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Research Article

On Refinements of Aczél, Popoviciu, Bellman's Inequalities and Related Results

G. Farid,[1] J. Pečarić,[1,2] and Atiq Ur Rehman[1]

[1] *Abdus Salam School of Mathematical Sciences, GC University, 68-B, New Muslim Town, Lahore 54600, Pakistan*
[2] *Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia*

Correspondence should be addressed to G. Farid, faridphdsms@hotmail.com

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We give some refinements of the inequalities of Aczél, Popoviciu, and Bellman. Also, we give some results related to power sums.

1. Introduction

The well-known Aczél's inequality [1] (see also [2, page 117]) is given in the following result.

Theorem 1.1. *Let n be a fixed positive integer, and let A, B, a_k, b_k $(k = 1, \dots, n)$ be real numbers such that*

$$A^2 - \sum_{k=1}^{n} a_k^2 > 0, \qquad B^2 - \sum_{k=1}^{n} b_k^2 > 0, \tag{1.1}$$

then

$$\left(A^2 - \sum_{k=1}^{n} a_k^2\right)^{1/2} \left(B^2 - \sum_{k=1}^{n} b_k^2\right)^{1/2} \leq AB - \sum_{k=1}^{n} a_k b_k, \tag{1.2}$$

with equality if and only if the sequences $A, a_1, \dots, .a_n$ and $B, b_1, \dots, b_n$ are proportional.

A related result due to Bjelica [3] is stated in the following theorem.

Theorem 1.2. *Let n be a fixed positive integer, and let p, A, B, a_k, b_k $(k = 1, \dots, n)$ be nonnegative real numbers such that*

$$A^p - \sum_{k=1}^{n} a_k^p > 0, \qquad B^p - \sum_{k=1}^{n} b_k^p > 0, \tag{1.3}$$

then, for $0 < p \le 2$, one has

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^p - \sum_{k=1}^{n} b_k^p\right)^{1/p} \le AB - \sum_{k=1}^{n} a_k b_k. \tag{1.4}$$

Note that quotation of the above result in [4, page 58] is mistakenly stated for all $p \ge 1$. In 1990, Bjelica [3] proved that the above result is true for $0 < p \le 2$. Mascioni [5], in 2002, gave the proof for $1 < p \le 2$ and gave the counter example to show that the above result is not true for $p > 2$. Díaz-Barreo et al. [6] mistakenly stated it for positive integer p and gave a refinement of the inequality (1.4) as follows.

Theorem 1.3. *Let n, p be positive integers, and let A, B, a_k, b_k, $(k = 1, \dots, n)$ be nonnegative real numbers such that* (1.3) *is satisfied, then for $1 \le j < n$, one has*

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)\left(B^p - \sum_{k=1}^{n} b_k^p\right) \le R(A, B, a_k, b_k) \le \left(AB - \sum_{k=1}^{n} a_k b_k\right)^p, \tag{1.5}$$

where

$$R(A, B, a_k, b_k) = \left(\sqrt[p]{A^p - \sum_{k=1}^{j} a_k^p}\,\sqrt[p]{B^p - \sum_{k=1}^{j} b_k^p} - \sum_{k=j+1}^{n} a_k b_k\right)^p. \tag{1.6}$$

Moreover, Díaz-Barreo et al. [6] stated the above result as Popoviciu's generalization of Aczél's inequality given in [7]. In fact, generalization of inequality (1.2) attributed to Popoviciu [7] is stated in the following theorem (see also [2, page 118]).

Theorem 1.4. *Let n be a fixed positive integer, and let p, q, A, B, a_k, b_k $(k = 1, \dots, n)$ be nonnegative real numbers such that*

$$A^p - \sum_{k=1}^{n} a_k^p > 0, \qquad B^q - \sum_{k=1}^{n} b_k^q > 0. \tag{1.7}$$

Also, let $1/p + 1/q = 1$, then, for $p > 1$, one has

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^q - \sum_{k=1}^{n} b_k^q\right)^{1/q} \le AB - \sum_{k=1}^{n} a_k b_k. \tag{1.8}$$

If $p < 1$ $(p \ne 0)$, then reverse of the inequality (1.8) *holds.*

The well-known Bellman's inequality is stated in the following theorem [8] (see also [2, pages 118-119]).

Theorem 1.5. *Let n be a fixed positive integer, and let p, A, B, a_k, b_k $(k = 1, \ldots, n)$ be nonnegative real numbers such that* (1.3) *is satisfied. If $p \geq 1$, then*

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(B^p - \sum_{k=1}^{n} b_k^p\right)^{1/p} \leq \left((A+B)^p - \sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/p}. \tag{1.9}$$

Díaz-Barreo et al. [6] gave a refinement of the above inequality for positive integer p. They proved the following result.

