LIMITING CASE OF THE BOUNDEDNESS OF FRACTIONAL INTEGRAL OPERATORS ON NONHOMOGENEOUS SPACE

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We show the boundedness of fractional integral operators by means of extrapolation. We also show that our result is sharp.

1. Introduction

Recently, harmonic analysis on $\mathbb{R}^d$ with nondoubling measures has been developed very rapidly; here, by a doubling measure, we mean a Radon measure μ on $\mathbb{R}^d$ satisfying $\mu(B(x,2r)) \le c_0\mu(B(x,r))$, $x \in \operatorname{supp}(\mu)$, $r > 0$. In what follows, $B(x,r)$ is the closed ball centered at x of radius r. In this paper, we deal with measures which do not necessarily satisfy the doubling condition.

We can list [7, 8, 11] as important works in this field. Tolsa proved subadditivity and bi-Lipschitz invariance of the analytic capacity [12, 13]. Many function spaces and many linear operators for such measures stem from their works. For example, Tolsa has defined the Hardy space $H^1(\mu)$ [11]. Han and Yang have defined the Triebel-Lizorkin spaces [3].

In the present paper, we mainly deal with the fractional integral operators. We occasionally postulate the growth condition on μ:

$$\mu \text{ is a Radon measure on } \mathbb{R}^d \text{ with } \mu\big(B(x,r)\big) \le c_0 r^n \quad \text{for some } c_0 > 0,\ 0 < n \le d. \tag{1.1}$$

A growth measure is a Radon measure μ satisfying (1.1). We define the fractional integral operator I_α associated with the growth measure μ as

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n\alpha}}\,d\mu(y), \quad 0 < \alpha < 1. \tag{1.2}$$

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Let $1/q = 1/p - (1-\alpha)$ with $1 < p < q < \infty$. $L^p(\mu)$-$L^q(\mu)$ boundedness of I_α in a more general form was proved by Kokilashvili [4]. On general nonhomogeneous spaces, that is, on metric measure spaces, it was also proved in [5] (see [1]). In [2], the limit case $p = 1/(1-\alpha)$ was considered. In general, the integral defining $I_\alpha f(x)$ does not converge absolutely for μ- a.e., if $f \in L^{1/(1-\alpha)}(\mu)$. García-Cuerva and Gatto considered some modified operator and showed its boundedness from $L^{1/(1-\alpha)}(\mu)$ to some BMO-like space defined in [11].

This paper deals mainly with the Morrey spaces. By a cube, we mean a set of the form

$$Q(x,r) := [x_1 - r, x_1 + r] \times \cdots \times [x_d - r, x_d + r], \quad x = (x_1,\ldots,x_d) \in \mathbb{R}^d,\ 0 < r \le \infty. \tag{1.3}$$

Given a cube $Q = Q(x,r)$, $\kappa > 0$, we denote $\kappa Q := Q(x,\kappa r)$ and $\ell(Q) = 2r$. We define $\mathfrak{Q}(\mu)$ by

$$\mathfrak{Q}(\mu) := \{Q \subset \mathbb{R}^d : Q \text{ is a cube with } 0 < \mu(Q) < \infty\}. \tag{1.4}$$

Now we are in the position of describing the Morrey spaces for nondoubling measures.

Definition 1.1 (see [10, Section 1]). Let $0 < q \le p < \infty$, $k > 1$. Denote by $\mathcal{M}_q^p(k,\mu)$ a set of $L^q_{\mathrm{loc}}(\mu)$ functions f for which the quasinorm

$$\|f : \mathcal{M}_q^p(k,\mu)\| := \sup_{Q \in \mathfrak{Q}(\mu)} \mu(kQ)^{1/p-1/q} \left(\int_Q |f(y)|^q d\mu(y) \right)^{1/q} < \infty. \tag{1.5}$$

Note that this definition does not involve the growth condition (1.1). So in this paper, we assume μ is just a Radon measure unless otherwise stated.

Key properties that we are going to use can be summarized as follows.

Proposition 1.2 (see [10, Proposition 1.1]). *Let $0 < q \le p < \infty$, $k_1 > k_2 > 1$. Then there exists $C_{d,k_1,k_2,q}$ so that, for every μ-measurable function f,*

$$\|f : \mathcal{M}_q^p(k_2,\mu)\| \le \|f : \mathcal{M}_q^p(k_1,\mu)\| \le C_{d,k_1,k_2,q} \|f : \mathcal{M}_q^p(k_2,\mu)\|. \tag{1.6}$$

The proof is omitted: interested readers may consult [10]. However, we deal with similar assertion whose proof is wholly included in this present paper.

Lemma 1.3 (see [10, Section 1]). (1) *Let $0 < q_1 \le q_2 \le p < \infty$ and $k > 1$. Then*

$$\|f : \mathcal{M}_{q_1}^p(k,\mu)\| \le \|f : \mathcal{M}_{q_2}^p(k,\mu)\| \le \|f : \mathcal{M}_p^p(k,\mu)\| = \|f : L^p(\mu)\|. \tag{1.7}$$

(2) *Let $\mu(\mathbb{R}^d) < \infty$ and $0 < q \le p_1 \le p_2 < \infty$. Then*

$$\|f : \mathcal{M}_q^{p_1}(k,\mu)\| \le \mu(\mathbb{R}^d)^{1/p_1 - 1/p_2} \|f : \mathcal{M}_q^{p_2}(k,\mu)\|. \tag{1.8}$$

Proof. Equation (1.7) is straightforward by using the Hölder inequality.

