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Research Article

A Limit Theorem for the Moment of Self-Normalized Sums

Qing-pei Zang

Department of Mathematics, Huaiyin Teachers College, Huaian 223300, China

Correspondence should be addressed to Qing-pei Zang, zqphunhu@yahoo.com.cn

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Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed (*i.i.d.*) random variables and X is in the domain of attraction of the normal law and $EX = 0$. For $1 \leq p < 2, b > -1$, we prove the precise asymptotics in Davis law of large numbers for $\sum_{n=1}^{\infty}((\log n)^b/n)E\{(|S_n|/V_n) - \varepsilon(2\log n)^{(2-p)/(2p)}\}_+$ as $\varepsilon \searrow 0$.

1. Introduction and Main Result

Throughout this paper, we let $\{X, X_n; n \geq 1\}$ be a sequence of *i.i.d.* random variables and X is in the domain of attraction of the normal law and $EX = 0$. Put

$$S_n = \sum_{k=1}^{n} X_k, \qquad V_n^2 = \sum_{i=1}^{n} X_i^2. \tag{1.1}$$

Also let $\log n = \ln(n \vee e)$. Then by the well-known Davis laws of large numbers [1],

$$\sum_{n=1}^{\infty} \frac{\log n}{n} P\left(|S_n| \geq \varepsilon\sqrt{n\log n}\right) < \infty, \quad \varepsilon > 0, \tag{1.2}$$

if and only if $EX = 0$ and $EX^2 < \infty$.

Gut and Spătaru [2] proved its precise asymptotics as follows.

Theorem A. *Suppose that* $EX_1 = 0$ *and* $EX_1^2 = \sigma^2 < \infty$. *Then for* $0 \le \delta \le 1$,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(\delta+1)} \sum_{n=1}^{\infty} \frac{(\log n)^{\delta}}{n} P\left(|S_n| \ge \varepsilon\sqrt{n \log n}\right) = \frac{\mu^{(2\delta+2)}}{\delta+1}\sigma^{2\delta+2}, \tag{1.3}$$

where $\mu^{(2\delta+2)}$ *stands for the* $(2\delta+2)th$ *absolute moment of the standard normal distribution.*

It is well known that, for $i.i.d.$ random variables, Chow [3] discussed the complete moment convergence, and got the following result.

Theorem B. *Let* $\{X, X_n; n \ge 1\}$ *be a sequence of* $i.i.d.$ *random variables with* $EX_1 = 0$. *Assume* $p \ge 1$, $\alpha > 1/2$, $p\alpha > 1$, *and* $E(|X|^p + |X| \log(1+|X|)) < \infty$. *Then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} E\left\{\max_{j \le n} |S_j| - \varepsilon n^{\alpha}\right\}_{+} < \infty. \tag{1.4}$$

On the other hand, the past decade has witnessed a significant development on the limit theorems for the so-called self-normalized sum $S_n/V_n, V_n = \sqrt{\sum_{i=1}^{n} X_i^2}$. Bentkus and Götze [4] obtained Berry-Esseen inequalities for self-normalized sums. Wang and Jing [5] derived exponential nonuniform Berry-Esseen bound. Giné et al. [6], established asymptotic normality of self-normalized sums.

Theorem C. *Let* $\{X, X_n; n \ge 1\}$ *be a sequence of* $i.i.d.$ *random variables with* $EX_1 = 0$. *Then for any* $x \in \mathbb{R}$,

$$\lim_{n \to \infty} P\left(\frac{S_n}{V_n} \le x\right) = \Phi(x) \tag{1.5}$$

holds, if and only if X *is in the domain of attraction of the normal law, where* $\Phi(x)$ *is the distribution function of the standard normal random variable.*

Shao [7] showed a self-normalization large deviation result for $P(S_n/V_n \ge x\sqrt{n})$ without any moment conditions.

