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Research Article

On Interpolation Functions of the Generalized Twisted (h, q)-Euler Polynomials

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The aim of this paper is to construct p-adic twisted two-variable Euler-(h,q)-L-functions, which interpolate generalized twisted (h,q)-Euler polynomials at negative integers. In this paper, we treat twisted (h,q)-Euler numbers and polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$. We will construct two-variable twisted (h,q)-Euler-zeta function and two-variable (h,q)-L-function in Complex s-plane.

1. Introduction

Tsumura and Young treated the interpolation functions of the Bernoulli and Euler polynomials in [1, 2]. Kim and Simsek studied on p-adic interpolation functions of these numbers and polynomials [3–48]. In [49], Carlitz originally constructed q-Bernoulli numbers and polynomials. Many authors studied these numbers and polynomials [4, 28, 38, 41, 50]. After that, twisted (h, q)-Bernoulli and Euler numbers(polynomials) were studied by several authors [1–32, 32–65]. In [62], Whashington constructed one-variable p-adic-L-function which interpolates generalized classical Bernoulli numbers at negative integers. Fox introduced the two-variable p-adi L-functions [53]. Young defined p-adic integral representation for the two-variable p-adic L-functions [64]. Furthermore, Kim constructed the two-variable p-adic q-L-function, which is interpolation function of the generalized q-Bernoulli polynomials [8]. This function is the q-extension of the two-variable p-adic L-function. Kim constructed q-extension of the generalized formula for two-variable of Diamond and Ferrero and Greenberg formula for two-variable p-adic L-function in the terms of the p-adic gamma and log-gamma functions [8]. Kim and Rim introduced twisted q-Euler numbers and polynomials associated with basic twisted q-ℓ-functions [28]. Also, Jang et al. investigated the p-adic analogue twisted q-ℓ-function, which interpolates generalized twisted

q-Euler numbers $E_{n,q,\xi,\chi}$ attached to Dirichlet's character χ [55]. Kim et al. have studied two-variable p-adic L-functions, which interpolate the generalized Bernoulli polynomials at negative integers. In this paper, we will construct two-variale p-adic twisted Euler (h,q)-L-functions. This functions interpolation functions of the generalized twisted (h,q)-Euler polynomials.

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of rational integers, the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ such that $|p|_p = p^{-v_p(p)} = p^{-1}$. If $s \in \mathbb{C}$, then $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume $|1-q|_p < p^{-(1/(p-1))}$, so that $q^x = \exp(\log q)$ for $|x|_p \leq 1$. Throughout this paper we use the following notations (cf. [1–32, 32–48, 50, 51, 54–65]):

$$[x]_q = [x:q] = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Hence, $\lim_{q\to 1}[x]_q = x$, for any x with $|x|_p \leq 1$ in the present p-adic case.

For d a fixed positive integer with $(p,d)=1$, set

$$\begin{gathered} \mathrm{X} = \mathrm{X}_d = \lim_{\overleftarrow{N}} \frac{\mathbb{Z}}{dp^N\mathbb{Z}}, \qquad \mathrm{X}_1 = \mathbb{Z}_p, \\ \mathrm{X}^* = \bigcup_{\substack{0<a<dp,\\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \left\{x \in \mathrm{X} \mid x \equiv a \ (\bmod\ dp^N)\right\}, \end{gathered} \tag{1.2}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. The distribution is defined by

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}. \tag{1.3}$$

We say that f is uniformly differential function at a point $a \in \mathbb{Z}_p$, and we write $f \in UD(\mathbb{Z}_p)$, if the difference quotients, $F_f(x,y) = (f(x)-f(y))/(x-y)$ have a limit $f'(a)$ as $(x,y) \to (a,a)$.

For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as [4, 18]

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x. \tag{1.4}$$

The fermionic p-adic q-measures on $\mathbb{Z}_p$ is defined as (cf. [14–16, 18, 22, 28])

$$\mu_{-q}(a + dp^N\mathbb{Z}_p) = \frac{(-q)^a}{[dp^N]_{-q}}, \tag{1.5}$$

for $f \in UD(\mathbb{Z}_p)$. For $f \in UD(\mathbb{Z}_p)$, the ferminoic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.6}$$

which has a sense as we see readily that the limit is convergent. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, we note that (cf. [14, 16, 18, 22, 28])

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \int_X f(x) d\mu_{-1}(x). \tag{1.7}$$

From the fermionic invariant integral on $\mathbb{Z}_p$, we derive the following integral equation (cf. [14, 35]):

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.8}$$

where $f_1(x) = f(x+1)$.