As for (1.8), thanks to the finiteness of μ writing out the left-hand side in full, we have

$$\begin{aligned} \| f : \mathcal{M}_q^{p_1}(k,\mu) \| &= \sup_{Q \in \mathcal{Q}(\mu)} \mu(kQ)^{1/p_1 - 1/q} \left(\int_Q |f(y)|^q d\mu(y) \right)^{1/q} \\ &\leq \sup_{Q \in \mathcal{Q}(\mu)} \mu(\mathbb{R}^d)^{1/p_1 - 1/p_2} \mu(kQ)^{1/p_2 - 1/q} \left(\int_Q |f(y)|^q d\mu(y) \right)^{1/q} \\ &= \mu(\mathbb{R}^d)^{1/p_1 - 1/p_2} \| f : \mathcal{M}_q^{p_2}(k,\mu) \|. \end{aligned} \tag{1.9}$$

Lemma 1.3 is therefore proved. □

Keeping Proposition 1.2 in mind, for simplicity, we denote

$$\mathcal{M}_q^p(\mu) := \mathcal{M}_q^p(2,\mu), \quad \| \cdot : \mathcal{M}_q^p(\mu) \| := \| \cdot : \mathcal{M}_q^p(2,\mu) \|. \tag{1.10}$$

In [10, Theorem 3.3], we showed that I_α is bounded from $\mathcal{M}_q^p(\mu)$ to $\mathcal{M}_t^s(\mu)$, if

$$\frac{q}{p} = \frac{t}{s}, \quad \frac{1}{s} = \frac{1}{p} - (1-\alpha), \quad 1 < q \leq p < \infty,\ 1 < t \leq s < \infty,\ 0 < \alpha < 1. \tag{1.11}$$

Having described the main function spaces, we present our problem. In the present paper, from the viewpoint different from [2], we will consider the limit case of the boundedness of I_α as "$p \to 1/(1-\alpha)$" or "$s \to \infty$," where p and s satisfy (1.11).

Problem 1.4. Let $0 < \alpha < 1$ and assume that μ is a finite growth measure. Find a nice function space X to which I_α sends $\mathcal{M}_q^{1/(1-\alpha)}(\mu)$ continuously, where $1 < q \leq 1/(1-\alpha)$.

Although the Morrey spaces are the function spaces coming with two parameters, we arrange $\mathcal{M}_q^p(\mu)$ to $\mathcal{M}_{\beta p}^p(\mu)$ with $\beta \in (0,1]$ fixed and regard them as a family of function spaces parameterized only by p. We turn our attention to the family of spaces $\{\mathcal{M}_{\beta p}^p(\mu)\}_{p \in (0,\infty)}$. We also consider the generalized version of Problem 1.4.

Problem 1.5. Let μ be finite and $0 < p_0 < p < r < \infty$, $0 < \beta \leq 1$, $1/s = 1/p - 1/r$. Suppose that we are given an operator T from $\bigcup_{p > p_0} \mathcal{M}_{\beta p}^p(\mu)$ to $\bigcup_{s>0} \mathcal{M}_{\beta s}^s(\mu)$. Assume, restricting T to $\mathcal{M}_{\beta p}^p(\mu)$, we have a precise estimate

$$\| Tf : \mathcal{M}_{\beta s}^s(\mu) \| \leq c(s) \| f : \mathcal{M}_{\beta p}^p(\mu) \|, \tag{1.12}$$

where $1/s = 1/p - 1/r$ with $p, r, s > 0$. Then what can we say about the boundedness of T on the limit function space $\mathcal{M}_{\beta r}^r(\mu)$?

Here we describe the organization of this paper. Section 2 is devoted to the definition of the function spaces to answer Problems 1.4 and 1.5. In Section 3, we give a general machinery for Problems 1.4 and 1.5. I_α appearing here will be an example of the theorem in Section 3. Besides I_α, we take up two types of other fractional integral operators. The task in Section 4 is to determine $c(s)$ in (1.12) precisely. We skillfully use two types of fractional integral operators as well as I_α to see the size of $c(s)$. In Section 5, we exhibit an example showing the sharpness of the estimate of $c(s)$ obtained in Section 4. The example will reveal us the difference between the Morrey spaces and the L^p spaces.

2. Orlicz-Morrey spaces $\mathcal{M}_\beta^\Phi(\mu)$

In this section, we introduce function spaces $\mathcal{M}_\beta^\Phi(\mu)$ to formulate our main results. E. Nakai defined $\mathcal{M}_\beta^\Phi(\mu)$ for Lebesgue measure $\mu = dx$. We denote by $|E|$ the volume of a measurable set E. Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function, that is, Φ is convex with $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$.

For $\beta \in (0,1]$, E. Nakai has defined the Orlicz-Morrey spaces: the space $\mathcal{M}_\beta^\Phi(dx)$ consists of all measurable functions f for which the norm

$$\|f : \mathcal{M}_\beta^\Phi(dx)\| := \inf\left\{\lambda > 0 : \sup_{Q\in\mathcal{Q}(dx)} |Q|^{\beta-1}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \leq 1\right\} < \infty. \tag{2.1}$$

For details, we refer to [6].

Motivated by this definition and that of $\mathcal{M}_q^p(\mu)$ with $0 < q \leq p < \infty$, we define the Orlicz-Morrey spaces $\mathcal{M}_\beta^\Phi(\mu)$ as follows.

Definition 2.1. Let $\beta \in (0,1]$, $k > 1$, and Φ be a Young function. Then define

$$\|f : \mathcal{M}_\beta^\Phi(k,\mu)\| := \inf\left\{\lambda > 0 : \sup_{Q\in\mathcal{Q}(\mu)} \mu(kQ)^{\beta-1}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)d\mu(y) \leq 1\right\}. \tag{2.2}$$

We define the function space $\mathcal{M}_\beta^\Phi(k,\mu)$ as a set of μ-measurable functions f for which the norm is finite.

The function space $\mathcal{M}_\beta^\Phi(k,\mu)$ is independent of $k > 1$. More precisely, we have the following.

Proposition 2.2. *Let $k_1 > k_2 > 1$. Then there exists constant C_{d,k_1,k_2} such that*

$$\|f : \mathcal{M}_\beta^\Phi(k_1,\mu)\| \leq \|f : \mathcal{M}_\beta^\Phi(k_2,\mu)\| \leq C_{d,k_1,k_2}\|f : \mathcal{M}_\beta^\Phi(k_1,\mu)\|. \tag{2.3}$$

Here, $C_{d,k_1,k_2} > 0$ is independent of f.

Proof. By the monotonicity of $\|f : \mathcal{M}_\beta^\Phi(k,\mu)\|$ with respect to k, the left inequality is obvious. What is essential in (2.3) is the right inequality. The monotonicity allows us to assume that $k_1 = 2k_2 - 1$. We take $Q \in \mathcal{Q}(\mu)$ arbitrarily. We have to majorize

$$\inf\left\{\lambda > 0 : \mu(k_2Q)^{\beta-1}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)d\mu(x) \leq 1\right\} \tag{2.4}$$

by $\lambda_0 := \|f : \mathcal{M}_\beta^\Phi(k_1,\mu)\|$ uniformly over Q.

