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A hybrid projection method for solving a common solution of a system of equilibrium problems and fixed point problems for asymptotically strict pseudocontractions in the intermediate sense in Hilbert spaces

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Abstract

In this paper, we introduce a new iterative algorithm which is constructed by using the hybrid projection method for finding a common solution of a system of equilibrium problems of bifunctions satisfying certain conditions and a common solution of fixed point problems of a family of uniformly Lipschitz continuous and asymptotically λ_i-strict pseudocontractive mappings in the intermediate sense. We prove the strong convergence theorem for a new iterative algorithm under some mild conditions in Hilbert spaces. Finally, we also give a numerical example which supports our results.

MSC: 47H05; 47H09; 47H10

Keywords: asymptotically strict pseudocontraction in the intermediate sense; hybrid projection method; system of equilibrium problems; fixed point problems

1 Introduction

Let C be a closed and convex subset of a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let $\{F_m\}_{m\in\Gamma}$ be a family of bifunctions from $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and Γ is an arbitrary index set. The *system of equilibrium problems* is to find $x\in C$ such that

$$F_m(x,y)\geq 0,\quad m\in\Gamma,\forall y\in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{SEP}(F_m)$, where $m\in\Gamma$, that is,

$$\mathrm{SEP}(F_m)=\{x\in C:F_m(x,y)\geq 0,\forall y\in C\}. \tag{1.2}$$

If Γ is a singleton, then the problem (1.1) is reduced to the *equilibrium problem* of finding $x\in C$ such that

$$F(x,y)\geq 0,\quad \forall y\in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{EP}(F)$.

Recall the following definitions.

A mapping $A : C \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{1.4}$$

A mapping A is called *α-inverse-strongly monotone* [1, 2], if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.5}$$

Clearly, if A is α-inverse-strongly monotone, then A is monotone.

A mapping A is called *β-strongly monotone* if there exists a positive real number β such that

$$\langle Ax - Ay, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in C. \tag{1.6}$$

A mapping A is called *L-Lipschitz continuous* if there exists a positive real number L such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{1.7}$$

It is easy to see that if A is an α-inverse-strongly monotone mapping from C into H, then A is $\frac{1}{\alpha}$-Lipschitz continuous.

In 2009, Qin *et al.* [3] introduced the following algorithm for a finite family of asymptotically λ_i-strictly pseudocontractions.

Let $x_0 \in C$ and $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $(0,1)$. The sequence $\{x_n\}$ is as follows:

$$\begin{cases} x_1 = \alpha_0 x_0 + (1-\alpha_0)S_1 x_0, \\ x_2 = \alpha_1 x_1 + (1-\alpha_1)S_2 x_1, \\ x_3 = \alpha_2 x_2 + (1-\alpha_2)S_3 x_2, \\ \vdots \\ x_N = \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})S_N x_{N-1}, \\ x_{N+1} = \alpha_N x_N + (1-\alpha_N)S_1^2 x_N, \\ x_{N+2} = \alpha_{N+1} x_{N+1} + (1-\alpha_{N+1})S_2^2 x_{N+1}, \\ \vdots \\ x_{2N} = \alpha_{2N-1} x_{2N-1} + (1-\alpha_{2N-1})S_N^2 x_{2N-1}, \\ x_{2N+1} = \alpha_{2N} x_{2N} + (1-\alpha_{2N})S_1^3 x_{2N}, \\ \vdots \end{cases} \tag{1.8}$$

It is called *the explicit iterative sequence* of a finite family of asymptotically λ_i-strictly pseudocontractions $\{S_1, S_2, \ldots, S_N\}$. Since for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, 3, \dots, N\}$, $h = h(n) \geq 1$ is a positive integer and $h(n) \to \infty$, as $n \to \infty$, we can rewrite the above table in the following compact form:

$$x_n = \alpha_{n-1} x_{n-1} + (1 - \alpha_{n-1}) S_{i(n)}^{h(n)} x_{x-1}, \quad \forall n \geq 1.$$

Next, Sahu *et al.* [4] introduced new iterative schemes for asymptotically strictly pseudocontractive mappings in the intermediate sense. To be more precise, they proved the following theorem.

Theorem (SXY) *Let C be a nonempty closed and convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically κ-strictly pseudocontractive mapping in the intermediate sense with a sequence $\{\gamma_n\}$ such that $F(T)$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa$ for all $n \in \mathbb{N}$. Let $\{x_n\} \subset C$ be a sequence generated by the following (CQ) algorithm:*

$$\begin{cases} u = x_1 \in C \text{ chosen arbitrarily}, \\ y_n = (1 - \alpha_n) x_n + \alpha_n T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, u - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(u), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.9}$$

where $\theta_n = c_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\{\|x_n - z\| : z \in F(T)\} < \infty$. Then $\{x_n\}$ converges strongly to $P_{F(T)}(u)$, where $P_{F(T)}$ is a metric projection from H into $F(T)$.

In 2010, Hu and Cai [5] considered the asymptotically strictly pseudocontractive mappings in the intermediate sense concerning the equilibrium problem. They obtained the following result in a real Hilbert space. Next, Ceng *et al.* [6] introduced the viscosity approximation method for a modified Mann iteration process for asymptotically strict pseudocontractive mappings in the intermediate sense and they proved the strong convergence of a general CQ-algorithm and extended the concept of asymptotically strictly pseudocontractive mappings in the intermediate sense to the Banach space setting called nearly asymptotically strictly pseudocontractive mappings in the intermediate sense. Finally, they established a weak convergence theorem for a fixed point of nearly asymptotically strictly pseudocontractive mappings in the intermediate sense which are not necessarily Lipschitz continuous mappings.