Theorem 1.6. *Let n, p be positive integers, and let A, B, a_k, b_k, $(k = 1, \ldots, n)$ be nonnegative real numbers such that* (1.3) *is satisfied, then for $1 \leq j < n$, one has*

$$\begin{aligned} &\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(B^p - \sum_{k=1}^{n} b_k^p\right)^{1/p} \\ &\quad \leq \widetilde{R}(A, B, a_k, b_k) \leq \left((A+B)^p - \sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/p}, \end{aligned} \tag{1.10}$$

where

$$\widetilde{R}(A, B, a_k, b_k) = \left[\left(\sqrt[p]{A^p - \sum_{k=1}^{j} a_k^p} + \sqrt[p]{B^p - \sum_{k=1}^{j} b_k^p}\right)^p - \sum_{k=j+1}^{n} (a_k + b_k)^p\right]^{1/p}. \tag{1.11}$$

In this paper, first we give a simple extension of a Theorem 1.2 with Aczél's inequality. Further, we give refinements of Theorems 1.2, 1.4, and 1.5. Also, we give some results related to power sums.

2. Main Results

To give extension of Theorem 1.2, we will use the result proved by Pečarić and Vasić in 1979 [9, page 165].

Lemma 2.1. *Let p, q, A, a_k $(k = 1, \ldots, n)$ be nonnegative real numbers such that $A^p - \sum_{k=1}^{n} a_k^p > 0$, then for $0 < p \leq q$, one has*

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \leq \left(A^q - \sum_{k=1}^{n} a_k^q\right)^{1/q}. \tag{2.1}$$

Theorem 2.2. *Let n be a fixed positive integer, and let p, A, B, a_k, b_k $(k = 1, \ldots, n)$ be nonnegative real numbers such that* (1.3) *is satisfied, then, for $0 < p \le 2$, one has*

$$\begin{aligned} &\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^p - \sum_{k=1}^{n} b_k^p\right)^{1/p} \\ &\quad \le \left(A^2 - \sum_{k=1}^{n} a_k^2\right)^{1/2} \left(B^2 - \sum_{k=1}^{n} b_k^2\right)^{1/2} \le AB - \sum_{k=1}^{n} a_k b_k. \end{aligned} \tag{2.2}$$

Proof. By using condition (1.3) in Lemma 2.1 for $0 < p \le 2$, we have

$$\begin{aligned} \left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} &\le \left(A^2 - \sum_{k=1}^{n} a_k^2\right)^{1/2}, \\ \left(B^p - \sum_{k=1}^{n} b_k^p\right)^{1/p} &\le \left(B^2 - \sum_{k=1}^{n} b_k^2\right)^{1/2}. \end{aligned} \tag{2.3}$$

These imply

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^p - \sum_{k=1}^{n} b_k^p\right)^{1/p} \le \left(A^2 - \sum_{k=1}^{n} a_k^2\right)^{1/2} \left(B^2 - \sum_{k=1}^{n} b_k^2\right)^{1/2}. \tag{2.4}$$

Now, applying Azcél's inequality on right-hand side of the above inequality gives us the required result. ☐

Let p and q be positive real numbers such that $1/p + 1/q = 1$, then the well-known Hölder's inequality states that

$$\sum_{k=1}^{n} a_k b_k \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}, \tag{2.5}$$

where a_k, b_k $(k = 1, \ldots, n)$ are positive real numbers.

If $0 < p \le q$, then the well-known inequality of power sums of order p and q states that

$$\left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \le \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}, \tag{2.6}$$

where b_k $(k = 1, \ldots, n)$ are positive real numbers (c.f [9, page 165]).

Now, if $1 < p \le 2$, then $q \ge 2$ and using inequality (2.6) in (2.5), we get

$$\sum_{k=1}^{n} a_k b_k \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}. \tag{2.7}$$

We use the inequality (2.7) and the Hölder's inequality to prove the further refinements of the Theorems 1.2 and 1.4.

Theorem 2.3. *Let j and n be fixed positive integers such that $1 \le j < n$, and let p, A, B, a_k, b_k ($k = 1, \dots, n$) be nonnegative real numbers such that* (1.3) *is satisfied. Let one denote*

$$M = \left(A^p - \sum_{k=1}^{j} a_k^p\right)^{1/p}, \qquad N = \left(B^p - \sum_{k=1}^{j} b_k^p\right)^{1/p}. \tag{2.8}$$

(i) *If $0 < p \le 2$, then*

$$\begin{aligned}\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^p - \sum_{k=1}^{n} b_k^p\right)^{1/p} \\ \le MN - \sum_{k=j+1}^{n} a_k b_k \le AB - \sum_{k=1}^{n} a_k b_k.\end{aligned} \tag{2.9}$$

(ii) *If $1 < p \le 2$, then*

$$\begin{aligned}&\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^p - \sum_{k=1}^{n} b_k^p\right)^{1/p} \\ &\le MN - \left(\sum_{k=j+1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=j+1}^{n} b_k^p\right)^{1/p} \le MN - \sum_{k=j+1}^{n} a_k b_k.\end{aligned} \tag{2.10}$$

Proof.