Bisect Q into 2^d cubes and label $Q_1, Q_2, \dots, Q_L$ to those in $\mathcal{Q}(\mu)$, then the distance between the boundary of $k_2 Q$ and the center of Q_j is

$$\left(\frac{k_2}{2} - \frac{1}{4}\right)\ell(Q) = \frac{k_1}{4}\ell(Q). \tag{2.5}$$

Consequently, we have $k_1 Q_j \subset k_2 Q$ for $j = 1, 2, \dots, L$. This inclusion gives us that

$$\mu(k_2Q)^{\beta-1}\int_Q \Phi\left(\frac{|f(x)|}{\lambda_0}\right)d\mu(x) \le \sum_{j=1}^{L}\mu(k_1Q_j)^{\beta-1}\int_{Q_j}\Phi\left(\frac{|f(x)|}{\lambda_0}\right)d\mu(x) \le 2^d. \tag{2.6}$$

Note that $\Phi(tx) \le t\Phi(x)$ for $0 \le t \le 1$ by convexity. As a result, we obtain

$$\sup_{Q\in\mathcal{Q}(\mu)} \mu(k_2Q)^{\beta-1}\int_Q \Phi\left(\frac{|f(x)|}{2^d\lambda_0}\right)d\mu(x) \le 1. \tag{2.7}$$

Thus we have obtained

$$\|f : \mathcal{M}_\beta^\Phi(k_2,\mu)\| \le 2^d\lambda_0 = 2^d\|f : \mathcal{M}_\beta^\Phi(k_1,\mu)\|. \tag{2.8}$$

Hence we have established that we can take $C_{d,2k_2-1,k_2} = 2^d$. □

Keeping this proposition in mind, we set $\mathcal{M}_\beta^\Phi(\mu) := \mathcal{M}_\beta^\Phi(2,\mu)$. The same argument as Proposition 2.2 works for Proposition 1.2.

3. Extrapolation theorem on the Morrey spaces

In this section, we will prove the key lemma dealing with an extrapolation theorem on the Morrey spaces. Assume that μ is finite and

$$0 < p_0 < p < r < \infty, \quad 0 < \beta \le 1, \quad \frac{1}{s} = \frac{1}{p} - \frac{1}{r}. \tag{3.1}$$

Let T be an operator from $\mathcal{M}_{\beta p}^p(\mu)$ to $\mathcal{M}_{\beta s}^s(\mu)$ with a precise estimate

$$\|Tf : \mathcal{M}_{\beta s}^s(\mu)\| \le cs^\rho \|f : \mathcal{M}_{\beta p}^p(\mu)\|, \quad \rho > 0. \tag{3.2}$$

Then we can say that the limit result of

$$T : \mathcal{M}_{\beta p}^p(\mu) \longrightarrow \mathcal{M}_{\beta s}^s(\mu), \quad p_0 < p < r, \quad \frac{1}{s} = \frac{1}{p} - \frac{1}{r}, \tag{3.3}$$

as $p \to r$, $s \to \infty$, is

$$T : \mathcal{M}_{\beta r}^r(\mu) \longrightarrow \mathcal{M}_\beta^\Phi(\mu), \tag{3.4}$$

where $\Phi(x) = \exp(x^{1/\rho}) - 1$. More precisely, our main extrapolation theorem is the following.

THEOREM 3.1. *Suppose $\mu(\mathbb{R}^d) < \infty$. Let $0 < p_0 < r$, $0 < \rho \le 1$, and $0 < \beta \le 1$. Suppose that the sublinear operator T satisfies*

$$||Tf : \mathcal{M}^s_{\beta s}(\mu)|| \le C_0 s^\rho ||f : \mathcal{M}^p_{\beta p}(\mu)|| \quad \forall f \in \mathcal{M}^p_{\beta p}(\mu) \tag{3.5}$$

for each $p_0 \le p < r$ with $1/s = 1/p - 1/r$. Here, $C_0 > 0$ is a constant independent of p and s. Then there exists a constant $\delta > 0$ such that

$$\sup_Q \left[\int_Q \left[\exp\left(\delta \left| \frac{Tf(x)}{||f : \mathcal{M}^r_{\beta r}(\mu)||} \right|^{1/\rho} \right) - 1 \right] \frac{d\mu(x)}{\mu(2Q)^{1-\beta}} \right] \le 1 \quad \forall f \in \mathcal{M}^r_{\beta r}(\mu) \tag{3.6}$$

or equivalently

$$||Tf : \mathcal{M}^\Phi_\beta(\mu)|| \le \delta^{-1/\rho} ||f : \mathcal{M}^r_{\beta r}(\mu)|| \quad \forall f \in \mathcal{M}^r_{\beta r}(\mu) \tag{3.7}$$

for $\Phi(t) = \exp(t^{1/\rho}) - 1$.

More can be said about this theorem: the case when $\beta = 1$ corresponds to the Zygmund-type extrapolation theorem (see [15]). Set $L^\Phi(\mu) = \mathcal{M}^\Phi_1(\mu)$.

COROLLARY 3.2. *Keep to the same assumption as Theorem 3.1 on μ, ρ, p_0, r, and T. Suppose*

$$||Tf : L^s(\mu)|| \le C_0 s^\rho ||f : L^p(\mu)|| \quad \forall f \in L^p(\mu) \tag{3.8}$$

for s, p with $1/s = 1/p - 1/r$. Here, $C_0 > 0$ is a constant independent of p and s. Then there exists some constant $\delta > 0$ such that

$$\int_{\mathbb{R}^d} \left[\exp\left(\delta \left| \frac{Tf(x)}{||f : L^r(\mu)||} \right|^{1/\rho} \right) - 1 \right] d\mu(x) \le 1 \quad \forall f \in L^r(\mu) \tag{3.9}$$

or equivalently

$$||Tf : L^\Phi(\mu)|| \le \delta^{-1/\rho} ||f : L^r(\mu)|| \quad \forall f \in L^r(\mu). \tag{3.10}$$

Before we come to the proof, a remark may be in order.

Remark 3.3. Suppose that Ω is a bounded open set in $\mathbb{R}^d$. Applying $T = I_\alpha$ with $\mu = dx|\Omega$, Lebesgue measure on Ω, we obtain a result corresponding to the one in [14].

The proof of Theorem 3.1 is after the one of Zygmund's extrapolation theorem in [15].

Proof of Theorem 3.1. By subadditivity, it can be assumed that $\| f : \mathcal{M}^r_{\beta r}(\mu)\| = 1$. From (3.5) and Lemma 1.3, we have $\| Tf : \mathcal{M}^s_{\beta s}(\mu)\| \le cs^\rho \| f : \mathcal{M}^p_{\beta p}(\mu)\| \le cs^\rho$.