Theorem (HC) *Let C be a nonempty closed and convex subset of a real Hilbert space H and $N \geq 1$ be an integer, $\phi : C \to C$ be a bifunction satisfying* (A1)-(A4), *and $A : C \to H$ be an α-inverse-strongly monotone mapping. Let for each $1 \leq i \leq N$, $T_i : C \to C$ be a uniformly continuous k_i-strictly asymptotically pseudocontractive mapping in the intermediate sense for some $0 \leq k_i < 1$ with sequences $\{\gamma_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} \gamma_{n,i} = 0$ and $\{c_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} c_{n,i} = 0$. Let $k = \max\{k_i : 1 \leq i \leq N\}$, $\gamma_n = \max\{\gamma_{n,i} : 1 \leq i \leq N\}$, and $c_n = \max\{c_{n,i} : 1 \leq i \leq N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap \mathrm{EP}(\phi)$ is nonempty and bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ such that $0 < a \leq \alpha_n \leq 1$, $0 < \delta \leq \beta_n \leq 1 - k$ for all $n \in \mathbb{N}$, and $0 < b \leq r_n \leq c < 2\alpha$.*

Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the following algorithm:

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ u_n \in C \text{ such that } \phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ \quad \forall y \in C, \\ z_n = (1-\beta_n)u_n + \beta_n T_{i(n)}^{h(n)} u_n, \\ y_n = (1-\alpha_n)u_n + \alpha_n z_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.10}$$

where $\theta_n = c_{h(n)} + \gamma_{h(n)}\rho_n^2 \to 0$, as $n \to \infty$, and $\rho_n = \sup\{\|x_n - v\| : v \in \mathcal{F}\} < \infty$. Then $\{x_n\}$ converges strongly to $P_{\mathcal{F}}(x_0)$.

In 2011, Duan and Zhao [7] introduced new iterative schemes for finding a common solution set of a system of equilibrium problems and a solution of a fixed point set of asymptotically strict pseudocontractions in the intermediate sense and they proved these schemes converge strongly.

In 2012, Shui Ge [8] introduced a new hybrid algorithm with variable coefficients for a fixed point problem of a uniformly Lipschitz continuous mapping and asymptotically pseudocontractive mapping in the intermediate sense on unbounded domains and he proved strong convergence in a real Hilbert space.

Theorem (Ge) *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, $T : C \to C$ be a uniformly L-Lipschitz continuous mapping and asymptotically pseudocontractive mapping in the intermediate sense with sequences $\{k_n\} \subset [1,\infty)$ and $\{v_n\} \subset [0,\infty)$. Let $q_n = 2k_n - 1$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be the sequence generated by the following hybrid algorithm with variable coefficients*:

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ z_n = (1-\hat{\beta}_n)x_n + \hat{\beta}_n T^n x_n, \\ y_n = (1-\hat{\alpha}_n)x_n + \hat{\alpha}_n T^n z_n, \\ C_n = \{u \in C_n : \|y_n - u\|^2 \leq \|x_n - u\|^2 \\ \qquad - \hat{\alpha}_n\hat{\beta}_n(1 - \hat{\beta}_n - \hat{\beta}_n^2 L^2 - q_n\hat{\beta}_n)\|x_n - T^n x_n\|^2 + \alpha_n\theta_n\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.11}$$

where

$$\theta_n = 2\left(1 + r_0^2\right)(q_n - 1)(1 + q_n\hat{\beta}_n) + 2(1 + q_n\hat{\beta}_n)v_n,$$

$$\hat{\alpha}_n = \frac{\alpha_n}{1 + \|x_n - x_1\|^2} \quad \text{and} \quad \hat{\beta}_n = \frac{\beta_n}{1 + \|x_n - x_1\|^2}.$$

Assume that the positive real number r_0 *is chosen so that* $B_{r_0}(x_1) \cap \mathrm{Fix}(T) \neq \emptyset$ *and that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *such that* $a \leq \alpha_n \leq \beta_n \leq b$ *for some* $a > 0$ *and for some* $b \in (0, \frac{1}{2+L})$.

Then $\{x_n\}$ *converges strongly to a fixed point of* T.

In this paper, motivated and inspired by the previously mentioned above results, we introduce a new iterative algorithm by the hybrid projection method for finding a common solution of a system of equilibrium problems of bifunctions satisfying certain conditions and a common solution of fixed point problems of a family of uniformly Lipschitz continuous and asymptotically λ_i-strict pseudocontractive mappings in the intermediate sense in a real Hilbert space. Then, we prove a strong convergence theorem of the iterative algorithm generated by this conditions. Finally, we also give a numerical example which supports our results. The results obtained in this paper extend and improve several recent results in this area.

2 Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a closed and convex subset of H. For any point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C defined by the following:

$$P_C(x) = \arg\min_{y \in C} \|x - y\|.$$

We know that P_C is a nonexpansive mapping H onto C. It is also known that P_C satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H,$$

and

$$\langle x - P_C x, z - P_C x \rangle \geq 0, \quad \forall z \in C.$$

We will adopt the following notations:

(1) $\rightarrow$ for strong convergence and $\rightharpoonup$ for weak convergence.
(2) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak w-limit set of $\{x_n\}$.
(3) A nonlinear mapping $S : C \to C$ is a self-mapping in C. We denote the set of fixed points of S by $F(S)$ (*i.e.*, $F(S) = \{x \in C : Sx = x\}$). Recall the following definitions.

Definition 2.1 Let S be a mapping from C to C. Then

(1) S is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.1}$$

(2) S is said to be *uniformly Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\left\|S^n x - S^n y\right\| \leq L\|x - y\|, \quad \text{for all integers } n \geq 1, \forall x, y \in C. \tag{2.2}$$

(3) S is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\left\|S^n x - S^n y\right\| \le k_n \|x - y\|, \quad \text{for all integers } n \ge 1, \forall x, y \in C. \tag{2.3}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk (see [9]) in 1972. It is known that if C is a nonempty, bounded, closed, and convex subset of a real Hilbert space H, then every asymptotically nonexpansive self-mapping has a fixed point. Further, the set $F(S)$ of fixed points of S is closed and convex.

(4) S is said to be *asymptotically nonexpansive* in the intermediate sense [10, 11] if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\|S^n x - S^n y\right\| - \|x - y\|\right) \le 0. \tag{2.4}$$

Putting $\xi_n = \max\{0, \sup_{x,y\in C}(\|S^n x - S^n y\| - \|x - y\|)\}$, we see that $\xi_n \to 0$ as $n \to \infty$. Then (2.4) is reduced to

$$\left\|S^n x - S^n y\right\| \le \|x - y\| + \xi_n, \quad \forall x, y \in C.$$

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Kirk and Bruck *et al.* (see [10, 11]) as a generalization of the class of asymptotically nonexpansive mappings. It is known that if C is a nonempty, bounded, closed, and convex subset of a real Hilbert space H, then every asymptotically nonexpansive self-mapping in the intermediate sense has a fixed point (see [12]).

