right) g\left(\frac{a+c}{2}, \frac{b+d}{2}\right). \end{aligned} \tag{3.7}$$

By applying the Hölder inequality to the left-hand side of (3.7) with $(1/p) + (1/q) = 1$, we get

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y) dx dy \leq f\left(\frac{a+c}{2}, \frac{b+d}{2}\right) g\left(\frac{a+c}{2}, \frac{b+d}{2}\right). \tag{3.8}$$

Remark 3.2 Let $f(x,y)$ and $g(x,y)$ change to $f(x)$ and $g(x)$, respectively, and with suitable changes in Theorem 3.1 and Remark 3.1, we have the following.

Corollary 3.1 Let $p, q > 1$ and let $f(x), g(x) : [a, b] \rightarrow \mathbb{R}$, $a < b$, be nonnegative functions such that $f(x)^p$ and $g(x)^q$ are concave on $[a, b]$. Then

$$\frac{f(a) + f(b)}{2} \cdot \frac{g(a) + g(b)}{2} \leq \frac{1}{(b-a)^{1/p+1/q}} B(p, q) \int_a^b f(x)g(x) dx,$$

and if $(1/p) + (1/q) = 1$, then one has

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \leq f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right).$$

This is just Theorem 2.1 established by Özdemir and Dragomir [11].

Theorem 3.2 Let $p \geq 1$ and let $\int_a^b \int_c^d f(x,y)^p dx dy < \infty$ and $\int_a^b \int_c^d g(x,y)^p dx dy < \infty$, and let $f(x,y), g(x,y) : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be positive functions with

$$0 < m \leq \frac{f(x,y)}{g(x,y)} \leq M, \quad \forall (x,y) \in [a, b] \times [c, d].$$

Then

$$\|f(x,y)\|_p^2 + \|g(x,y)\|_p^2 \geq \left(\frac{(M+1)(m+1)}{M} - 2\right) \|f(x,y)\|_p \|g(x,y)\|_p. \tag{3.9}$$

Proof Since $f(x,y), g(x,y)$ are positive, as in the proof of Lemma 2.3 (see [13, p.2]), we have

$$\left(\int_a^b \int_c^d f(x,y)^p dx dy\right)^{1/p} \leq \frac{M}{M+1} \left(\int_a^b \int_c^d (f(x,y) + g(x,y))^p dx dy\right)^{1/p}$$

and

$$\left(\int_a^b \int_c^d g(x,y)^p dx dy\right)^{1/p} \leq \frac{1}{m+1} \left(\int_a^b \int_c^d (f(x,y) + g(x,y))^p dx dy\right)^{1/p}.$$

By multiplying the above inequalities and in view of the Minkowski inequality, we get

$$\begin{aligned} & \left(\int_a^b \int_c^d f(x,y)^p dx dy\right)^{1/p} \left(\int_a^b \int_c^d g(x,y)^p dx dy\right)^{1/p} \\ & \leq \frac{M}{(M+1)(m+1)} \left(\int_a^b \int_c^d (f(x,y) + g(x,y))^p dx dy\right)^{2/p} \\ & \leq \frac{M}{(M+1)(m+1)} \left(\left(\int_a^b \int_c^d f(x,y)^p dx dy\right)^{1/p} \right. \\ & \quad \left. + \left(\int_a^b \int_c^d g(x,y)^p dx dy\right)^{1/p}\right)^2. \end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned} & \left(\int_a^b \int_c^d f(x,y)^p dx dy\right)^{2/p} + \left(\int_a^b \int_c^d g(x,y)^p dx dy\right)^{2/p} \\ & \geq \left(\frac{(M+1)(m+1)}{M} - 2\right) \left(\int_a^b \int_c^d f(x,y)^p dx dy\right)^{1/p} \left(\int_a^b \int_c^d g(x,y)^p dx dy\right)^{1/p}. \end{aligned}$$

This proof is complete. □

Remark 3.3 Let $f(x,y)$ and $g(x,y)$ change to $f(x)$ and $g(x)$, respectively, and with suitable changes in (3.9), (3.9) reduces to an inequality established by Özdemir and Dragomir [11].