Let $Q \in \mathfrak{Q}(\mu)$. Then by Taylor's expansion,

$$\begin{aligned}
&\int_Q \left\{\exp\left(\delta \,|\, Tf(x)\,|^{1/\rho}\right) - 1\right\} \frac{d\mu(x)}{\mu(2Q)^{1-\beta}} \\
&\quad = \sum_{k=1}^{\infty} \frac{\delta^k}{k!} \int_Q |\, Tf(x)\,|^{k/\rho} \frac{d\mu(x)}{\mu(2Q)^{1-\beta}} \le \sum_{k=1}^{\infty} \frac{\delta^k}{k!} \left\| Tf : \mathcal{M}_{k/\rho}^{k/\rho\beta}(\mu)\right\|^{k/\rho} \\
&\quad = \sum_{k=1}^{L} \frac{\delta^k}{k!} \left\| Tf : \mathcal{M}_{k/\rho}^{k/\rho\beta}(\mu)\right\|^{k/\rho} + \sum_{k=L+1}^{\infty} \frac{\delta^k}{k!} \left\| Tf : \mathcal{M}_{k/\rho}^{k/\rho\beta}(\mu)\right\|^{k/\rho},
\end{aligned} \tag{3.11}$$

where L is the largest integer not exceeding $\beta\rho p_0$. If we invoke Lemma 1.3, we see

$$\sum_{k=1}^{L} \frac{\delta^k}{k!} \left\| Tf : \mathcal{M}_{k/\rho}^{k/\rho\beta}(\mu)\right\|^{k/\rho} \le c \sum_{k=1}^{L} \frac{\delta^k}{k!} \left\| Tf : \mathcal{M}_{L/\rho}^{L/\rho\beta}(\mu)\right\|^{k/\rho} \le c \sum_{k=1}^{L} \delta^k. \tag{3.12}$$

By (3.5), we have

$$\sum_{k=L+1}^{\infty} \frac{\delta^k}{k!} \left\| Tf : \mathcal{M}_{k/\rho}^{k/\rho\beta}(\mu)\right\|^{k/\rho} \le \sum_{k=L+1}^{\infty} \frac{(c\delta)^k k^k}{k!}. \tag{3.13}$$

We put (3.12) and (3.13) together,

$$\int_Q \left\{\exp\left(\delta \,|\, Tf(x)\,|^{1/\rho}\right) - 1\right\} \frac{d\mu(x)}{\mu(2Q)^{1-\beta}} \le \sum_{k=1}^{\infty} \frac{(c\delta)^k k^k}{k!}. \tag{3.14}$$

$\lim_{k\to\infty}(k^k/k!)^{1/k} = e$ implies that the function $\psi(\delta) := \sum_{k=1}^{\infty}((C_0\delta)^k k^k/k!)$ is a continuous function in the neighborhood of 0 in $[0,1)$ with $\psi(0) = 0$. Consequently, if δ is small enough, then

$$\int_Q \left\{\exp\left(\delta \,|\, Tf(x)\,|^{1/\rho}\right) - 1\right\} \frac{d\mu(x)}{\mu(2Q)^{1-\beta}} \le \psi(\delta) \le 1 \tag{3.15}$$

for all $f \in \mathcal{M}_{\beta r}^{r}(\mu)$ with $\| f : \mathcal{M}_{\beta r}^{r}(\mu)\| = 1$. Theorem 3.1 is therefore proved. □

Remark 3.4. To obtain Theorem 3.1, the growth condition is unnecessary. Thus, the proof is still available, if μ is just a finite Radon measure.

4. Precise estimate of the fractional integrals

Our task in this section is to see the size of $c(s)$ in (1.12) with $T = I_\alpha$. The estimates involve the modified uncentered maximal operator given by

$$M_\kappa f(x) := \sup_{x \in Q \in \mathfrak{Q}(\mu)} \frac{1}{\mu(\kappa Q)} \int_Q |\, f(y)\,|\, d\mu(y), \quad \kappa > 1. \tag{4.1}$$

We make a quick view of the size of the constant. First, we see that

$$\mu\{x \in \mathbb{R}^d : M_\kappa f(x) > \lambda\} \le \frac{C_{d,\kappa}}{\lambda} \int_{\mathbb{R}^d} |\, f(x)\,|\, d\mu(x) \tag{4.2}$$

by Besicovitch's covering lemma. Then thanks to Marcinkiewicz's interpolation theorem, we obtain a precise estimate of the operator norm of M_κ:

$$\|M_\kappa\|_{L^p(\mu)\to L^p(\mu)} \le \frac{C_{d,\kappa}\, p}{p-1}. \tag{4.3}$$

Finally, examining the proof in [10, Theorem 2.3] gives us the estimate of the operator norm on $\mathcal{M}^p_q(\mu)$:

$$\|M_\kappa\|_{\mathcal{M}^p_q(\mu)\to \mathcal{M}^p_q(\mu)} \le \frac{C_{d,\kappa} q}{q-1}. \tag{4.4}$$

We will make use of (4.3) and (4.4) in this section.

4.1. Fractional integral operators $J_{\alpha,\kappa}$ and $I^\flat_{\alpha,\kappa}$. For the definition of I_α, the growth condition on μ is indispensable. However, in [9], the theory of fractional integral operators without the growth condition was developed. The construction of the fractional integral operators without the growth condition involves a covering lemma. In this present paper, we intend to define another substitute. We take advantage of the simple definition of the new fractional integral operator.

Definition 4.1 (see [9, Definitions 13, 14]). Let $\alpha \in (0,1)$ and $\kappa > 1$. For $k \in \mathbb{Z}$, take $\mathfrak{Q}^{(k)} \subset \mathfrak{Q}(\mu)$ that satisfies the following.

(1) For all $Q \in \mathfrak{Q}^{(k)}$, $2^k < \mu(\kappa^2 Q) \le 2^{k+1}$.

(2) $\sup_{x\in\mathbb{R}^d} \sum_{Q\in\mathfrak{Q}^{(k)}} \chi_{\kappa Q}(x) \le N_\kappa < \infty$, where N_κ depends only on κ and d.

(3) For any cube with $2^{k-1} < \mu(\kappa^2 Q') \le 2^k$, find $Q \in \mathfrak{Q}^{(k)}$ such that $Q' \subset \kappa Q$.