(5) S is said to be *contractive* if there exists a coefficient $k \in (0,1)$ such that

$$\|Sx - Sy\| \le k\|x - y\|, \quad \forall x, y \in C. \tag{2.5}$$

(6) S is said to be a *λ-strict pseudocontraction* if there exists a coefficient $\lambda \in [0,1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \lambda\left\|(I - S)x - (I - S)y\right\|^2, \quad \forall x, y \in C. \tag{2.6}$$

The class of strict pseudocontractions was introduced by Brower and Petryshyn (see [1]) in 1967. Clearly, if S is a nonexpansive mapping, then S is a strict pseudocontraction with $\lambda = 0$. We also remark that if $\lambda = 1$, then S is called a pseudocontractive mapping.

(7) S is said to be an *asymptotically λ-strict pseudocontraction* with the sequence $\{d_n\}$ (see also [13]) if there exists a sequence $\{d_n\} \subset [0,\infty)$ with $d_n \to 0$ as $n \to \infty$ and a constant $\lambda \in [0,1)$ such that

$$\begin{aligned} &\left\|S^n x - S^n y\right\|^2 \le (1 + d_n)\|x - y\|^2 + \lambda\left\|\left(x - S^n x\right) - \left(y - S^n y\right)\right\|^2, \\ &\quad \forall x, y \in C, \forall n \in \mathbb{N}. \end{aligned} \tag{2.7}$$

The class of asymptotically strict pseudocontractions was introduced by Qihou [14] in 1996. Clearly, if S is an asymptotically nonexpansive mapping, then S is an asymptotically strict pseudocontraction with $\lambda = 0$. We also remark that if $\lambda = 1$, then S is said to be an asymptotically pseudocontractive mapping which was introduced by Schu [15] in 1991.

(8) S is said to be an *asymptotically λ-strict pseudocontraction* in the intermediate sense with the sequence $\{d_n\}$ [4, 5] if there exists a sequence $\{d_n\} \subset [0,\infty)$ with $d_n \to 0$ as $n \to \infty$ and a constant $\lambda \in [0,1)$ such that

$$\begin{aligned}&\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\| S^n x - S^n y\right\|^2 - (1+d_n)\|x-y\|^2 - \lambda \left\| \left(x - s^n x\right) - \left(y - s^n y\right)\right\|^2 \right) \le 0,\\ &\forall x,y \in C, \forall n \in \mathbb{N}.\end{aligned} \tag{2.8}$$

Putting $c_n = \max\{0, \sup_{x,y\in C}(\|S^n x - S^n y\|^2 - (1+d_n)\|x-y\|^2 - \lambda\|(x - s^n x) - (y - s^n y)\|^2)\}$, we see that $c_n \to 0$ as $n \to \infty$. Then (2.8) is reduced to

$$\left\| S^n x - S^n y\right\|^2 \le (1+d_n)\|x-y\|^2 + \lambda \left\| \left(x - s^n x\right) - \left(y - s^n y\right)\right\|^2 + c_n, \quad \forall x,y \in C, \forall n \in \mathbb{N}.$$

The class of asymptotically strict pseudocontractions in the intermediate sense was introduced by Sahu, Xu, and Yao [4] as a generalization of a class of asymptotically strict pseudocontractions.

For solving the equilibrium problem, let us give the following assumptions for the bifunction F and the set C:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \le 0$, for all $x,y \in C$;
(A3) for each $x,y,z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \le F(x,y)$;
(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.2 ([16]) *Let C be a nonempty closed and convex subset of a real Hilbert space H. For any $x,y,z \in H$ and given also a real number $a \in \mathbb{R}$, the set*

$$\left\{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\right\}$$

is closed and convex.

Lemma 2.3 ([17]) *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ satisfy* (A1)-(A4), *and let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.4 ([18]) *Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4). *For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\}. \tag{2.9}$$

Then the following hold:

(1) *T_r is single-valued*;

(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.10}$$

(3) $F(T_r) = \mathrm{EP}(F)$; *and*
(4) $\mathrm{EP}(F)$ *is closed and convex*.

Lemma 2.5 ([7, 19]) *Let H be a real Hilbert space. Then the following identities hold*:
(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, $\forall x, y \in H$.
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, $\forall t \in [0, 1]$, $\forall x, y \in H$.
(iii) $\|x + y\|^2 = \|x\|^2 + 2\langle y, x + y\rangle$.

Lemma 2.6 ([4]) *Let C be a nonempty closed and convex subset of a real Hilbert space H, and* $S : C \to C$ *be a uniformly L-Lipschitz continuous and asymptotically λ-strict pseudocontraction in the intermediate sense. Then* $F(S)$ *is closed and convex.*

Lemma 2.7 ([4]) *Let C be a nonempty closed and convex subset of a real Hilbert space H and* $S : C \to C$ *be a uniformly L-Lipschitz continuous and asymptotically λ-strict pseudocontraction in the intermediate sense. Then the mapping* $I - S$ *is demiclosed at zero, that is, if the sequence* $\{x_n\}$ *in C is such that* $x_n \rightharpoonup \overline{x}$ *and* $x_n - Sx_n \to 0$, *then* $\overline{x} \in F(S)$.

Lemma 2.8 ([20]) *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let* $\{x_n\}$ *be a sequence in H and* $u \in H$, *and let* $q = P_C u$. *Suppose that* $\{x_n\}$ *is such that* $\omega_n(x_n) \subset C$ *and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n \in \mathbb{N}.$$

Then $x_n \to q$.

Lemma 2.9 ([4]) *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let* $S : C \to C$ *be an asymptotically λ-strict pseudocontractive mapping in the intermediate sense with the sequence* γ_n. *Then*

$$\|S^n x - S^n y\| \leq \frac{1}{1-\lambda}\Big(\lambda\|x - y\| + \sqrt{\big(1 + (1-\lambda)\gamma_n\big)\|x - y\|^2 + (1-\lambda)c_n}\Big),$$

for all $x, y \in C$ *and* $n \in \mathbb{N}$.

3 Main results

In this section, we prove a strong convergence theorem which solves the problem of finding a common solution of a system of equilibrium problems and a common solution of fixed point problems in Hilbert spaces.