(i) First of all, we observe that $M, N > 0$ and also $0 < p \le 2$, therefore by Theorem 1.2, we have

$$MN \le AB - \sum_{k=1}^{j} a_k b_k. \tag{2.11}$$

We can write

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} = \left(M^p - \sum_{k=j+1}^{n} a_k^p\right)^{1/p} \left(N^p - \sum_{k=j+1}^{n} b_k^p\right)^{1/p}. \tag{2.12}$$

By applying Theorem 1.2 for $0 < p \leq 2$ on right-hand side of the above equation, we get

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \leq MN - \sum_{k=j+1}^{n} a_k b_k. \tag{2.13}$$

By using inequality (2.11) on right-hand side of the above expression follows the required result.

(ii) Since

$$A^p - \sum_{k=1}^{n} a_k^p = A^p - \sum_{k=1}^{j} a_k^p - \sum_{k=j+1}^{n} a_k^p \tag{2.14}$$

and denoting $a = (\sum_{k=j+1}^{n} a_k^p)^{1/p}$, $b = (\sum_{k=j+1}^{n} b_k^p)^{1/p}$,
then

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} = (M^p - a^p)^{1/p}(N^p - b^p)^{1/p}. \tag{2.15}$$

It is given that $M^p - a^p > 0$ and $N^p - b^p > 0$, therefore by using Theorem 1.2, for $n = 1$, on right-hand side of the above equation, we get

$$\begin{aligned} &\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \\ &\quad \leq MN - ab = MN - \left(\sum_{k=j+1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=j+1}^{n} b_k^p\right)^{1/p}, \end{aligned} \tag{2.16}$$

since $1 < p \leq 2$, so by using (2.7)

$$\leq MN - \sum_{k=j+1}^{n} a_k b_k. \tag{2.17}$$

□

Theorem 2.4. *Let j and n be fixed positive integers such that $1 \leq j < n$, and let p, q, A, B, a_k, b_k $(k = 1, \ldots, n)$ be nonnegative real numbers such that* (1.7) *is satisfied. Also let $1/p+1/q = 1$, M be defined in* (2.8) *and*

$$\widetilde{N} = \left(B^q - \sum_{k=1}^{j} b_k^q\right)^{1/q}, \tag{2.18}$$

then, for $p > 1$, one has

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^q - \sum_{k=1}^{n} b_k^q\right)^{1/q} \leq M\widetilde{N} - \left(\sum_{k=j+1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=j+1}^{n} b_k^q\right)^{1/q}$$
$$\leq M\widetilde{N} - \sum_{k=j+1}^{n} a_k b_k \tag{2.19}$$
$$\leq AB - \sum_{k=1}^{n} a_k b_k.$$

Proof. First of all, note that $M, \widetilde{N} > 0$, therefore by generalized Aczél's inequality, we have

$$M\widetilde{N} \leq A\mathrm{B} - \sum_{k=1}^{j} a_k b_k. \tag{2.20}$$

Now,

$$A^p - \sum_{k=1}^{n} a_k^p = A^p - \sum_{k=1}^{j} a_k^p - \sum_{k=j+1}^{n} a_k^p, \tag{2.21}$$

and denote $a = (\sum_{k=j+1}^{n} a_k^p)^{1/p}$, $b = (\sum_{k=j+1}^{n} b_k^q)^{1/p}$.

Then

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^p - \sum_{k=1}^{n} b_k^q\right)^{1/q} = (M^p - a^p)^{1/p} \left(\widetilde{N}^q - b^q\right)^{1/q}. \tag{2.22}$$

It is given that $M^p - a^p > 0$ and $\widetilde{N}^q - b^q > 0$, therefore by using Theorem 1.4, for $n = 1$, on right-hand side of the above equation, we get

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} \left(B^q - \sum_{k=1}^{n} b_k^q\right)^{1/q}$$
$$\leq M\widetilde{N} - ab = M\widetilde{N} - \left(\sum_{k=j+1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=j+1}^{n} b_k^q\right)^{1/q}, \tag{2.23}$$

by applying Hölder's inequality

$$\leq M\widetilde{N} - \sum_{k=j+1}^{n} a_k b_k, \tag{2.24}$$

by using inequality (2.20)

$$\begin{aligned} &\le AB - \sum_{k=1}^{j} a_k b_k - \sum_{k=j+1}^{n} a_k b_k \\ &= AB - \sum_{k=1}^{n} a_k b_k. \end{aligned} \tag{2.25}$$

□

In [6], a refinement of Bellman's inequality is given for positive integer p; here, we give further refinements of Bellman's inequality for real $p \ge 1$. We will use Minkowski's inequality in the proof and recall that, for real $p \ge 1$ and for positive reals a_k, b_k $(k = 1, \ldots, n)$, the Minkowski's inequality states that

$$\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/p} \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}. \tag{2.26}$$