Theorem D. *Let* $\{x_n; n \ge 1\}$ *be a sequence of positive numbers with* $x_n \to \infty$ *and* $x_n = o(\sqrt{n})$ *as* $n \to \infty$. *If* $EX = 0$ *and* $EX^2 I(|X| \le x)$ *is slowly varying as* $x \to \infty$, *then*

$$\lim_{n \to \infty} x_n^{-2} \ln P\left(\frac{S_n}{V_n} \ge x_n\right) = -\frac{1}{2}. \tag{1.6}$$

Since then, many subsequent developments of self-normalized sums have been obtained. For example, Csörgő et al. [8] have established Darling-Erdös theorem for self-normalized sums, and they [9] have also obtained Donsker's theorem for self-normalized partial sums processes.

Inspired by the above results, in this note we study the precise asymptotics in Davis law of large numbers for the moment of self-normalized sums. Our main result is as follows.

Theorem 1.1. *Suppose X is in the domain of attraction of the normal law and $EX = 0$. Then, for $b > -1$ and $1 \le p < 2$, one has*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2p(b+1)/(2-p)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} E\left\{ \frac{|S_n|}{V_n} - \varepsilon (2\log n)^{(2-p)/(2p)} \right\}_+ = \frac{2^{-b-1}(2-p)}{(b+1)(2pb+p+2)} E|N|^{(2pb+p+2)/(2-p)}, \tag{1.7}$$

here and in the sequel, N is the standard normal random variable.

Remark 1.2. If $p = 1$ and $0 < \sigma^2 = EX^2 < \infty$, by the strong law of large numbers, we have $V_n^2/n \to \sigma^2$, *a.s.* Then, we can easily obtain the following result:

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n^{3/2}} E\left\{ |S_n| - \varepsilon\sigma\sqrt{2n\log n} \right\}_+ = \frac{\sigma 2^{-b-1}}{(b+1)(2b+3)} E|N|^{2b+3}. \tag{1.8}$$

Remark 1.3. As is well known, the strong approximation method is taken in order to obtain such an analogous result, however, this method is not applicable here.

2. Proof of Theorem 1.1

In this section, we set $A(\varepsilon) = \exp(M/\varepsilon^{2p/(2-p)})$, for $M > 1$ and $\varepsilon > 0$. Here and in the sequel, C will denote positive constants, possibly varying from place to place, and $[x]$ means the largest integer $\le x$. The proof of Theorem 1.1 is based on the following propositions.

Proposition 2.1. *For $b > -1$, one has*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2p(b+1)/(2-p)} \sum_{n=1}^{\infty} \frac{(\log n)^b}{n} E\left\{ |N| - \varepsilon (2\log n)^{(2-p)/(2p)} \right\}_+ = \frac{2^{-b-1}(2-p)}{(b+1)(2pb+p+2)} E|N|^{(2pb+p+2)/(2-p)}. \tag{2.1}$$

Proof. Via the change of variable $y = \varepsilon(2\log t)^{(2-p)/(2p)}$, we have

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{2p(b+1)/(2-p)}\sum_{n=1}^{\infty}\frac{(\log n)^b}{n}E\left\{|N|-\varepsilon\left(2\log n\right)^{(2-p)/(2p)}\right\}_+\\
&\quad=\lim_{\varepsilon\searrow 0}\varepsilon^{2p(b+1)/(2-p)}\sum_{n=1}^{\infty}\frac{(\log n)^b}{n}\int_{\varepsilon(2\log n)^{(2-p)/(2p)}}^{\infty}P\left(|N|\geq x\right)dx\\
&\quad=\lim_{\varepsilon\searrow 0}\varepsilon^{2p(b+1)/(2-p)}\int_{e}^{\infty}\frac{(\log t)^b}{t}\int_{\varepsilon(2\log t)^{(2-p)/(2p)}}^{\infty}P\left(|N|\geq x\right)dx\,dt\\
&\quad=\lim_{\varepsilon\searrow 0}\frac{p2^{-b}}{2-p}\int_{\varepsilon 2^{(2-p)/(2p)}}^{\infty}y^{(2p/(2-p))(b+1)-1}\int_{y}^{\infty}P\left(|N|\geq x\right)dx\,dy\\
&\quad=\lim_{\varepsilon\searrow 0}\frac{p2^{-b}}{2-p}\int_{\varepsilon 2^{(2-p)/(2p)}}^{\infty}P\left(|N|\geq x\right)\int_{\varepsilon 2^{(2-p)/(2p)}}^{x}y^{(2p/(2-p))(b+1)-1}dy\,dx\\
&\quad=\lim_{\varepsilon\searrow 0}\frac{2^{-b-1}}{(b+1)}\int_{\varepsilon 2^{(2-p)/(2p)}}^{\infty}P\left(|N|\geq x\right)\left(x^{(2p/(2-p))(b+1)}-\varepsilon^{(2p/(2-p))(b+1)}\cdot 2^{b+1}\right)dx\\
&\quad=\lim_{\varepsilon\searrow 0}\frac{2^{-b-1}}{(b+1)}\int_{\varepsilon 2^{(2-p)/(2p)}}^{\infty}x^{(2p/(2-p))(b+1)}P\left(|N|\geq x\right)dx\\
&\quad=\frac{2^{-b-1}\left(2-p\right)}{(b+1)\left(2pb+p+2\right)}E|N|^{(2pb+p+2)/(2-p)}.
\end{aligned}\tag{2.2}$$