Theorem 3.1 *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let* $M \geq 1$ *be a positive integer. Let* $\{F_m\}_{m=1}^{M} : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)-(A4). *Let* $\{S_i\}_{i=1}^{N} : C \to C$ *be a uniformly Lipschitz continuous and asymptotically* λ_i*-strict pseudocontractive mapping in the intermediate sense for some* $0 \leq \lambda_i < 1$ *with the sequences* $\{c_{n,i}\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} c_{n,i} = 0$ *and* $\{d_{n,i}\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} d_{n,i} = 0$. *Let*

$\lambda = \max\{\lambda_i : 1 \le i \le N\}$, $c_n = \max\{c_{n,i} : 1 \le i \le N\}$ *and* $d_n = \max\{d_{n,i} : 1 \le i \le N\}$. *Assume that* $\Omega := (\bigcap_{m=1}^{M} \mathrm{SEP}(F_m)) \cap (\bigcap_{i=1}^{N} F(S_i))$ *is nonempty and bounded. Let* $\{\alpha_n\}$, $\{\beta_n\}$ *be sequences in* $[0,1]$ *such that* $0 < a \le \alpha_n \le 1$, $0 < b \le \beta_n \le 1 - \lambda$, $a, b \in \mathbb{R}$, $\forall n \in \mathbb{N}$ *and* $\{r_{m,n}\}$ *be a sequence in* $(0,\infty)$ *such that* $\lim_{n\to\infty} r_{m,n} > 0$.

Let $\{x_n\}$ *be a sequence generated by the following algorithm*:

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = H, \\ u_n = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n, \\ z_n = (1-\beta_n)u_n + \beta_n S_{i(n)}^{h(n)} u_n, \\ y_n = (1-\alpha_n)u_n + \alpha_n z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \le \|x_n - w\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\theta_n = c_{h(n)} + d_{h(n)}\rho_n^2 \to 0$, *as* $n \to \infty$ *and* $\rho_n = \sup\{\|x_n - w\| : w \in \Omega\} < \infty$ *and* $n = (h(n)-1)N + i(n)$, *where* $i(n) \in \{1,2,3,\ldots,N\}$. *Then* $\{x_n\}$ *converges strongly to some point* p^*, *where* $p^* = P_\Omega(x_1)$.

Proof The proof is split into seven steps.

Step 1. We will show that P_Ω is well defined.

From Lemma 2.4, we get $\bigcap_{m=1}^{M} \mathrm{SEP}(F_m)$ is closed and convex. From the assumption of $\{S_i\}_{i=1}^{N}$ and Lemma 2.6, it follows that $\bigcap_{i=1}^{N} F(S_i)$ is closed and convex.

Therefore, $\Omega := (\bigcap_{m=1}^{M} \mathrm{SEP}(F_m)) \cap (\bigcap_{i=1}^{N} F(S_i))$ is closed and convex. Hence, P_Ω is well defined.

Step 2. We will show that C_n is closed and convex for each $n \ge 1$.

By the assumption of C_{n+1}, it is easy to see that C_n is closed for each $n \ge 1$. We only show that C_n is convex for each $n \ge 1$.

Note that $C_1 = H$ is convex. Suppose that C_k is convex for some $k \ge 1$. Next, we show that C_{k+1} is convex for the same k. For each $w \in C_k$, we see that

$$\|y_k - w\|^2 \le \|x_k - w\|^2 + \theta_k$$

is equivalent to

$$2\langle x_k - y_k, w\rangle \le \|x_k\|^2 - \|y_k\|^2 + \theta_k. \tag{3.2}$$

Taking w_1 and w_2 in C_{k+1} and putting $\overline{w} = tw_1 + (1-t)w_2$, it follows that $w_1, w_2 \in C_k$, and so

$$2\langle x_k - y_k, w_1\rangle \le \|x_k\|^2 - \|y_k\|^2 + \theta_k \tag{3.3}$$

and

$$2\langle x_k - y_k, w_2\rangle \le \|x_k\|^2 - \|y_k\|^2 + \theta_k. \tag{3.4}$$

Combining (3.3) with (3.4), we obtain that

$$2\langle x_k - y_k, \overline{w}\rangle \le \|x_k\|^2 - \|y_k\|^2 + \theta_k.$$

That is,

$$\|y_k - \overline{w}\|^2 \leq \|x_k - \overline{w}\|^2 + \theta_k.$$

In view of the convexity of C_k, we see that $\overline{w} \in C_k$. This implies that $\overline{w} \in C_{k+1}$. Therefore, C_{k+1} is convex. Hence, C_n is closed and convex for each $n \geq 1$.

Step 3. We will show that $\Omega \subset C_n$ for each $n \geq 1$.

Put $\Theta_n^m := T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n$ for every $m \in \{1,2,3,\dots,M\}$ and $\Theta_n^0 = I$ for all $n \in \mathbb{N}$. Therefore, $u_n = \Theta_n^M x_n$. It is obvious that $\Omega \subset C_1 = H$. Suppose that $\Omega \subset C_k$ for some $k \geq 1$.

Next, we show that $\Omega \subset C_{k+1}$ for the same k. Taking $p \in \Omega$ and for each $m \in \{1,2,3,\dots,M\}$, we see that $T_{r_{m,n}}^{F_m}$ is nonexpansive and $T_{r_{m,n}}^{F_m} p = p$. We note that

$$\|u_n - p\| = \left\|\Theta_n^M x_n - \Theta_n^M p\right\| \leq \|x_n - p\|, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

We observe that

$$\begin{aligned} \|z_n - p\|^2 &= \left\|(1-\beta_n)(u_n - p) + \beta_n\big(S_{i(n)}^{h(n)}(u_n - p)\big)\right\|^2 \\ &= (1-\beta_n)\|u_n - p\|^2 + \beta_n\left\|S_{i(n)}^{h(n)}u_n - p\right\|^2 - \beta_n(1-\beta_n)\left\|S_{i(n)}^{h(n)}u_n - u_n\right\|^2 \\ &\leq (1-\beta_n)\|u_n - p\|^2 + \beta_n\Big[(1+d_{h(n)})\|u_n - p\|^2 + \lambda\left\|S_{i(n)}^{h(n)}u_n - u_n\right\|^2 + c_{h(n)}\Big] \\ &\quad - \beta_n(1-\beta_n)\left\|S_{i(n)}^{h(n)}u_n - u_n\right\|^2 \\ &\leq (1+d_{h(n)})\|u_n - p\|^2 - \beta_n(1-\beta_n-\lambda)\left\|S_{i(n)}^{h(n)}u_n - u_n\right\|^2 + \beta_n c_{h(n)} \\ &\leq (1+d_{h(n)})\|u_n - p\|^2 + \beta_n c_{h(n)}. \end{aligned} \tag{3.6}$$