2. Twisted (h, q)-Euler Numbers and Polynomials

In this section, we will treat some properties of twisted (h, q)-Euler numbers and polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$. From now on, we take $h \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-(1/(p-1))}$. Let C_{p^n} be the space of primitive p^nth root of unity,

$$C_{p^n} = \{ w \in \mathbb{C}_{p^n} \mid w^{p^n} = 1 \}. \tag{2.1}$$

Then, we denote

$$T_p = \lim_{n \to \infty} C_{p^n} = \bigcup_{n \geq 0} C_{p^n}. \tag{2.2}$$

Hence T_p is a p-adic locally constant space. For $\xi \in T_p$, we denote by $\phi_\xi : \mathbb{Z}_p \to \mathbb{C}_p$ defined by $\phi_\xi(x) = \xi^x$, the locally constant function. If we take $f(x) = \xi^x e^{xt}$, then we have (cf. [35])

$$E_{n,\xi} = \int_{\mathbb{Z}_p} x^n \xi^n d\mu_{-1}(x). \tag{2.3}$$

By induction in (1.8), Kim constructed the following useful identity (cf. [14, 28]):

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} f(\ell), \tag{2.4}$$

where $n \in \mathbb{N}$, $f_n = f(x+n)$. From (2.4), if n is odd, then we have

$$I_{-1}(f_n) + I_{-1}(f) = 2\sum_{\ell=0}^{n-1}(-1)^{\ell} f(\ell). \tag{2.5}$$

If we replace n by d (= odd) into (2.5), we obtain

$$I_{-1}(f_d) + I_{-1}(f) = 2\sum_{\ell=0}^{d-1}(-1)^{\ell} f(\ell). \tag{2.6}$$

Let $\xi \in T_p$. Let χ be a Dirichlet's character of conductor d, which d is any multiple of p with $p \equiv 1 \pmod 2$. By substituting $f(x) = \chi(x)\xi^x e^{xt}$ into (2.6), we have

$$I_{-1}\left(\chi(x)\xi^x e^{xt}\right) = \sum_{n=0}^{\infty} E_{n,\xi,\chi}\frac{t^n}{n!}. \tag{2.7}$$

Remark 2.1. In complex case, the generating function of the Euler numbers $E_{n,\xi,\chi}$ is given by (cf. [28])

$$\frac{2\sum_{\ell=0}^{d-1}(-1)^{\ell}\chi(\ell)\xi^{\ell}e^{\ell t}}{\xi^d e^{dt}+1} = \sum_{n=0}^{\infty} E_{n,\xi,\chi}\frac{t^n}{n!}, \quad |t| < \frac{\pi}{d}. \tag{2.8}$$

By using Taylor series of e^{xt}, then we can define the generalized twisted Euler numbers $E_{n,\xi,\chi}$ attached to χ as follows (cf. [55]):

$$E_{n,\xi,\chi} = \int_X \xi^n x^n \chi(x)\, d\mu_{-1}(x). \tag{2.9}$$

In [8], (h,q)-Euler numbers were defined by

$$E_{n,q}^{(h,1)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)y}[x+y]_q^n\, d\mu_{-q}(y), \tag{2.10}$$

where $h \in \mathbb{Z}$ and $x \in \mathbb{Z}_p$. In particular, if we take $x = 0$, then $E_{n,q}^{(h,1)}(0) = E_{n,q}^{(h,1)}$. These numbers are called (h,q)-Euler numbers.

By using iterative method of p-adic invariant integral on $\mathbb{Z}_p$ in the sense of fermionic, we define twisted (h,q)-Euler numbers as follows (cf. [55]):

$$E_{n,q,\xi}^{(h,1)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)y}\phi_{\xi}(y)[x+y]_q^n\, d\mu_{-q}(y). \tag{2.11}$$

For $h \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have that (cf. [55])

$$E_{n,q,\xi}^{(h,1)}(x) = \frac{1+q}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i q^{xi} \frac{1}{1+\xi q^{h+i}}, \tag{2.12}$$

$$E_{n,q,\xi}^{(h,1)}(x) = \frac{1+q}{1+q^d} \sum_{a=0}^{d-1} (-1)^a q^{ha} \xi^a E_{n,\xi^d,q^d}^{(h,1)}\left(\frac{x+a}{d}\right)[d]_q^n, \tag{2.13}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$.

Let $F_{q,\xi}^{(h,1)}(t,x)$ be the generating function of $E_{n,q,\xi}^{(h,1)}(x)$ in complex plane as follows (cf. [55]):

$$\begin{aligned} F_{q,\xi}^{(h,1)}(t,x) &= (1+q) \sum_{n=0}^{\infty} (-1)^n q^{hn} \xi^n e^{t[n+x]_q} \\ &= \sum_{n=0}^{\infty} E_{n,q,\xi}^{(h,1)}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.14}$$

Let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then the generalized twisted (h,q)-Euler polynomials attached to χ is given by as follows:

For $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$,

$$E_{n,q,\xi,\chi}^{(h,1)}(x) = \int_X \chi(y) q^{(h-1)y} \xi^y [x+y]_q^n d\mu_{-q}(y), \tag{2.15}$$

where $h \in \mathbb{Z}$, d is any multiple of p with $p \equiv 1 \pmod 2$ and $x \in \mathbb{C}_p$.