Theorem 3.3 *If $f^p(x, y)$ and $g^q(x, y)$ are as in Theorem 3.1, then the following inequality holds:*

$$\frac{1}{(b-a)(d-c)} \|f(x, y)\|_p^p \cdot \|g(x, y)\|_q^q \geq \frac{(f(a, b) + f(c, d))^p (g(a, b) + g(c, d))^q}{2^{p+q}}. \quad (3.11)$$

Proof If $f^p(x, y)$ and $g^q(x, y)$ are concave on $[a, b] \times [c, d]$, then from Lemma 2.2, we get

$$\frac{f(a, b)^p + f(c, d)^p}{2} \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)^p dx dy$$

and

$$\frac{g(a, b)^q + g(c, d)^q}{2} \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x, y)^q dx dy,$$

which imply that

$$\begin{aligned} & \frac{[f(a, b)^p + f(c, d)^p][g(a, b)^q + g(c, d)^q]}{4} \\ & \leq \frac{1}{[(b-a)(d-c)]^2} \int_a^b \int_c^d f(x, y)^p dx dy \int_a^b \int_c^d g(x, y)^q dx dy. \end{aligned} \quad (3.12)$$

On the other hand, if $p, q \geq 1$, from (2.3) we get

$$\frac{f(a, b)^p + f(c, d)^p}{2} \leq 2^{-p} [f(a, b) + f(c, d)]^p$$

and

$$\frac{g(a, b)^q + g(c, d)^q}{2} \leq 2^{-q} [g(a, b) + g(c, d)]^q,$$

which imply that

$$\begin{aligned} & \frac{[f(a, b)^p + f(c, d)^p][g(a, b)^q + g(c, d)^q]}{4} \\ & \geq 2^{-p-q} [f(a, b) + f(c, d)]^p [g(a, b) + g(c, d)]^q. \end{aligned} \quad (3.13)$$

Combining (3.12) and (3.13), we obtain the desired inequality as

$$\begin{aligned} & 2^{-p-q} [f(a, b) + f(c, d)]^p [g(a, b) + g(c, d)]^q \\ & \leq \frac{1}{[(b-a)(d-c)]^2} \|f(x, y)\|_p^p \cdot \|g(x, y)\|_q^q. \end{aligned}$$

This proof is complete. □

Remark 3.4 Let $f(x, y)$ and $g(x, y)$ change to $f(x)$ and $g(x)$, respectively, and with suitable changes in (3.11), (3.11) reduces to an inequality established by Özdemir and Dragomir [11].

Theorem 3.4 Let $f(x, y), g(x, y) : [a, b] \times [c, d] \rightarrow \mathbb{R}^+$ be functions such that $f(x, y)^p, g(x, y)^q$ and $f(x, y)g(x, y)$ are in $L_1([a, b] \times [c, d])$, and

$$0 < m \leq \frac{f(x, y)}{g(x, y)} \leq M, \quad \forall (x, y) \in [a, b] \times [c, d], a, b, c, d \in [0, \infty).$$

Then

$$\int_a^b \int_c^d f(x, y)g(x, y) dx dy \leq c_1 \left(\frac{\|f(x, y)\|_p^p + \|g(x, y)\|_p^p}{2} \right) + c_2 \left(\frac{\|f(x, y)\|_q^q + \|g(x, y)\|_q^q}{2} \right), \quad (3.14)$$

where

$$c_1 = \frac{2^p}{p} \left(\frac{M}{M+1} \right)^p, \quad c_2 = \frac{2^q}{q} \left(\frac{1}{m+1} \right)^q,$$

and $(1/p) + (1/q) = 1$ with $p > 1$.

Proof Since $0 < m \leq \frac{f(x, y)}{g(x, y)} \leq M, \forall (x, y) \in [a, b] \times [c, d]$, we have

$$f(x, y) \leq \frac{M}{M+1} (f(x, y) + g(x, y))$$

and

$$g(x, y) \leq \frac{1}{m+1} (f(x, y) + g(x, y)).$$

In view of the Young-type inequality and using the elementary inequality

$$(a + b)^p \leq 2^{p-1}(a^p + b^p), \quad p > 1, a, b \in \mathbb{R}^+,$$

we have

$$\begin{aligned} & \int_a^b \int_c^d f(x, y)g(x, y) dx dy \\ & \leq \frac{1}{p} \left(\frac{M}{M+1} \right)^p \int_a^b \int_c^d (f(x, y) + g(x, y))^p dx dy \\ & \quad + \frac{1}{q} \left(\frac{1}{m+1} \right)^q \int_a^b \int_c^d (f(x, y) + g(x, y))^q dx dy \\ & \leq \frac{1}{p} \left(\frac{M}{M+1} \right)^p 2^{p-1} \int_a^b \int_c^d [f(x, y)^p + g(x, y)^p] dx dy \\ & \quad + \frac{1}{q} \left(\frac{1}{m+1} \right)^q 2^{q-1} \int_a^b \int_c^d [f(x, y)^q + g(x, y)^q] dx dy. \end{aligned}$$

This completes the proof. □

Remark 3.5 Let $f(x, y)$ and $g(x, y)$ change to $f(x)$ and $g(x)$, respectively, and with suitable changes in (3.14), (3.14) reduces to an inequality established by Özdemir and Dragomir [11].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

C-JZ, W-SC and X-YL jointly contributed to the main results Theorems 3.1-3.4. All authors read and approved the final manuscript.

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