By virtue of convexity of $\|\cdot\|^2$, one has

$$\begin{aligned} \|y_n - p\|^2 &= \left\|(1-\alpha_n)(u_n - p) + \alpha_n(z_n - p)\right\|^2 \\ &\leq (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\|z_n - p\|^2. \end{aligned} \tag{3.7}$$

Substituting (3.5) and (3.6) into (3.7), we obtain

$$\begin{aligned} \|y_n - p\|^2 &\leq (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\|z_n - p\|^2 \\ &\leq (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\big[(1+d_{h(n)})\|u_n - p\|^2 + \beta_n c_{h(n)}\big] \\ &\leq \|u_n - p\|^2 + d_{h(n)}\|u_n - p\|^2 + \beta_n c_{h(n)} \\ &\leq \|u_n - p\|^2 + d_{h(n)}\|x_n - p\|^2 + c_{h(n)} \\ &= \|u_n - p\|^2 + \theta_n & (3.8) \\ &\leq \|x_n - p\|^2 + \theta_n. & (3.9) \end{aligned}$$

Therefore, $p \in C_{k+1}$, and so $\Omega \subset C_n$ for each $n \geq 1$. Hence, $\{x_n\}$ is well defined.

Step 4. We will show that $\{x_n\}$ is bounded.

Since Ω is a nonempty closed and convex subset of H, there exists a unique $q \in \Omega$ such that $q = P_\Omega x_1$. By the assumption, we have $x_n = P_{C_n}x_1$ for any $q \in \Omega \subset C_n$. Then

$$\|x_n - x_1\| \leq \|q - x_1\| = \|P_\Omega x_1 - x_1\|.$$

This implies that $\{x_n\}$ is bounded. Therefore, $\{u_n\}$, $\{z_n\}$, and $\{y_n\}$ are also bounded.

Step 5. We will show that $\|u_n - S_i u_n\| \to 0$ and $\|x_n - S_i x_n\| \to 0$ as $n \to \infty$, $\forall i \in \{1, 2, 3, \ldots, N\}$.

Since $x_n = P_{C_n} x_1$ and $x_n = P_{C_n} x_1 \in C_{n+1} \subset C_n$, we have

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1} \rangle \\ &\leq -\|x_1 - x_n\|^2 + \|x_1 - x_n\| \|x_1 - x_{n+1}\|. \end{aligned} \tag{3.10}$$

Therefore, $\|x_1 - x_n\|^2 \leq \|x_1 - x_n\| \|x_1 - x_{n+1}\|$, and so

$$\|x_n - x_1\| = \|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|. \tag{3.11}$$

Thus, the sequence $\{\|x_n - x_1\|\}$ is nondecreasing. Since $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_1\|$ exists. On the other hand, from (3.10), we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2. \end{aligned} \tag{3.12}$$

The fact that $\lim_{n\to\infty} \|x_n - x_1\|$ exists implies that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.13}$$

It is easy to see that

$$\lim_{n\to\infty} \|x_n - x_{n+i}\| = 0, \quad \forall i = 1, 2, 3, \ldots, N.$$

Since $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n.$$

It follows that

$$\begin{aligned} \|y_n - x_n\|^2 &= \|y_n - x_{n+1} + x_{n+1} - x_n\|^2 \\ &= \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 + 2\langle y_n - x_{n+1}, x_{n+1} - x_n \rangle \\ &\leq \|x_n - x_{n+1}\|^2 + \theta_n + \|x_{n+1} - x_n\|^2 + 2\langle y_n - x_{n+1}, x_{n+1} - x_n \rangle \\ &\leq 2\|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\| \|x_{n+1} - x_n\| + \theta_n. \end{aligned} \tag{3.14}$$

Since $\theta_n \to 0$ as $n \to \infty$ and from (3.13), we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.15}$$

For each $p \in \Omega$, it follows from the firmly nonexpansive $T^{F_m}_{r_{m,n}}$ that for each $m \in \{1,2,3,\dots,M\}$, we have

$$\begin{aligned}\left\|\Theta_n^m x_n - p\right\|^2 &= \left\|T^{F_m}_{r_{m,n}}\Theta_n^{m-1}x_n - T^{F_m}_{r_{m,n}}p\right\|^2\\ &\le \langle \Theta_n^m x_n - p, \Theta_n^{m-1}x_n - p\rangle\\ &= \frac{1}{2}\left(\left\|\Theta_n^m x_n - p\right\|^2 + \left\|\Theta_n^{m-1}x_n - p\right\|^2 - \left\|\Theta_n^m x_n - \Theta_n^{m-1}x_n\right\|^2\right),\\ &\quad \text{for all } 1 \le m \le M.\end{aligned}$$

Thus, we get

$$\left\|\Theta_n^m x_n - p\right\|^2 \le \left\|\Theta_n^{m-1}x_n - p\right\|^2 - \left\|\Theta_n^m x_n - \Theta_n^{m-1}x_n\right\|^2, \quad \text{for all } 1 \le m \le M. \tag{3.16}$$

This implies that for each $m \in \{1,2,3,\dots,M\}$,

$$\begin{aligned}\left\|\Theta_n^m x_n - p\right\|^2 &\le \left\|\Theta_n^0 x_n - p\right\|^2 - \left\|\Theta_n^m x_n - \Theta_n^{m-1}x_n\right\|^2 - \left\|\Theta_n^{m-1}x_n - \Theta_n^{m-2}x_n\right\|^2\\ &\quad - \cdots - \left\|\Theta_n^2 x_n - \Theta_n^1 x_n\right\|^2 - \left\|\Theta_n^1 x_n - \Theta_n^0 x_n\right\|^2\\ &\le \|x_n - p\|^2 - \left\|\Theta_n^m x_n - \Theta_n^{m-1}x_n\right\|^2.\end{aligned}$$