□

Proposition 2.2. *For $b > -1$, one has*

$$\lim_{\varepsilon\searrow 0}\varepsilon^{2p(b+1)/(2-p)}\sum_{n\leq A(\varepsilon)}\frac{(\log n)^b}{n}\left|E\left\{\frac{|S_n|}{V_n}-\varepsilon\left(2\log n\right)^{(2-p)/(2p)}\right\}_+-E\left\{|N|-\varepsilon\left(2\log n\right)^{(2-p)/(2p)}\right\}_+\right|=0.\tag{2.3}$$

Proof. Set $\Delta_n = \sup_{x\in\mathbb{R}}|P((|S_n|)/V_n \geq x) - P(|N| \geq x)|$. Then, by (1.5), it is easy to see $\Delta_n \to 0$ as $n \to \infty$. Observe that

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\varepsilon^{2p(b+1)/(2-p)}\sum_{n\leq A(\varepsilon)}\frac{(\log n)^b}{n}\left|E\left\{\frac{|S_n|}{V_n}-\varepsilon\left(2\log n\right)^{(2-p)/(2p)}\right\}_+-E\left\{|N|-\varepsilon\left(2\log n\right)^{(2-p)/(2p)}\right\}_+\right|\\
&\quad=\lim_{\varepsilon\searrow 0}\varepsilon^{2p(b+1)/(2-p)}\sum_{n\leq A(\varepsilon)}\frac{(\log n)^b}{n}\\
&\qquad\times\left|\int_0^{\infty}P\left(\frac{|S_n|}{V_n}\geq x+\varepsilon\left(2\log n\right)^{(2-p)/(2p)}\right)dx-\int_0^{\infty}P\left(|N|\geq x+\varepsilon\left(2\log n\right)^{(2-p)/(2p)}\right)dx\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{\varepsilon\searrow 0}\varepsilon^{2p(b+1)/(2-p)}\sum_{n\leq A(\varepsilon)}\frac{(\log n)^{b}}{n}\int_{0}^{\infty}\left|P\left(\frac{|S_n|}{V_n}\geq x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)\right.\\
&\qquad\qquad\left.-\int_{0}^{\infty}P\left(|N|\geq x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)\right|dx\\
&\leq \lim_{\varepsilon\searrow 0}\varepsilon^{2p(b+1)/(2-p)}\sum_{n\leq A(\varepsilon)}\frac{(\log n)^{b}}{n}\left(\Delta_{n1}+\Delta_{n2}+\Delta_{n3}+\Delta_{n4}\right),
\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}
\Delta_{n1}&=\int_{0}^{\min(\log n,1/\sqrt{\Delta_n})}\left|P\left(\frac{|S_n|}{V_n}\geq x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)\right.\\
&\qquad\left.-P\left(|N|\geq x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)\right|dx,\\
\Delta_{n2}&=\int_{\min(\log n,1/\sqrt{\Delta_n})}^{n^{1/4}}\left|P\left(\frac{|S_n|}{V_n}\geq x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)\right.\\
&\qquad\left.-P\left(|N|\geq x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)\right|dx,\\
\Delta_{n3}&=\int_{n^{1/4}}^{n^{1/2}}\left|P\left(\frac{|S_n|}{V_n}\geq x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)-P\left(|N|\geq x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)\right|dx,\\
\Delta_{n4}&=\int_{n^{1/2}}^{\infty}\left|P\frac{|S_n|}{V_n}\geq x+\varepsilon(2\log n)^{(2-p)/(2p)}-P\left(|N|\geq x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)\right|dx.
\end{aligned}\tag{2.5}$$