By the way of $\{\mathfrak{Q}^{(k)}\}_{k\in\mathbb{Z}}$, for $f \in L^1_{\mathrm{loc}}(\mu)$, define the operator $J_{\alpha,\kappa}$ as

$$J_{\alpha,\kappa} f(x) := \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathfrak{Q}^{(k)}} \frac{\chi_{\kappa Q}(x)\chi_{\kappa Q}(y)}{2^{k\alpha}} f(y)\,d\mu(y). \tag{4.5}$$

If

$$j_{\alpha,\kappa}(x,y) := \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathfrak{Q}^{(k)}} \frac{\chi_{\kappa Q}(x)\chi_{\kappa Q}(y)}{2^{k\alpha}}, \tag{4.6}$$

then one can write $J_{\alpha,\kappa} f(x) = \int_{\mathbb{R}^d} j_{\alpha,\kappa}(x,y) f(y)\,d\mu(y)$ in terms of the integral kernel.

What is important about $J_{\alpha,\kappa}$ is that it is linear, it can be defined for any Radon measure μ and, if μ satisfies the growth condition, it plays a role of the majorant operator of I_α. We give a more simpler fractional maximal operator which substitutes for $J_{\alpha,\kappa}$.

Definition 4.2. Let $\alpha \in (0,1)$ and $\kappa > 1$. For $x, y \in \mathbb{R}^d \in \operatorname{supp}(\mu)$, set

$$K^\flat_{\alpha,\kappa}(x,y) = \sup_{x,y\in Q\in\mathfrak{Q}(\mu)} \mu(\kappa Q)^{-\alpha}. \tag{4.7}$$

It will be understood that $K^{\flat}_{\alpha,\kappa}(x,y)=0$ unless $x,y\in\operatorname{supp}(\mu)$. For a positive μ-measurable function f, set

$$I^{\flat}_{\alpha,\kappa}f(x)=\int_{\mathbb{R}^d}K^{\flat}_{\alpha,\kappa}(x,y)f(y)d\mu(y). \tag{4.8}$$

Suppose that μ satisfies the growth condition (1.1). Then the comparison of the kernel reveals us that $I_\alpha f(x)\le cI^{\flat}_{\alpha,\kappa}f(x)$ μ- a.e. for all positive μ-measurable functions f.

$I^{\flat}_{\alpha,\kappa}$ and $J_{\alpha,\kappa}$ are comparable in the following sense.

Lemma 4.3. *Let $\alpha\in(0,1)$ and $\kappa>1$. There exists constant $C>0$ so that, for every positive μ-measurable function f,*

$$I^{\flat}_{\alpha,\kappa^2}f(x)\le J_{\alpha,\kappa}f(x)\le CI^{\flat}_{\alpha,\kappa}f(x). \tag{4.9}$$

Proof. It suffices to compare the kernel.

First, we will deal with the left inequality. Suppose that $Q\in\mathcal{Q}(\mu)$ contains x, y and satisfies

$$2^{k_0}<\mu(\kappa^2Q)\le 2^{k_0+1},\quad k_0\in\mathbb{Z}. \tag{4.10}$$

Then by Definition 4.1, we can find $Q^*\in\mathcal{Q}^{(k_0)}$ such that $Q\subset\kappa Q^*$. Since κQ^* contains both x and y, we obtain

$$\mu(\kappa^2Q)^{-\alpha}\le 2^{-k_0\alpha}=\frac{\chi_{\kappa Q^*}(x)\chi_{\kappa Q^*}(y)}{2^{k_0\alpha}}\le j_{\alpha,\kappa}(x,y). \tag{4.11}$$

Consequently, the left inequality is established.

We turn to the right inequality. Assume that

$$2^{-\alpha(k_1+1)}\le K^{\flat}_{\alpha,\kappa}(x,y)<2^{-\alpha k_1},\quad k_1\in\mathbb{Z}. \tag{4.12}$$

Let $Q\in\mathcal{Q}^{(k)}$. Suppose that κQ contains x,y. Then by definition,

$$\mu(\kappa^2Q)^{-\alpha}\le K^{\flat}_{\alpha,\kappa}(x,y)<2^{-\alpha k_1} \tag{4.13}$$

and hence $\mu(\kappa^2Q)>2^{k_1}$. Since $Q\in\mathcal{Q}^{(k)}$, we have $k\ge k_1$. Thus if $Q\in\mathcal{Q}^{(k)}$ and κQ contains x, y, then $k\ge k_1$. From the definition of $j_{\alpha,\kappa}$, it follows that

$$j_{\alpha,\kappa}(x,y)=\sum_{k\ge k_1}\sum_{Q\in\mathcal{Q}^{(k)}}\frac{\chi_{\kappa Q}(x)\chi_{\kappa Q}(y)}{2^{k\alpha}}\le cN_\kappa\sum_{k\ge k_1}\frac{1}{2^{k\alpha}}=c2^{-k_1\alpha}\le cK^{\flat}_{\alpha,\kappa}(x,y). \tag{4.14}$$

As a result, the right inequality is proved. □

We summarize the relations between three operators.

Corollary 4.4. *If μ satisfies the growth condition (1.1), then, for every positive μ-measurable function f,*

$$I_\alpha f(x)\lesssim J_{\alpha,\kappa}f(x)\sim I^{\flat}_{\alpha,\kappa}f(x), \tag{4.15}$$

and μ- a.e. $x \in \mathbb{R}^d$, where the implicit constants in $\lesssim$ and $\sim$ depend only on α, κ, and c_0 in (1.1).