Therefore, by the convexity of $\|\cdot\|^2$ and (3.8) and the nonexpansivity of $T^{F_m}_{r_{m,n}}$, we get

$$\begin{aligned}\|y_n - p\|^2 &\le \|u_n - p\|^2 + \theta_n\\ &= \left\|\Theta_n^M x_n - \Theta_n^M p\right\|^2 + \theta_n\\ &\le \left\|\Theta_n^m x_n - p\right\|^2 + \theta_n\\ &\le \|x_n - p\|^2 - \left\|\Theta_n^m x_n - \Theta_n^{m-1}x_n\right\|^2 + \theta_n.\end{aligned}$$

It follows that

$$\begin{aligned}\left\|\Theta_n^m x_n - \Theta_n^{m-1}x_n\right\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n\\ &\le \|x_n - y_n\|\left(\|x_n - p\| + \|y_n - p\|\right) + \theta_n.\end{aligned} \tag{3.17}$$

From (3.15) and (3.17), we obtain

$$\lim_{n\to\infty}\left\|\Theta_n^m x_n - \Theta_n^{m-1}x_n\right\| = 0, \quad \forall m \in \{1,2,3,\dots,M\}. \tag{3.18}$$

Then we have

$$\begin{aligned}\|u_n - x_n\| &\le \left\|u_n - \Theta_n^{M-1}x_n\right\| + \left\|\Theta_n^{M-1}x_n - \Theta_n^{M-2}x_n\right\|\\ &\quad + \cdots + \left\|\Theta_n^1 x_n - \Theta_n^0 x_n\right\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.19}$$

From (3.13) and (3.19), we get

$$\|u_{n+1}-u_n\| \le \|u_{n+1}-x_{n+1}\| + \|x_{n+1}-x_n\| + \|x_n-u_n\| \to 0, \quad \text{as } n\to\infty. \tag{3.20}$$

It follows that

$$\lim_{n\to\infty}\|u_{n+i}-u_n\| = 0, \quad \forall i\in\{1,2,3,\dots,N\}. \tag{3.21}$$

Since for any positive integer $n \ge N$, we can write $n = (h(n)-1)N + i(n)$, where $i(n) \in \{1,2,3,\dots,N\}$, note that

$$\begin{aligned}\|u_n - S_n u_n\| &\le \big\|u_n - S_{i(n)}^{h(n)}u_n\big\| + \big\|S_{i(n)}^{h(n)}u_n - S_n u_n\big\| \\ &= \big\|u_n - S_{i(n)}^{h(n)}u_n\big\| + \big\|S_{i(n)}^{h(n)}u_n - S_{i(n)} u_n\big\|.\end{aligned} \tag{3.22}$$

From the conditions $0 < a \le \alpha_n \le 1$ and $0 < b \le \beta_n \le 1-\lambda$, we get

$$\begin{aligned}\big\|u_n - S_{i(n)}^{h(n)}u_n\big\| &= \frac{1}{\beta_n}\|z_n - u_n\| \\ &= \frac{1}{\alpha_n\beta_n}\|y_n - u_n\| \\ &\le \frac{1}{ab}\big(\|y_n - x_n\| + \|x_n - u_n\|\big).\end{aligned}$$

From (3.15) and (3.19), we obtain

$$\lim_{n\to\infty}\big\|u_n - S_{i(n)}^{h(n)}u_n\big\| = 0. \tag{3.23}$$

It is obvious that the relations $h(n) = h(n-N)+1$ and $i(n) = i(n-N)$ hold.

Therefore, we compute

$$\begin{aligned}\big\|S_{i(n)}^{h(n)-1}u_n - u_n\big\| &\le \big\|S_{i(n)}^{h(n)-1}u_n - S_{i(n-N)}^{h(n)-1}u_{n-N+1}\big\| + \big\|S_{i(n-N)}^{h(n)-1}u_{n-N+1} - S_{i(n-N)}^{h(n-N)}u_{n-N}\big\| \\ &\quad + \big\|S_{i(n-N)}^{h(n-N)}u_{n-N} - u_{n-N}\big\| + \|u_{n-N} - u_{n-N+1}\| + \|u_{n-N+1} - u_n\| \\ &= \big\|S_{i(n)}^{h(n)-1}u_n - S_{i(n)}^{h(n)-1}u_{n-N+1}\big\| + \big\|S_{i(n-N)}^{h(n-N)}u_{n-N+1} - S_{i(n-N)}^{h(n-N)}u_{n-N}\big\| \\ &\quad + \big\|S_{i(n-N)}^{h(n-N)}u_{n-N} - u_{n-N}\big\| + \|u_{n-N} - u_{n-N+1}\| + \|u_{n-N+1} - u_n\|.\end{aligned}$$

Applying Lemma 2.9 and (3.21), we get

$$\lim_{n\to\infty}\big\|u_n - S_{i(n)}^{h(n)-1}u_n\big\| = 0. \tag{3.24}$$

From (3.22) and (3.24), it follows that

$$\lim_{n\to\infty}\|u_n - S_n u_n\| = 0. \tag{3.25}$$

Since

$$\|u_n - S_{n+i}u_n\| \le \|u_n - u_{n+i}\| + \|u_{n+i} - S_{n+i}u_{n+i}\| + \|S_{n+i}u_{n+i} - S_{n+i}u_n\| \to 0, \quad \text{as } n\to\infty$$

for any $i \in \{1,2,3,\dots,N\}$, which gives that

$$\lim_{n\to\infty} \|u_n - S_i u_n\| = 0, \quad \forall i \in \{1,2,3,\dots,N\}. \tag{3.26}$$

Moreover, for each $i \in \{1,2,3,\dots,N\}$, we obtain

$$\|x_n - S_i x_n\| \le \|x_n - u_n\| + \|u_n - S_i u_n\| + \|S_i u_n - S_i x_n\| \to 0, \quad \text{as } n \to \infty.$$

This implies that

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0, \quad \forall i \in \{1,2,3,\dots,N\}. \tag{3.27}$$

Step 6. We will show that $p^* \in \Omega := (\bigcap_{i=1}^N F(S_i)) \cap (\bigcap_{m=1}^M \mathrm{SEP}(F_m))$.
(6.1) We will show that $p^* \in \bigcap_{i=1}^N F(S_i)$.
We take $p^* \in \omega_w(x_n)$ and assume that $x_{n_j} \rightharpoonup p^*$ for some subsequence $\{x_{n_j}\}$ of $\{x_n\}$.
Note that S_i is uniformly Lipschitz continuous and (3.27), we obtain

$$\lim_{n\to\infty} \left\|x_n - S_i^k x_n\right\| = 0, \quad \forall k \in \mathbb{N}. \tag{3.28}$$