Theorem 2.5. *Let j and n be fixed positive integers such that $1 \le j < n$, and let p, A, B, a_k, b_k $(k = 1, \ldots, n)$ be nonnegative real numbers such that* (1.3) *is satisfied. Also let M and N be defined in* (2.8). *If $p \ge 1$, then*

$$\begin{aligned} &\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(B^p - \sum_{k=1}^{n} b_k^p\right)^{1/p} \\ &\quad \le \left[(M+N)^p - \left\{\left(\sum_{k=j+1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=j+1}^{n} b_k^p\right)^{1/p}\right\}^p\right]^{1/p} \\ &\quad \le \left((M+N)^p - \sum_{k=j+1}^{n}(a_k + b_k)^p\right)^{1/p} \\ &\quad \le \left((A+B)^p - \sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/p}. \end{aligned} \tag{2.27}$$

Proof. First of all, note that $M, N > 0$ and $p \ge 1$, therefore by using Bellman's inequality, we have

$$M + N \le \left((A+B)^p - \sum_{k=1}^{j}(a_k + b_k)^p\right)^{1/p}. \tag{2.28}$$

Now,

$$A^p - \sum_{k=1}^{n} a_k^p = A^p - \sum_{k=1}^{j} a_k^p - \sum_{k=j+1}^{n} a_k^p, \tag{2.29}$$

and denote $a = (\sum_{k=j+1}^{n} a_k^p)^{1/p}$, $b = (\sum_{k=j+1}^{n} b_k^p)^{1/p}$.

Then

$$\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(B^p - \sum_{k=1}^{n} b_k^p\right)^{1/p} = (M^p - a^p)^{1/p} + (N^p - b^p)^{1/p}. \tag{2.30}$$

It is given that $M^p - a^p > 0$ and $N^p - b^p > 0$, therefore by using Bellman's inequality, for $n = 1$, on right-hand side of the above equation, we get

$$\begin{aligned} &\left(A^p - \sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(B^q - \sum_{k=1}^{n} b_k^q\right)^{1/q} \\ &\quad\le \left[(M+N)^p - (a+b)^p\right]^{1/p} = \left[(M+N)^p - \left\{\left(\sum_{k=j+1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=j+1}^{n} b_k^p\right)^{1/p}\right\}^p\right]^{1/p}, \end{aligned} \tag{2.31}$$

by applying Minkowski's inequality

$$\le \left[(M+N)^p - \sum_{k=j+1}^{n} (a_k + b_k)^p\right]^{1/p}, \tag{2.32}$$

and by using inequality (2.28)

$$\begin{aligned} &\le \left[(A+B)^p - \sum_{k=1}^{j} (a_k + b_k)^p - \sum_{k=j+1}^{n} (a_k + b_k)^p\right]^{1/p} \\ &= \left[(A+B)^p - \sum_{k=1}^{n} (a_k + b_k)^p\right]^{1/p}. \end{aligned} \tag{2.33}$$

□

Remark 2.6. In [10], Hu and Xu gave the generalized results related to Theorems 2.4 and 2.5.

3. Some Further Remarks on Power Sums

The following theorem [9, page 152] is very useful to give results related to power sums in connection with results given in [11, 12].

Theorem 3.1. *Let $(x_1,\ldots,x_n)\in I^n$, where $I=(0,a]$ is interval in $\mathbb{R}$ and $x_1-x_2-\cdots-x_n\in I$. Also let $f: I\to\mathbb{R}$ be a function such that $f(x)/x$ is increasing on I, then*

$$f\left(x_1-\sum_{i=2}^{n}x_i\right)\le f(x_1)-\sum_{i=2}^{n}f(x_i). \tag{3.1}$$

Remark 3.2. If $f(x)/x$ is strictly increasing on I, then strict inequality holds in (3.1).

Here, it is important to note that if we consider

$$f(x)=x^{q/p},\quad p,q\in\mathbb{R},\ p\neq 0, \tag{3.2}$$

then $f(x)/x$ is increasing on $(0,\infty)$ for $0<p\le q$. By using it in Theorem 3.1, we get

$$\left(x_1-\sum_{i=2}^{n}x_i\right)^{q/p}\le x_1^{q/p}-\sum_{i=2}^{n}x_i^{q/p}. \tag{3.3}$$

This implies Lemma 2.1 by substitution, $x_i\to x_i^p$.

In this section, we use Theorem 3.1 to give some results related to power sums as given in [11–13], but here we will discuss only the nonweighted case.

In [11], we introduced Cauchy means related to power sums; here, we restate the means without weights.