Thus for Δ_{n1}, it is easy to see

$$\Delta_{n1}\leq\sqrt{\Delta_n}\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.\tag{2.6}$$

Now we are in a position to estimate Δ_{n2}. From (1.6), and by applying $-X_i's$ to it, we can obtain that for large enough n and any $0<a\leq 1/4$, there exist C and b such that $P(|S_n|/V_n>x)\leq Ce^{-((1/2)-a)x^2}$ for $b<x<n^{1/2}/b$. In particular, for $b<x<n^{1/2}/b$, there exists $C>0$ such that

$$P\left(\frac{|S_n|}{V_n}>x\right)\leq Ce^{-x^2/4}.\tag{2.7}$$

Hence, by Markov's inequality and (2.7), we have

$$\begin{aligned}\Delta_{n2} &\le \int_{\min(\log n, 1/\sqrt{\Delta_n})}^{n^{1/4}} e^{-(x+\varepsilon(2\log n)^{(2-p)/2p})^2/4}dx + \int_{\min(\log n, 1/\sqrt{(\Delta_n)})}^{n^{1/4}} \frac{C}{\left(x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)^2}dx \\ &\le \int_{\min(\log n, 1/\sqrt{\Delta_n})}^{n^{1/4}} e^{-x^2/4}dx + \int_{\min(\log n, 1/\sqrt{(\Delta_n)})}^{n^{1/4}} \frac{C}{x^2}dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{2.8}$$

For Δ_{n3}, by Markov's inequality and (2.7), we have

$$\begin{aligned}\Delta_{n3} &\le \int_{n^{1/4}}^{n^{1/2}} P\left(\frac{|S_n|}{V_n} \ge n^{1/4}\right)dx + \int_{n^{1/4}}^{n^{1/2}} \frac{C}{\left(x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)^2}dx \\ &\le e^{-\sqrt{n}/4}\left(n^{1/2}-n^{1/4}\right) + \int_{n^{1/4}}^{n^{1/2}} \frac{C}{x^2}dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{2.9}$$

From Cauchy inequality, it follows that

$$\frac{|S_n|}{V_n} \le \sqrt{n}. \tag{2.10}$$

Therefore

$$\begin{aligned}\Delta_{n4} &= \int_{n^{1/2}}^{\infty} P\left(|N| \ge x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)dx \\ &\le \int_{n^{1/2}}^{\infty} \frac{C}{\left(x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)^2}dx \\ &\le \int_{n^{1/2}}^{\infty} \frac{C}{x^2}dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{2.11}$$

Denote $\Delta_n' = \Delta_{n1} + \Delta_{n2} + \Delta_{n3} + \Delta_{n4}$, then, since the weighted average of a sequence that converges to 0 also converges to 0, it follows that, for any $M > 1$,

$$\begin{aligned}&\lim_{\varepsilon\searrow 0} \varepsilon^{2p(b+1)/(2-p)} \sum_{n\le A(\varepsilon)} \frac{(\log n)^b}{n} \left| E\left\{\frac{|S_n|}{V_n} - \varepsilon(2\log n)^{(2-p)/(2p)}\right\}_+ - E\left\{|N| - \varepsilon(2\log n)^{(2-p)/(2p)}\right\}_+ \right| \\ &\quad \le \lim_{\varepsilon\searrow 0} \varepsilon^{2p(b+1)/(2-p)} \sum_{n\le A(\varepsilon)} \frac{(\log n)^b}{n} \Delta_n' \longrightarrow 0, \quad \text{as } \varepsilon \searrow 0.\end{aligned} \tag{2.12}$$