4.2. L^p-estimates. Here we will prove the L^p-estimates associated with fractional integral operators.

THEOREM 4.5. *Let $\kappa > 1$, $0 < \alpha < 1$, and $p_0 > 1$. Assume that $p, s > 1$ satisfy*

$$p_0 \leq p, \quad \frac{1}{s} = \frac{1}{p} - (1-\alpha). \tag{4.16}$$

Then there exists a constant $C > 0$ depending only on α and p_0 so that, for every $f \in L^p(\mu)$,

$$\|J_{\alpha,\kappa} f : L^s(\mu)\| \leq Cs^{\alpha} \|f : L^p(\mu)\|, \tag{4.17}$$

$$\|I^{\flat}_{\alpha,\kappa} f : L^s(\mu)\| \leq Cs^{\alpha} \|f : L^p(\mu)\|. \tag{4.18}$$

If μ additionally satisfies the growth condition (1.1), then

$$\|I_{\alpha} f : L^s(\mu)\| \leq Cs^{\alpha} \|f : L^p(\mu)\|. \tag{4.19}$$

Proof. We have only to prove (4.18). The rest is immediate once we prove it. We may assume that f is positive. Let $R > 0$ be fixed. We will split $I^{\flat}_{\alpha,\kappa} f(x)$. For fixed $x \in \operatorname{supp}(\mu)$, let us set

$$\mathcal{D}_j := \Big\{ y \in \mathbb{R}^d \setminus \{x\} : 2^{j-1}R < \inf_{x,y \in Q \in \mathcal{Q}(\mu)} \mu(\kappa Q) \leq 2^j R \Big\}, \quad j \in \mathbb{Z}. \tag{4.20}$$

We decompose $I^{\flat}_{\alpha,\kappa} f(x)$ by using the partition $\{\mathcal{D}_j\}_{j=-\infty}^{\infty} \cup \{x\}$ of $\operatorname{supp}(\mu)$. For the time being, we assume that μ charges $\{x\}$. By definition, we have

$$I^{\flat}_{\alpha,\kappa} f(x) = \sum_{j=-\infty}^{0} \int_{\mathcal{D}_j} K^{\flat}_{\alpha,\kappa}(x,y) f(y)\,d\mu(y) + \int_{\bigcup_{j=1}^{\infty} \mathcal{D}_j} K^{\flat}_{\alpha,\kappa}(x,y) f(y)\,d\mu(y) + \mu(\{x\})^{1-\alpha} f(x). \tag{4.21}$$

Suppose that $\mathcal{D}_j$ is nonempty. By the Besicovitch covering lemma, we can find $N \in \mathbb{N}$, independent of x, j, and R, and a collection of cubes $Q_1^j, Q_2^j, \ldots, Q_N^j$ which contain x such that $\mathcal{D}_j \subset \sqrt{\kappa} Q_1^j \cup \sqrt{\kappa} Q_2^j \cup \cdots \cup \sqrt{\kappa} Q_N^j$ and $\mu(\kappa Q_l^j) \leq 2^{j+1} R$ for all $1 \leq l \leq N$ and $j \in \mathbb{Z}$.

From this covering and the definition of $\mathcal{D}_j$, we obtain $\mu(\mathcal{D}_j) \leq c2^j R$. With these observations, it follows that

$$\sum_{j=-\infty}^{0} \int_{\mathcal{D}_j} K^{\flat}_{\alpha,\kappa}(x,y) f(y)\,d\mu(y) \leq c \sum_{j=-\infty}^{0} \sum_{l=1}^{N} \frac{1}{2^{j\alpha} R^{\alpha}} \int_{\sqrt{\kappa} Q_l^j} f(y)\,d\mu(y) \leq cR^{1-\alpha} M_{\sqrt{\kappa}} f(x). \tag{4.22}$$

The estimate of the second term will be accomplished by the Hölder inequality,

$$\begin{aligned}
&\int_{\bigcup_{j=1}^{\infty}\mathcal{D}_j} K^{\flat}_{\alpha,\kappa}(x,y)f(y)d\mu(y)\\
&\quad\le \left(\int_{\bigcup_{j=1}^{\infty}\mathcal{D}_j} K^{\flat}_{\alpha,\kappa}(x,y)^{p'}d\mu(y)\right)^{1/p'} \|f : L^p(\mu)\|\\
&\quad= \left(\sum_{j=1}^{\infty}\int_{\mathcal{D}_j} K^{\flat}_{\alpha,\kappa}(x,y)^{p'}d\mu(y)\right)^{1/p'} \|f : L^p(\mu)\|\\
&\quad\le c\left(\sum_{j=1}^{\infty}(2^jR)^{1-\alpha p'}\right)^{1/p'} \|f : L^p(\mu)\| \le c\left(\alpha-\frac{1}{p'}\right)^{-1/p'} R^{1/p'-\alpha}\|f : L^p(\mu)\|,
\end{aligned} \tag{4.23}$$

where we use an inequality $1/(2^a-1) \le 1/(\log 2\cdot a)$, $a>0$. Taking into account these estimates, we obtain

$$\begin{aligned}
&\sum_{j=-\infty}^{0}\int_{\mathcal{D}_j} K^{\flat}_{\alpha,\kappa}(x,y)f(y)d\mu(y) + \int_{\bigcup_{j=1}^{\infty}\mathcal{D}_j} K^{\flat}_{\alpha,\kappa}(x,y)f(y)d\mu(y)\\
&\qquad\le C_{\alpha,\kappa}\left(R^{1-\alpha}M_{\sqrt{\kappa}}f(x) + R^{-(\alpha-1/p')}\left(\alpha-\frac{1}{p'}\right)^{-1/p'}\|f : L^p(\mu)\|\right).
\end{aligned} \tag{4.24}$$

We have to deal with $\mu(\{x\})^{1-\alpha}f(x)$. If $\mu(\{x\})\le R$, then $\mu(\{x\})^{1-\alpha}f(x)\le R^{1-\alpha}M_{\sqrt{\kappa}}f(x)$. Conversely, if $\mu(\{x\})\ge R$, then $\mu(\{x\})^{1-\alpha}f(x)\le R^{-(\alpha-1/p')}\|f : L^p(\mu)\|$. As a result, we can incorporate $\mu(\{x\})^{1-\alpha}f(x)$ to the above formula. The result is

$$I^{\flat}_{\alpha,\kappa}f(x) \le C_{\alpha,\kappa}\left(R^{1-\alpha}M_{\sqrt{\kappa}}f(x) + R^{-(\alpha-1/p')}\left(\alpha-\frac{1}{p'}\right)^{-1/p'}\|f : L^p(\mu)\|\right) \tag{4.25}$$

for all $R\in(0,\infty)$. Taking

$$R = \left(\frac{(\alpha-1/p')^{-1/p'}\|f : L^p(\mu)\|}{M_{\sqrt{\kappa}}f(x)}\right)^{p}, \tag{4.26}$$

we have

$$I^{\flat}_{\alpha,\kappa}f(x) \le C_{\alpha,\kappa}\left(\alpha-\frac{1}{p'}\right)^{-(1-\alpha)(p-1)} M_{\sqrt{\kappa}}f(x)^{p(\alpha-1/p')}\|f : L^p(\mu)\|^{1-p(\alpha-1/p')}. \tag{4.27}$$

Recall that $1/s=\alpha-1/p'$ by assumption. Thus the above estimate can be restated as

$$I^{\flat}_{\alpha,\kappa}f(x) \le C_{\alpha,\kappa}s^{(1-\alpha)(p-1)}M_{\sqrt{\kappa}}f(x)^{p/s}\|f : L^p(\mu)\|^{1-p/s}. \tag{4.28}$$

Inserting $p(1-\alpha)-1=-p/s$, we see $s^{(1-\alpha)(p-1)}=s^{\alpha-p/s}\leq s^{\alpha}$. As a consequence, we have

$$\|I^{\flat}_{\alpha,\kappa}f:L^s(\mu)\|\leq C_{\alpha,\kappa,p_0}s^{\alpha}\|f:L^p(\mu)\|. \tag{4.29}$$

This is the desired estimate. □

Consequently, if we use Theorem 3.1, then we obtain the following.