Let $\mathbf{x}=(x_1,\ldots,x_n)$ be a positive n-tuple, then for $r,s,t\in(0,\infty)$ we defined

$$\begin{gathered}
A^s_{t,r}(\mathbf{x})=\left\{\frac{(r-s)}{(t-s)}\frac{\left(\sum_{i=1}^n x_i^s\right)^{t/s}-\sum_{i=1}^n x_i^t}{\left(\sum_{i=1}^n x_i^s\right)^{r/s}-\sum_{i=1}^n x_i^r}\right\}^{1/(t-r)},\quad t\neq r,\ r\neq s,\ t\neq s,\\
A^s_{s,r}(\mathbf{x})=A^s_{r,s}(\mathbf{x})=\left\{\frac{(r-s)}{s}\frac{\left(\sum_{i=1}^n x_i^s\right)\log\sum_{i=1}^n x_i^s-s\sum_{i=1}^n x_i^s\log x_i}{\left(\sum_{i=1}^n x_i^s\right)^{r/s}-\sum_{i=1}^n x_i^r}\right\}^{1/(s-r)},\quad s\neq r,\\
A^s_{r,r}(\mathbf{x})=\exp\left(\frac{1}{(s-r)}+\frac{\left(\sum_{i=1}^n x_i^s\right)^{r/s}\log\sum_{i=1}^n x_i^s-s\sum_{i=1}^n x_i^r\log x_i}{s\left\{\left(\sum_{i=1}^n x_i^s\right)^{r/s}-\sum_{i=1}^n x_i^r\right\}}\right),\quad s\neq r,\\
A^s_{s,s}(\mathbf{x})=\exp\left(\frac{\left(\sum_{i=1}^n x_i^s\right)\left(\log\sum_{i=1}^n x_i^s\right)^2-s^2\sum_{i=1}^n x_i^s(\log x_i)^2}{2s\left\{\left(\sum_{i=1}^n x_i^s\right)\log\left(\sum_{i=1}^n x_i^s\right)-s\sum_{i=1}^n x_i^s\log x_i\right\}}\right).
\end{gathered} \tag{3.4}$$

We proved that $A^s_{t,r}(\mathbf{x})$ is monotonically increasing with respect to t and r.

In this section, we give exponential convexity of a positive difference of the inequality (3.1) by using parameterized class of functions. We define new means and discuss their relation to the means defined in [11]. Also, we prove mean value theorem of Cauchy type.

It is worthwhile to recall the following.

Definition 3.3. A function $h : (a,b) \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} u_i u_j h(x_i + x_j) \geq 0, \tag{3.5}$$

for all $n \in \mathbb{N}$ and all choices $u_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, and $x_i \in (a,b)$, such that $x_i + x_j \in (a,b)$, $1 \leq i,j \leq n$.

Proposition 3.4. *Let $f : (a,b) \to \mathbb{R}$. The following propositions are equivalent:*

(i) *f is exponentially convex,*

(ii) *f is continuous and*

$$\sum_{i,j=1}^{n} v_i v_j f\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{3.6}$$

for every $v_i \in \mathbb{R}$ and for every $x_i \in (a,b)$, $1 \leq i \leq n$.

Corollary 3.5. *If $h : (a,b) \to (0,\infty)$ is exponentially convex function, then h is a log-convex function.*

3.1. Exponential Convexity

Lemma 3.6. *Let $t \in \mathbb{R}$ and $\varphi_t : (0,\infty) \to \mathbb{R}$ be the function defined as*

$$\varphi_t(x) = \begin{cases} \dfrac{x^t}{(t-1)}, & t \neq 1, \\ x \log x, & t = 1, \end{cases} \tag{3.7}$$

then $\varphi_t(x)/x$ is strictly increasing function on $(0,\infty)$ for each $t \in \mathbb{R}$.

Proof. Since

$$\left(\frac{\varphi_t(x)}{x}\right)' = x^{t-2} > 0, \quad \forall x \in (0,\infty), \tag{3.8}$$

therefore $\varphi_t(x)/x$ is strictly increasing function on $(0,\infty)$ for each $t \in \mathbb{R}$. □

Theorem 3.7. *Let $\mathbf{x} = (x_1,\ldots,x_n)$ be a positive n-tuple ($n \geq 2$) such that $x_1 - x_2 - \cdots - x_n > 0$, and let*

$$\Lambda_t(\mathbf{x}) = \varphi_t(x_1) - \sum_{i=2}^{n} \varphi_t(x_i) - \varphi_t\left(x_1 - \sum_{i=2}^{n} x_i\right). \tag{3.9}$$

(a) *For $m \in \mathbb{N}$, let $p_1, \dots, p_m$ be arbitrary real numbers, then the matrix*

$$\left[\Lambda_{(p_i+p_j)/2}\right] \quad \text{where } 1 \le i, j \le m \tag{3.10}$$

is a positive semidefinite matrix.