The proof is completed. □

Proposition 2.3. *For $b > -1$, one has*

$$\lim_{M\to\infty}\lim_{\varepsilon\searrow 0}\varepsilon^{2p(b+1)/(2-p)}\sum_{n>A(\varepsilon)}\frac{(\log n)^b}{n}E\left\{|N|-\varepsilon(2\log n)^{(2-p)/(2p)}\right\}_+=0. \tag{2.13}$$

Proof. Note that

$$\begin{aligned}
&\varepsilon^{2p(b+1)/(2-p)}\sum_{n>A(\varepsilon)}\frac{(\log n)^b}{n}E\left\{|N|-\varepsilon(2\log n)^{(2-p)/(2p)}\right\}_+\\
&\quad\le \varepsilon^{2p(b+1)/(2-p)}\int_{A(\varepsilon)}^{\infty}\frac{(\log n)^b}{t}\int_{\varepsilon(2\log t)^{(2-p)/(2p)}}^{\infty}P(|N|\ge x)\,dx\,dt\\
&\quad\le \int_{\sqrt{2M}}^{\infty}y^{(2p/(2-p))(b+1)-1}\int_{y}^{\infty}P(|N|\ge x)\,dx\,dy\\
&\quad= \int_{\sqrt{2M}}^{\infty}P(|N|\ge x)\int_{\sqrt{2M}}^{x}y^{(2p/(2-p))(b+1)-1}dy\,dx\\
&\quad\le C\int_{\sqrt{2M}}^{\infty}x^{(2p/(2-p))(b+1)}P(|N|\ge x)\,dx\longrightarrow 0,\quad \text{as } M\longrightarrow\infty.
\end{aligned} \tag{2.14}$$

So this proposition is proved now. □

Proposition 2.4. *For $b > -1$, one has*

$$\lim_{M\to\infty}\lim_{\varepsilon\searrow 0}\varepsilon^{2p(b+1)/(2-p)}\sum_{n>A(\varepsilon)}\frac{(\log n)^b}{n}E\left\{\frac{|S_n|}{V_n}-\varepsilon(2\log n)^{(2-p)/(2p)}\right\}_+=0. \tag{2.15}$$

Proof. Note that

$$\begin{aligned}
&\varepsilon^{2p(b+1)/(2-p)}\sum_{n>A(\varepsilon)}\frac{(\log n)^b}{n}E\left\{\frac{|S_n|}{V_n}-\varepsilon(2\log n)^{(2-p)/(2p)}\right\}_+\\
&\quad=\varepsilon^{2p(b+1)/(2-p)}\sum_{n>A(\varepsilon)}\frac{(\log n)^b}{n}\int_0^{\infty}P\left(\frac{|S_n|}{V_n}\ge x+\varepsilon(2\log n)^{(2-p)/(2p)}\right)dx\\
&\quad=B_1+B_2+B_3,
\end{aligned} \tag{2.16}$$

where

$$B_1 = \varepsilon^{2p(b+1)/(2-p)} \sum_{n>A(\varepsilon)} \frac{(\log n)^b}{n} \int_0^{n^{1/4}} P\left(\frac{|S_n|}{V_n} \geq x + \varepsilon(2\log n)^{(2-p)/(2p)}\right) dx,$$

$$B_2 = \varepsilon^{2p(b+1)/(2-p)} \sum_{n>A(\varepsilon)} \frac{(\log n)^b}{n} \int_{n^{1/4}}^{n^{1/2}} P\left(\frac{|S_n|}{V_n} \geq x + \varepsilon(2\log n)^{(2-p)/(2p)}\right) dx, \tag{2.17}$$

$$B_3 = \varepsilon^{2p(b+1)/(2-p)} \sum_{n>A(\varepsilon)} \frac{(\log n)^b}{n} \int_{n^{1/2}}^{\infty} P\left(\frac{|S_n|}{V_n} \geq x + \varepsilon(2\log n)^{(2-p)/(2p)}\right) dx.$$