Theorem 4.6. *Assume that μ is a finite Radon measure. Let T be either $J_{\alpha,\kappa}$ or $I^{\flat}_{\alpha,\kappa}$ with $0<\alpha<1$ and $\kappa>1$. Then there exists $C>0$ so that, for every $f\in L^{1/(1-\alpha)}(\mu)$,*

$$\|Tf:L^{\Phi}(\mu)\|\leq C\|f:L^{1/(1-\alpha)}(\mu)\|, \tag{4.30}$$

where $\Phi(x)=\exp(x^{1/\alpha})-1$. If μ satisfies the growth condition (1.1), then (4.30) is still available for $T=I_{\alpha}$.

4.3. Morrey estimates. Now we will prove the Morrey estimates associated with fractional integral operators.

Theorem 4.7. *Let $0<\alpha<1$, $0<\beta\leq 1$, $\kappa>1$, and $p_0>1/\beta$. Assume that p and s satisfy*

$$p_0\leq p<\infty,\quad 1<s<\infty,\quad \frac{1}{s}=\frac{1}{p}-(1-\alpha). \tag{4.31}$$

Then there exists a constant $C>0$ depending only on α, β and p_0 so that, for every $f\in \mathcal{M}^p_{\beta p}(\mu)$,

$$\|J_{\alpha,\kappa}f:\mathcal{M}^s_{\beta s}(\mu)\|\leq Cs\|f:\mathcal{M}^p_{\beta p}(\mu)\|, \tag{4.32}$$

$$\|I^{\flat}_{\alpha,\kappa}f:\mathcal{M}^s_{\beta s}(\mu)\|\leq Cs\|f:\mathcal{M}^p_{\beta p}(\mu)\|. \tag{4.33}$$

If μ additionally satisfies the growth condition (1.1), then

$$\|I_{\alpha}f:\mathcal{M}^s_{\beta s}(\mu)\|\leq Cs\|f:\mathcal{M}^p_{\beta p}(\mu)\|. \tag{4.34}$$

Proof. It is enough to prove (4.33) for a positive μ-measurable function f. We have only to make a minor change of the proof of Theorem 4.5. So we indicate the necessary change. Under the notation in the proof of Theorem 4.5, we change the estimate of

$$\int_{\bigcup_{j=1}^{\infty}\mathcal{D}_j}K^{\flat}_{\alpha,\kappa}(x,y)f(y)\,d\mu(y). \tag{4.35}$$

By using the Morrey norm, we obtain

$$
\begin{aligned}
\int_{\bigcup_{j=1}^{\infty}\mathcal{D}_j} & K^{\flat}_{\alpha,\kappa}(x,y)f(y)\,d\mu(y) \\
&= \sum_{j=1}^{\infty}\int_{\mathcal{D}_j} K^{\flat}_{\alpha,\kappa}(x,y)f(y)\,d\mu(y) \\
&\leq c\sum_{j=1}^{\infty}\sum_{l=1}^{N}\frac{1}{2^{j\alpha}R^{\alpha}}\int_{\sqrt{\kappa}Q_l^j} f(y)\,d\mu(y) \\
&\leq c\sum_{j=1}^{\infty}\sum_{l=1}^{N}2^{-j(\alpha-1/p')}R^{-(\alpha-1/p')}||f:\mathcal{M}_1^p(\mu)|| \\
&\leq cR^{-(\alpha-1/p')}\left(\alpha-\frac{1}{p'}\right)||f:\mathcal{M}_{\beta p}^p(\mu)||.
\end{aligned}
\tag{4.36}
$$

Proceeding in the same way as Theorem 4.5, we obtain

$$
I^{\flat}_{\alpha,\kappa}f(x) \leq C_{\alpha,\kappa}\left(R^{1-\alpha}M_{\sqrt{\kappa}}f(x)+R^{1/p'-\alpha}\left(\alpha-\frac{1}{p'}\right)||f:\mathcal{M}_{\beta p}^p(\mu)||\right). \tag{4.37}
$$

Now R is still at our disposal again. Thus, if we put

$$
R=\left(\frac{(\alpha-1/p')||f:\mathcal{M}_{\beta p}^p(\mu)||}{M_{\sqrt{\kappa}}f(x)}\right)^p, \tag{4.38}
$$

we have the pointwise estimate

$$
I^{\flat}_{\alpha,\kappa}f(x) \leq C_{\alpha,\kappa}\left(\alpha-\frac{1}{p'}\right)^{-p(1-\alpha)}M_{\sqrt{\kappa}}f(x)^{p(\alpha-1/p')}||f:\mathcal{M}_{\beta p}^p(\mu)||^{1-p(\alpha-1/p')}. \tag{4.39}
$$

Using $\alpha-1/p'=1/s$, we have $(\alpha-1/p')^{-p(1-\alpha)}=s^{1-p(\alpha-1/p')}=s^{1-p/s}\leq s$. If we insert this estimate, (4.39) is simplified to $I^{\flat}_{\alpha,\kappa}f(x)\leq C_{\alpha,\kappa}sM_{\sqrt{\kappa}}f(x)^{p/s}\|f:\mathcal{M}_{\beta p}^p(\mu)\|^{1-p/s}$. By using the boundedness of $M_{\sqrt{\kappa}}$, we finally have

$$
||I^{\flat}_{\alpha,\kappa}f:\mathcal{M}_{\beta s}^s(\mu)|| \leq C_{\alpha,\kappa,p_0}s||f:\mathcal{M}_{\beta p}^p(\mu)||. \tag{4.40}
$$

This is the desired result. □

If we use our extrapolation machinery, we obtain the following.