(b) *The function $t \mapsto \Lambda_t,\ t \in \mathbb{R}$ is exponentially convex.*

(c) *The function $t \mapsto \Lambda_t,\ t \in \mathbb{R}$ is log convex.*

Proof. (a) Define a function

$$F(x) = \sum_{i,j=1}^{n} u_i u_j \varphi_{p_{ij}}(x), \quad \text{where } p_{ij} = \frac{(p_i + p_j)}{2}, \tag{3.11}$$

then

$$\left(\frac{F(x)}{x}\right)' = \left(\sum_{i=1}^{n} u_i x^{(p_i-2)/2}\right)^2 \ge 0 \quad \forall x \in (0, \infty). \tag{3.12}$$

This implies that $F(x)/x$ is increasing function on $(0, \infty)$. So using F in the place of f in (3.1), we have

$$\sum_{i,j=1}^{n} u_i u_j \Lambda_{\varphi_{p_{ij}}} \ge 0. \tag{3.13}$$

Hence, the given matrix is positive semidefinite.

(b) Since after some computation we have that $\lim_{t\to 1}\Lambda_t = \Lambda_1$ so $t \mapsto \Lambda_t$ is continuous on $\mathbb{R}$, then by Proposition 3.4, we have that $t \mapsto \Lambda_t$ is exponentially convex.

(c) Since $\varphi_t(x)/x$ is strictly increasing function on $(0, \infty)$, so by Remark 3.2, we have

$$\varphi_t\left(x_1 - \sum_{i=2}^{n} x_i\right) < \varphi_t(x_1) - \sum_{i=2}^{n} \varphi_t(x_i), \tag{3.14}$$

it follows that $\Lambda_t(\mathbf{x}) > 0$. Now, by Corollary 3.5, we have that $t \mapsto \Lambda_t$ is log convex. □

Let us introduce the following.

Definition 3.8. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a positive n-tuple ($n \geq 2$) such that $x_1^s - x_2^s - \cdots - x_n^s > 0$ for $s \in (0, \infty)$, then for $t, r, s \in (0, \infty)$, we define

$$C_{t,r}^s(\mathbf{x}) = \left\{ \frac{(r-s)}{(t-s)} \frac{x_1^t - \sum_{i=2}^n x_i^t - \left(x_1^s - \sum_{i=2}^n x_i^s\right)^{t/s}}{x_1^r - \sum_{i=2}^n x_i^r - \left(x_1^s - \sum_{i=2}^n x_i^s\right)^{r/s}} \right\}^{1/(t-r)}, \quad t \neq r,\ r \neq s,\ t \neq s,$$

$$\begin{aligned} &C_{s,r}^s(\mathbf{x}) \\ &= C_{r,s}^s(\mathbf{x}) \\ &= \left(\frac{(r-s)}{s} \frac{s x_1^s \log x_1 - s \sum_{i=2}^n x_i^s \log x_i - \left(x_1^s - \sum_{i=2}^n x_i^s\right) \log\left(x_1^s - \sum_{i=2}^n x_i^s\right)}{x_1^r - \sum_{i=2}^n x_i^r - \left(x_1^s - \sum_{i=2}^n x_i^s\right)^{r/s}} \right)^{1/(s-r)}, \\ &\qquad s \neq r, \\ &C_{r,r}^s(\mathbf{x}) \\ &= \exp\left(\frac{1}{(s-r)} + \frac{s x_1^r \log x_1 - s \sum_{i=2}^n x_i^r \log x_i - \left(x_1^s - \sum_{i=2}^n x_i^s\right)^{r/s} \log\left(x_1^s - \sum_{i=1}^n x_i^s\right)}{s\left\{x_1^r - \sum_{i=2}^n x_i^r - \left(x_1^s - \sum_{i=2}^n x_i^s\right)^{r/s}\right\}} \right), \\ &\qquad s \neq r, \\ &C_{s,s}^s(\mathbf{x}) \\ &= \exp\left(\frac{s^2 x_1^s (\log x_1)^2 - s^2 \sum_{i=2}^n x_i^s (\log x_i)^2 - \left(x_1^s - \sum_{i=2}^n x_i^s\right)\left(\log\left(x_1^s - \sum_{i=2}^n x_i^s\right)\right)^2}{2s\left\{s x_1^s \log x_1 - s \sum_{i=2}^n x_i^s \log x_i - \left(x_1^s - \sum_{i=2}^n x_i^s\right) \log\left(x_1^s - \sum_{i=2}^n x_i^s\right)\right\}} \right). \end{aligned} \tag{3.15}$$

Remark 3.9. Let us note that $C_{s,r}^s(\mathbf{x}) = C_{r,s}^s(\mathbf{x}) = \lim_{t \to s} C_{t,r}^s(\mathbf{x}) = \lim_{t \to s} C_{\mathrm{r},t}^s(\mathbf{x})$, $C_{r,r}^s(\mathbf{x}) = \lim_{t \to r} C_{t,r}^s(\mathbf{x})$, and $C_{s,s}^s(\mathbf{x}) = \lim_{r \to s} C_{r,r}^s(\mathbf{x})$.