It follows from Lemma 2.7 that

$$p^* \in \bigcap_{i=1}^N F(S_i). \tag{3.29}$$

(6.2) We will show that $p^* \in \bigcap_{m=1}^M \mathrm{SEP}(F_m)$.
By Lemma 2.3, for each $m \in \{1,2,3,\dots,M\}$, we have

$$F_m\left(\Theta_n^m x_n, y\right) + \frac{1}{r_n}\left\langle y - \Theta_n^m x_n, \Theta_n^m x_n - \Theta_n^{m-1} x_n\right\rangle \ge 0, \quad \forall y \in C.$$

From (A2), we get

$$\frac{1}{r_n}\left\langle y - \Theta_n^m x_n, \Theta_n^m x_n - \Theta_n^{m-1} x_n\right\rangle \ge F_m\left(y, \Theta_n^m x_n\right), \quad \forall y \in C.$$

Taking $n = n_j$, we get

$$\left\langle y - \Theta_{n_j}^m x_{n_j}, \frac{\Theta_{n_j}^m x_{n_j} - \Theta_{n_j}^{m-1} x_{n_j}}{r_{n_j}}\right\rangle \ge F_m\left(y, \Theta_{n_j}^m x_{n_j}\right), \quad \forall y \in C.$$

From (3.18), we obtain that $\Theta_{n_j}^m x_{n_j} \rightharpoonup p^*$ as $j \to \infty$ for each $m \in \{1,2,3,\dots,M\}$ (especially $u_{n_j} = \Theta_{n_j}^M x_{n_j}$). Considering this together with (3.18) and (A4), we have for each $m \in \{1,2,3,\dots,M\}$ that

$$0 \ge F_m\left(y, p^*\right), \quad \forall y \in C.$$

For any $0 < t \leq 1$ and $y \in C$, we let $y_t = ty + (1-t)p^*$. Since $y \in C$ and $p^* \in C$, we obtain that $y_t \in C$, and so $F_m(y_t, p^*) \leq 0$. It follows that

$$0 = F_m(y_t, y_t) \leq tF_m(y_t, y) + (1-t)F_m(y_t, p^*) \leq tF_m(y_t, y).$$

Dividing by t, for each $m \in \{1, 2, 3, \dots, M\}$, we get

$$F_m(y_t, y) \geq 0, \quad \forall y \in C.$$

Letting $t \to 0$, from (A3), we get

$$F_m(p^*, y) \geq 0, \quad \forall y \in C.$$

Therefore, $p^* \in \bigcap_{m=1}^{M} \mathrm{SEP}(F_m)$, and so $p^* \in \Omega$.

Step 7. We will show that $\{x_n\}$ converges strongly to $P_\Omega x_1$.

Set $p^* = P_\Omega(x_1)$, then

$$\|x_{n+1} - x_1\| \leq \|p^* - x_1\|, \quad \forall n \in \mathbb{N}.$$

Since Ω is a nonempty closed and convex subset of H, there exists a unique $p^* \in \Omega$ such that $p^* = P_\Omega(x_1)$. It follows from Lemma 2.8 that $x_n \to p^*$, where $p^* = P_\Omega(x_1)$. This completes proof. □

4 Deduced theorems

If we take $M = 1$ in Theorem 3.1, then we obtain the following result.

Theorem 4.1 *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $M \geq 1$ be a positive integer. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $\{S_i\}_{i=1}^N : C \to C$ be a uniformly Lipschitz continuous and asymptotically λ_i-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \lambda_i < 1$ with the sequences $\{c_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} c_{n,i} = 0$ and $\{d_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} d_{n,i} = 0$. Let $\lambda = \max\{\lambda_i : 1 \leq i \leq N\}$, $c_n = \max\{c_{n,i} : 1 \leq i \leq N\}$ and $d_n = \max\{d_{n,i} : 1 \leq i \leq N\}$. Assume that $\Omega := \mathrm{EP}(F) \cap (\bigcap_{i=1}^N F(S_i))$ is nonempty and bounded. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \alpha_n \leq 1$, $0 < b \leq \beta_n \leq 1 - \lambda$, $a, b \in \mathbb{R}$, $\forall n \in \mathbb{N}$, $\{r_{m,n}\}$ be a sequence in $(0, \infty)$ such that $\lim_{n\to\infty} r_{m,n} > 0$.*

Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = H, \\ u_n = T_{r_n}^F x_n, \\ z_n = (1-\beta_n)u_n + \beta_n S_{i(n)}^{h(n)} u_n, \\ y_n = (1-\alpha_n)u_n + \alpha_n z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\theta_n = c_{h(n)} + d_{h(n)}\rho_n^2 \to 0$, *as* $n \to \infty$ *and* $\rho_n = \sup\{\|x_n - w\| : w \in \Omega\} < \infty$ *and* $n = (h(n) - 1)N + i(n)$, *where* $i(n) \in \{1, 2, 3, \dots, N\}$. *Then* $\{x_n\}$ *converges strongly to some point* p^*, *where* $p^* = P_\Omega(x_1)$.

Remark 4.2 Theorem 4.1 improves and extends the theorem of Tada and Takahashi [21] and the corollary of Duan and Zhao [7].

If we set $F_m \equiv 0$ and $r_{m,n} = 1$ for all $m \in \{1, 2, 3, \dots, N\}$ in Theorem 3.1, then we obtain the following result.

Theorem 4.3 *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $M \ge 1$ *be a positive integer. Let* $\{S_i\}_{i=1}^N : C \to C$ *be a uniformly Lipschitz continuous and asymptotically* λ_i*-strict pseudocontractive mapping in the intermediate sense for some* $0 \le \lambda_i < 1$ *with the sequences* $\{c_{n,i}\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} c_{n,i} = 0$ *and* $\{d_{n,i}\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} d_{n,i} = 0$. *Let* $\lambda = \max\{\lambda_i : 1 \le i \le N\}$, $c_n = \max\{c_{n,i} : 1 \le i \le N\}$ *and* $d_n = \max\{d_{n,i} : 1 \le i \le N\}$. *Assume that* $\Omega := \bigcap_{i=1}^N F(S_i)$ *is nonempty and bounded. Let* $\{\alpha_n\}$, $\{\beta_n\}$ *be sequences in* $[0, 1]$ *such that* $0 < a \le \alpha_n \le 1$, $0 < b \le \beta_n \le 1 - \lambda$, $a, b \in \mathbb{R}$, $\forall n \in \mathbb{N}$, $\{r_{m,n}\}$ *be a sequence in* $(0, \infty)$ *such that* $\lim_{n\to\infty} r_{m,n} > 0$.