Theorem 4.8. *Assume that μ is a finite Radon measure. Let T be either $J_{\alpha,\kappa}$ or $I^{\flat}_{\alpha,\kappa}$ with $0<\alpha<1$, $1-\alpha<\beta\leq 1$, and $\kappa>1$. Then there exists $C>0$ such that*

$$
||Tf:\mathcal{M}_{\beta}^{\Phi}(\mu)|| \leq C\left|\left|f:\mathcal{M}_{\beta/(1-\alpha)}^{1/(1-\alpha)}(\mu)\right|\right| \tag{4.41}
$$

for all $f\in L^{1/(1-\alpha)}(\mu)$, where $\Phi(x)=\exp(x)-1$. If μ satisfies the growth condition (1.1), then (4.41) is still valid for $T=I_{\alpha}$.

5. Sharpness of the results

Finally, we show that Theorems 4.7 and 4.8 are sharp. The notations in this section are valid here only.

Example 5.1. Let $\mu = dx|(0,1)$ be the restriction of one-dimensional Lebesgue measure to $(0,1)$, $n = 1$, $\alpha = 1/2$, and $f(x) = |x|^{-1/2}$.

We claim the following.

Claim 5.2. $f \in \mathcal{M}^2_{2\beta}(\mu)$ for all $0 < \beta < 1$.

Claim 5.3. $I_{1/2} f(x)$ differs from $\log(1/x)$ by some constant C_1 independent of x. In particular,

$$||I_{1/2} f : \mathcal{M}^s_{\beta s}(\mu)|| \geq ||I_{1/2} f : L^{\beta s}(\mu)|| \geq c_\beta s - C_1 \tag{5.1}$$

for all $s \geq 1/\beta$.

Proof of Claim 5.2. By definition of the Morrey norm $\|\cdot : \mathcal{M}^2_{2\beta}(\mu)\|$, we have

$$||f : \mathcal{M}^2_{2\beta}(\mu)|| = \sup_{\substack{Q \in \mathcal{Q}(\mu) \\ Q \subset [0,1]}} \mu(2Q)^{1/2 - 1/2\beta} \left(\int_Q |f(y)|^{2\beta} d\mu(y) \right)^{1/2\beta}. \tag{5.2}$$

Writing it out in full, we obtain

$$||f : \mathcal{M}^2_{2\beta}(\mu)|| \leq \sup_{0 \leq a \leq b \leq 1} (b-a)^{1/2 - 1/2\beta} \left(\int_a^b |x|^{-\beta} dx \right)^{1/2\beta}. \tag{5.3}$$

If $0 \leq a \leq b \leq 1$ satisfies $b - a = h$, then $\int_a^b |x|^{-\beta} dx$ attains its maximum at $a = 0$ and $b = h$. Consequently, we have

$$||f : \mathcal{M}^2_{2\beta}(\mu)|| \leq \sup_{0 \leq h \leq 1} h^{1/2 - 1/2\beta} \left(\int_0^h |x|^{-\beta} dx \right)^{1/2\beta} = (1-\beta)^{-1/2\beta} < \infty. \tag{5.4}$$

Thus Claim 5.2 is proved. □

Proof of Claim 5.3. By definition of $I_{1/2} f$, we have $I_{1/2} f(x) = \int_0^1 (dy/\sqrt{y|x-y|})$. Changing the variables, we can rewrite the integral as $I_{1/2} f(x) = \int_0^{1/x} (dz/\sqrt{z|1-z|})$. With $x < 1$ in mind, we decompose

$$\begin{aligned} I_{1/2} f(x) &= \int_0^1 \frac{dz}{\sqrt{z(1-z)}} + \int_1^{1/x} \frac{dz}{\sqrt{z(z-1)}} \\ &= \int_0^1 \frac{dz}{\sqrt{z(1-z)}} + \int_1^{1/x} \left(\frac{1}{\sqrt{z(z-1)}} - \frac{1}{z} \right) dz + \int_1^{1/x} \frac{dz}{z} \\ &= \int_0^1 \frac{dz}{\sqrt{z(1-z)}} + \int_1^{1/x} \frac{dz}{\sqrt{z^2(z-1)}(\sqrt{z} + \sqrt{z-1})} + \log \frac{1}{x}. \end{aligned} \tag{5.5}$$

The integrals of the last formula remain bounded since

$$\frac{1}{\sqrt{z(1-z)}}, \qquad \frac{1}{\sqrt{z^2(z-1)}(\sqrt{z}+\sqrt{z-1})} \tag{5.6}$$

are Lebesgue-integrable on $(0,1)$ and $(1,\infty)$, respectively. As a consequence, $\log(1/x)$ and $I_{1/2}f(x)$ differ by some absolute constant for all $x \in (0,1)$.

Finally, let us see (5.1). By virtue of the triangle inequality, $(\int_0^1 I_{1/2}f(x)^{\beta s}dx)^{1/\beta s}$ can be bounded from below by

$$\left(\int_0^1 \left(\log\frac{1}{x}\right)^{\beta s} dx\right)^{1/\beta s} - C_1 \geq \left(\int_0^{e^{-s}} \left(\log\frac{1}{x}\right)^{\beta s} dx\right)^{1/\beta s} - C_1 \geq c_\beta s - C_1. \tag{5.7}$$

As a result, Claim 5.3 is proved. □

Corollary 5.4. (1) *One has*

$$||I_{1/2}||_{\mathcal{M}^p_{\beta p}(\mu)\to\mathcal{M}^s_{\beta s}(\mu)} \sim s, \tag{5.8}$$

where the parameters p, s, β satisfy

$$0<\beta<1, \quad 0<p<2, \quad 0<s<\infty, \quad \frac{1}{s}=\frac{1}{p}-\frac{1}{2}, \tag{5.9}$$

where the implicit constants in $\sim$ depend only on β.

(2) *Let $0<\beta,\rho<1$, and $\lambda>0$. Then*

$$\sup_Q \left[\int_Q \left[\exp\left(\lambda\left|\frac{I_{1/2}f(x)}{||f:\mathcal{M}^2_{\beta 2}(\mu)||}\right|^{1/\rho}\right)-1\right]\frac{d\mu(x)}{\mu(2Q)^{1-\beta}}\right]=\infty. \tag{5.10}$$

In particular, Theorem 4.8 is sharp in the sense that the conclusion of Theorem 4.8 fails if Φ is replaced by $\Psi(x)=\exp(x^{1/\rho})-1$.

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