Remark 3.10. If in $C_{t,r}^s(\mathbf{x})$ we substitute x_1 by $\left(\sum_{i=1}^n x_i^s\right)^{1/s}$, then we get $A_{t,r}^s(\mathbf{x})$, and if we substitute x_1 by $\left(x_i^s - \sum_{i=2}^n x_i^s\right)^{1/s}$ in $A_{t,r}^s(\mathbf{x})$, we get $C_{t,r}^s(\mathbf{x})$.

In [11], we have the following lemma.

Lemma 3.11. *Let f be a log-convex function and assume that if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)} \right)^{1/(x_2 - x_1)} \leq \left(\frac{f(y_2)}{f(y_1)} \right)^{1/(y_2 - y_1)}. \tag{3.16}$$

Theorem 3.12. *Let $\mathbf{x} = (x_1, \ldots, x_n)$ be positive n-tuple ($n \geq 2$) such that $x_1^s - x_2^s - \cdots - x_n^s > 0$ for $s \in (0, \infty)$, then for $r, t, u, v \in (0, \infty)$ such that $r \leq u$, $t \leq v$, one has*

$$C_{t,r}^s(\mathbf{x}) \leq C_{v,u}^s(\mathbf{x}). \tag{3.17}$$

Proof. Let Λ_s be defined by (3.9). Now taking $x_1 = r$, $x_2 = t$, $y_1 = u$, $y_2 = v$, where $r \neq t$, $u \neq v$, $r, t, u, v \neq 1$, and $f(s) = \Lambda_s$ in Lemma 3.11, we have

$$\begin{aligned}&\left(\frac{(r-1)}{(t-1)}\frac{x_1^t - \sum_{i=2}^n x_i^t - \left(x_1 - \sum_{i=2}^n x_i\right)^t}{x_1^r - \sum_{i=2}^n x_i^r - \left(x_1 - \sum_{i=2}^n x_i\right)^r}\right)^{1/(t-r)} \\ &\quad\leq \left(\frac{(u-1)}{(v-1)}\frac{x_1^v - \sum_{i=2}^n x_i^v - \left(x_1 - \sum_{i=2}^n x_i\right)^v}{x_1^u - \sum_{i=2}^n x_i^u - \left(x_1 - \sum_{i=2}^n x_i\right)^u}\right)^{1/(v-u)}.\end{aligned} \tag{3.18}$$

Since $s > 0$, by substituting $x_i = x_i^s$, $t = t/s$, $r = r/s$, $u = u/s$, and $v = v/s$, where $r, t, u, v \neq s$, in above inequality, we get

$$\begin{aligned}&\left(\frac{(r-s)}{(t-s)}\frac{x_1^t - \sum_{i=2}^n x_i^t - \left(x_1^s - \sum_{i=2}^n x_i^s\right)^{t/s}}{x_1^r - \sum_{i=2}^n x_i^r - \left(x_1^s - \sum_{i=2}^n x_i^s\right)^{r/s}}\right)^{s/(t-r)} \\ &\quad\leq \left(\frac{(u-s)}{(v-s)}\frac{x_1^v - \sum_{i=2}^n x_i^v - \left(x_1^s - \sum_{i=2}^n x_i^s\right)^{v/s}}{x_1^u - \sum_{i=2}^n x_i^u - \left(x_1^s - \sum_{i=2}^n x_i^s\right)^{u/s}}\right)^{s/(v-u)}.\end{aligned} \tag{3.19}$$

By raising power $1/s$, we get (3.17) for $r, t, u, v \neq s$, $r \neq t$ and $u \neq v$.

From Remark 3.9, we get that (3.17) is also valid for $r = t$ or $u = v$ or $r, t, u, v = s$. □

Remark 3.13. If we substitute x_1 by $\left(\sum_{i=1}^n x_i^s\right)^{1/s}$, then monotonicity of $C_{t,r}^s(\mathbf{x})$ implies the monotonicity of $A_{t,r}^s(\mathbf{x})$, and if we substitute x_1 by $\left(x_i^s - \sum_{i=2}^n x_i^s\right)^{1/s}$, then monotonicity of $A_{t,r}^s(\mathbf{x})$ implies monotonicity of $C_{t,r}^s(\mathbf{x})$.

3.2. Mean Value Theorems

We will use the following lemma [11] to prove the related mean value theorems of Cauchy type.

Lemma 3.14. *Let $f \in C^1(I)$, where $I = (0, a]$ such that*

$$m \leq \frac{x f'(x) - f(x)}{x_2} \leq M. \tag{3.20}$$

Consider the functions ϕ_1, ϕ_2 defined as

$$\begin{aligned}\phi_1(x) &= Mx^2 - f(x), \\ \phi_2(x) &= f(x) - mx^2,\end{aligned} \tag{3.21}$$

then $\phi_i(x)/x$ for $i = 1, 2$ are monotonically increasing functions.