Let $\{x_n\}$ *be a sequence generated by the following algorithm*:

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = H, \\ z_n = (1 - \beta_n)x_n + \beta_n S_{i(n)}^{h(n)} x_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \le \|x_n - w\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $\theta_n = c_{h(n)} + d_{h(n)}\rho_n^2 \to 0$, *as* $n \to \infty$ *and* $\rho_n = \sup\{\|x_n - w\| : w \in \Omega\} < \infty$ *and* $n = (h(n) - 1)N + i(n)$, *where* $i(n) \in \{1, 2, 3, \dots, N\}$. *Then* $\{x_n\}$ *converges strongly to some point* p^*, *where* $p^* = P_\Omega(x_1)$.

Remark 4.4 Theorem 4.1 improves and extends the theorem of Sahu, Xu, and Yao [4], the theorem of Qin, Cho, Kang, and Shang [3] and the corollary of Duan and Zhao [7].

5 Numerical examples

In this section, in order to demonstrate the effectiveness, realization and convergence of algorithm of Theorem 3.1, we consider the following simple example that was presented in reference [4].

Example 5.1 Let $x \in \mathbb{R}$ and $C = [0, 1]$. For each $x \in C$, we define

$$Sx = \begin{cases} kx, & \text{if } x \in [0, \frac{1}{2}]; \\ 0, & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

where $0 < k < 1$.

It is easy to see that $S : C \to C$ is discontinuous at $x = \frac{1}{2}$ and S is not Lipschitz continuous. Set $C_1 = [0, \frac{1}{2}]$ and $C_2 = (\frac{1}{2}, 1]$.

For each $x, y \in C_1$, we have

$$\left|S^n x - S^n y\right| = k^n |x - y| \leq |x - y|, \quad \forall x, y \in C_1 \text{ and } \forall n \in \mathbb{N}.$$

For each $x, y \in C_2$, we have

$$\left|S^n x - S^n y\right| = 0 \leq |x - y|, \quad \forall x, y \in C_2 \text{ and } \forall n \in \mathbb{N}.$$

For each $x \in C_1$ and $y \in C_2$, we have

$$\begin{aligned} \left|S^n x - S^n y\right| &= \left|k^n x - 0\right| \\ &\leq \left|k^n (x - y) + k^n y\right| \\ &\leq k^n |x - y| + k^n |y| \\ &\leq |x - y| + k^n, \quad \forall n \in \mathbb{N}. \end{aligned}$$

It follows that

$$\begin{aligned} \left|S^n x - S^n y\right|^2 &\leq \left(|x - y| + k^n\right)^2 \\ &\leq |x - y|^2 + k\left|x - S^n x - \left(y - S^n y\right)\right|^2 + k^n K, \end{aligned}$$

for all $x, y \in C$ and $n \in \mathbb{N}$ and for some $K > 0$.

Therefore, S is an asymptotically k-strict pseudocontractive mapping in the intermediate sense.

In Theorem 3.1, we set $N = 1$, $F_m \equiv 0$, $\beta_n = 1 - k$, $\alpha_n = \frac{n+1}{2n}$. We apply it to find the fixed point of S of Example 5.1.

Under the above assumption in Theorem 3.1 is simplified as follows:

$$\begin{cases} x_1 \in H \text{ chosen arbitrarily}, \\ C_1 = H, \\ z_n = k x_n + (1 - k) S^n x_n, \\ y_n = (\frac{n-1}{2n}) x_n + (\frac{n+1}{2n}) z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}. \end{cases} \tag{5.1}$$

In fact, in one-dimensional case, C_{n+1} is a closed interval. If we set $[a_{n+1}, b_{n+1}] := C_{n+1}$, then the projection point x_{n+1} of $x_1 \in C$ onto C_{n+1} can be expressed as

$$x_{n+1} = P_\Omega(x_1) \begin{cases} x_1, & \text{if } x_1 \in [a_{n+1}, b_{n+1}]; \\ b_{n+1}, & \text{if } x_1 > b_{n+1}; \\ a_{n+1}, & \text{if } x_1 < a_{n+1}. \end{cases}$$

Table 1 The numerical results for an initial guess $x_1 = 0.2, 0.5, 0.8$

n **(iterative number)**	**Initial guess**		
	$x_1 = 0.2$	$x_1 = 0.5$	$x_1 = 0.8$
10	0.1467	0.2049	0.2105
20	0.0163	0.0205	0.0209
30	0.0016	0.0019	0.0020
40	1.5149×10^{-4}	1.8819×10^{-4}	1.9196×10^{-4}
50	1.4889×10^{-5}	1.8494×10^{-5}	1.8864×10^{-5}

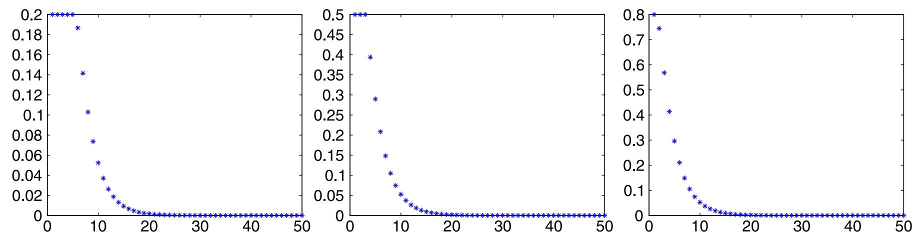

Figure 1 The convergence comparison of different initial values x_1.

The numerical results for an initial guess $x_1 = 0.2, 0.5, 0.8$ are shown in Table 1. From the table, we see that the iterations converge to 0 which is the unique fixed point of S. The convergence of each iteration is also shown in Figure 1 for comparison.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in this research. All authors read and approved the final manuscript.

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