For B_1, by (2.7), we have

$$\begin{aligned}
B_1 &\leq C\varepsilon^{2p(b+1)/(2-p)} \sum_{n>A(\varepsilon)} \frac{(\log n)^b}{n} \int_0^{n^{1/4}} e^{-(x+\varepsilon(2\log n)^{(2-p)/(2p)})^2/4} dx \\
&\leq C\varepsilon^{2p(b+1)/(2-p)} \sum_{n>A(\varepsilon)} \frac{(\log n)^b}{n} \int_0^{\infty} e^{-(x+\varepsilon(2\log n)^{(2-p)/(2p)})^2/4} dx \\
&= C\varepsilon^{2p(b+1)/(2-p)} \sum_{n>A(\varepsilon)} \frac{(\log n)^b}{n} \int_{\varepsilon(2\log n)^{(2-p)/(2p)}}^{\infty} e^{-x^2/4} dx \\
&\leq C\varepsilon^{2p(b+1)/(2-p)} \int_{A(\varepsilon)}^{\infty} \frac{(\log n)^b}{t} \int_{\varepsilon(2\log t)^{(2-p)/(2p)}}^{\infty} e^{-x^2/4} dx dt \\
&\leq C \int_{\sqrt{2M}}^{\infty} y^{(2p/(2-p))(b+1)-1} \int_y^{\infty} e^{-x^2/4} dx dy \\
&= C \int_{\sqrt{2M}}^{\infty} e^{-x^2/4} \int_{\sqrt{2M}}^{x} y^{(2p/(2-p))(b+1)-1} dy dx \\
&\leq C \int_{\sqrt{2M}}^{\infty} x^{(2p/(2-p))(b+1)} e^{-x^2/4} dx \longrightarrow 0, \quad \text{as } M \longrightarrow \infty.
\end{aligned} \tag{2.18}$$

For B_2, using (2.7) again, we have

$$\begin{aligned}
B_2 &\leq \varepsilon^{2p(b+1)/(2-p)} \sum_{n>A(\varepsilon)} \frac{(\log n)^b}{n} \left(n^{1/2} - n^{1/4}\right) P\left(\frac{|S_n|}{V_n} \geq n^{1/4} + \varepsilon(2\log n)^{(2-p)/(2p)}\right) \\
&\leq C\varepsilon^{2p(b+1)/(2-p)} \sum_{n>A(\varepsilon)} \frac{(\log n)^b}{n} \left(n^{1/2} - n^{1/4}\right) e^{-(n^{1/4}+\varepsilon(2\log n)^{(2-p)/(2p)})^2/4}
\end{aligned}$$

$$
\begin{aligned}
&\le C\varepsilon^{2p(b+1)/(2-p)} \sum_{n>A(\varepsilon)} \frac{(\log n)^b}{n} \left(n^{1/2}-n^{1/4}\right) e^{-\sqrt{n}/4} e^{-\varepsilon^2(2\log n)^{(2-p)/p}/4} \\
&\le C\varepsilon^{2p(b+1)/(2-p)} \sum_{n>A(\varepsilon)} \frac{(\log n)^b}{n} e^{-\varepsilon^2(2\log n)^{(2-p)/p}/4} \\
&\le C\varepsilon^{2p(b+1)/(2-p)} \int_{A(\varepsilon)}^{\infty} \frac{(\log n)^b}{t} e^{-\varepsilon^2(2\log t)^{(2-p)/p}/4} dt \\
&\quad \left(\text{by letting } z = \frac{\varepsilon^2(2\log t)^{(2-p)/p}}{4}\right) \\
&\le C\int_{(2M)^{(2-p)/p}/4}^{\infty} z^{(p(b+1))/(2-p)-1} e^{-z} dz \longrightarrow 0, \quad \text{as } M \longrightarrow \infty.
\end{aligned}
\tag{2.19}
$$

By noting that (2.10), it is easily seen that

$$
B_3 = 0. \tag{2.20}
$$

Combining (2.18), (2.19), and (2.20), the proposition is proved. □

Our main result follows from the propositions using the triangle inequality.

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