Theorem 3.15. *Let $(x_1,\ldots,x_n)\in I^n$, where I is a compact interval such that $I\subseteq(0,\infty)$ and $x_1-x_2-\cdots-x_n\in I$. If $f\in C^1(I)$, then there exists $\xi\in I$ such that*

$$\begin{aligned} f(x_1)-\sum_{i=2}^{n}f(x_i)-f\left(x_1-\sum_{i=2}^{n}f(x_i\ )\right) \\ =\frac{\xi f'(\xi)-f(\xi)}{\xi^2}\left\{x_1^2-\sum_{i=2}^{n}x_i^2-\left(x_1-\sum_{i=2}^{n}x_i\right)^2\right\}. \end{aligned}\tag{3.22}$$

Proof. Since I is compact and $f\in C(I)$, therefore let

$$M=\max\left\{\frac{xf'(x)-f(x)}{x^2}:x\in I\right\},\qquad m=\min\left\{\frac{xf'(x)-f(x)}{x^2}:x\in I\right\}.\tag{3.23}$$

In Theorem 3.1, setting $f=\phi_1$ and $f=\phi_2$, respectively, as defined in Lemma 3.14, we get the following inequalities:

$$\begin{aligned} f(x_1)\ -\sum_{i=2}^{n}f(x_i)-f\left(x_1-\sum_{i=2}^{n}x_i\right)\le M\left\{x_1^2-\sum_{i=2}^{n}x_i^2-\left(x_1-\sum_{i=2}^{n}x_i\right)^2\right\}, \\ f(x_1)-\sum_{i=2}^{n}f(x_i)-f\left(x_1-\sum_{i=2}^{n}x_i\right)\ge m\left\{x_1^2-\sum_{i=2}^{n}x_i^2-\left(x_1-\sum_{i=2}^{n}x_i\right)^2\right\}. \end{aligned}\tag{3.24}$$

If $f(x)=x^2$, then $f(x)/x$ is strictly increasing function on I, therefore by Theorem 3.1, we have

$$x_1^2-\sum_{i=2}^{n}x_i^2-\left(x_1-\sum_{i=2}^{n}x_i\right)^2>0.\tag{3.25}$$

Now, by combining inequalities (3.24), we get

$$m\le\frac{f(x_1)-\sum_{i=2}^{n}f(x_i)-f\left(x_1-\sum_{i=2}^{n}x_i\right)}{x_1^2-\sum_{i=2}^{n}x_i^2-\left(x_1-\sum_{i=2}^{n}x_i\right)^2}\le M.\tag{3.26}$$

Finally, by condition (3.20), there exists $\xi\in I$, such that

$$\frac{f(x_1)-\sum_{i=2}^{n}f(x_i)-f\left(x_1-\sum_{i=2}^{n}x_i\right)}{x_1^2-\sum_{i=2}^{n}x_i^2-\left(x_1-\sum_{i=2}^{n}x_i\right)^2}=\frac{\xi f'(\xi)-f(\xi)}{\xi^2}\tag{3.27}$$

as required. □

Theorem 3.16. *Let $(x_1, \ldots, x_n) \in I^n$, where I is a compact interval such that $I \subseteq (0,\infty)$ and $x_1 - x_2 - \cdots - x_n \in I$. If $f, g \in C^1(I)$, then there exists $\xi \in I$ such that the following equality is true:*

$$\frac{f(x_1) - \sum_{i=2}^{n} f(x_i) - f\left(x_1 - \sum_{i=2}^{n} x_i\right)}{g(x_1) - \sum_{i=2}^{n} g(x_i) - g\left(x_1 - \sum_{i=2}^{n} x_i\right)} = \frac{\xi f'(\xi) - f(\xi)}{\xi g'(\xi) - g(\xi)} \tag{3.28}$$

provided that the denominators are nonzero.

Proof. Let a function $k \in C^1(I)$ be defined as

$$k = c_1 f - c_2 g, \tag{3.29}$$

where c_1 and c_2 are defined as

$$\begin{aligned} c_1 &= g(x_1) - \sum_{i=2}^{n} g(x_i) - g\left(x_1 - \sum_{i=2}^{n} x_i\right), \\ c_2 &= f(x_1) - \sum_{i=2}^{n} f(x_i) - f\left(x_1 - \sum_{i=2}^{n} x_i\right). \end{aligned} \tag{3.30}$$

Then, using Theorem 3.15, with $f = k$, we have

$$0 = \left(\frac{c_1(\xi f'(\xi) - f(\xi))}{\xi^2} - \frac{c_2(\xi g'(\xi) - g(\xi))}{\xi^2}\right)\left\{x_1^2 - \sum_{i=2}^{n} x_i^2 - \left(x_1 - \sum_{i=2}^{n} x_i\right)^2\right\}. \tag{3.31}$$

Since $x_1^2 - \sum_{i=2}^{n} x_i^2 - (x_1 - \sum_{i=2}^{n} x_i)^2 > 0$, therefore (3.31) gives

$$\frac{c_2}{c_1} = \frac{\xi f'(\xi) - f(\xi)}{\xi g'(\xi) - g(\xi)}. \tag{3.32}$$

Putting in (3.30), we get (3.28). □

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