Then the distribution relation of the generalized twisted (h,q)-Euler polynomials is given by as follows (cf. [14]):

$$E_{n,q,\xi,\chi}^{(h,1)}(x) = \frac{1+q}{1+q^d} \sum_{a=1}^{d} \chi(a)(-1)^a q^{ha} \xi^a E_{n,q^d,\xi^d}^{(h,1)}\left(\frac{x+a}{d}\right)[d]_q^n. \tag{2.16}$$

3. Two-Variable Twisted (h,q)-Euler-Zeta Function and (h,q)-L-Function

In this section, we will construct two-variable twisted (h,q)-Euler-zeta function and two-variable (h,q)-L-function in Complex s-plane. We assume $q \in \mathbb{C}$ with $|q| < 1$.

Firstly, we consider twisted q-Euler numbers and polynomials in $\mathbb{C}$ as follows (cf. [55]):

$$\begin{aligned} F_{q,\xi}^{(h,1)}(t,x) &= (1+q) \sum_{n=0}^{\infty} (-1)^n q^{hn} \xi^n e^{t[n+x]_q} \\ &= \sum_{n=0}^{\infty} E_{n,q,\xi}^{(h,1)}(x) \frac{t^n}{n!}, \end{aligned} \tag{3.1}$$

where $q, x \in \mathbb{C}$, $r \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ and ξ is rth root of unity. In particular, if we take $x = 0$, then we have $E_{n,q,\xi}^{(h,1)}(0) = E_{n,q,\xi}^{(h,1)}$. These numbers are called twisted Euler numbers. By using derivative operator, we have $(d^k/dt^k)F_{q,\xi}(t,x)|_{t=0} = E_{n,q,\xi}^{(h,1)}(x)$.

From (3.1), we can define Hurwitz-type twisted (h,q)-Euler-zeta function as follows (cf. [55]):

$$\zeta_{E,q,\xi}^{(h,1)}(s,x) = (1+q)\sum_{k=0}^{\infty}\frac{(-1)^k q^{hk}\xi^k}{[x+k]_q^s}, \tag{3.2}$$

where $q \in \mathbb{C}$, $|q| < 1$, $s \in \mathbb{C}$, $h \in \mathbb{Z}$ and $x \in \mathbb{R}$, $0 < x \le 1$. Note that if $x = 1$ in (3.2), then we see that the twisted (h,q)-Euler-zeta function is defined by (cf. [28, 55])

$$\zeta_{E,q,\xi}^{(h,1)}(s) = (1+q)\sum_{k=1}^{\infty}\frac{(-1)^k q^{hk}\xi^k}{[k]_q^s}, \quad s \in \mathbb{C},\ \operatorname{Re}(s) > 1. \tag{3.3}$$

For $n \in \mathbb{N}$, we know (cf. [28])

$$\zeta_{E,q,\xi}^{(h,1)}(-n,x) = E_{n,q,\xi}^{(h,1)}(x). \tag{3.4}$$

From now on, we will define the two-variable (h,q)-L-functions $L_{E,q,\xi}^{(h,1)}(s,x:\chi)$ which interpolates the generalized (h,q)-Euler polynomials.

Definition 3.1. Let χ be the Dirichlet's character with conductor d with $d \equiv 1 \pmod 2$. For $s \in \mathbb{C}$, $h \in \mathbb{Z}$ and $x \in \mathbb{R}$, $0 < x \le 1$, we define

$$L_{E,q,\xi}^{(h,1)}(s,x:\chi) = (1+q)\sum_{n=0}^{\infty}\frac{\chi(n)(-1)^n q^{hn}\xi^n}{[n+x]_q^s}. \tag{3.5}$$

By substituting $n = a + jd$, $d \equiv 1 \pmod 2$, $1 \le a \le d$ and $n = 0,1,2,\ldots$ into (3.5), then using (3.2), we have

$$\begin{aligned} L_{E,q,\xi}^{(h,1)}(s,x:\chi)&(1+q)\sum_{a=1}^{d}\sum_{j=0}^{\infty}\frac{\chi(a+jd)(-1)^{a+jd}q^{h(a+jd)}\xi^{a+jd}}{[a+jd+x]_q^s} \\ &= (1+q)\sum_{a=1}^{d}\frac{\chi(a)(-1)^a q^{ha}\xi^a}{[d]_q^s}\sum_{j=0}^{\infty}\frac{(-1)^{jd}q^{hjd}}{[j+((a+x)/d)]_{q^d}^s} \\ &= \frac{1+q}{1+q^d}\sum_{a=1}^{d}\chi(a)(-1)^a q^{ha}\xi^a\zeta_{E,q^d,\xi^d}^{(h,1)}\left(s,\frac{a+x}{d}\right)[d]_q^{-s}. \end{aligned} \tag{3.6}$$

Thus, we see the function $L_{E,q,\xi}^{(h,1)}(s,x:\chi)$ which interpolates the generalized (h,q)-Euler polynomials as follows.

Theorem 3.2. *For $s \in \mathbb{C}$, $h \in \mathbb{Z}$, let χ be the Dirichlet's character with conductor d with $d \equiv 1 \pmod 2$. Then one has*

$$L_{E,q,\xi}^{(h,1)}(s,x:\chi) = \frac{1+q}{1+q^d}\sum_{a=1}^{d}\chi(a)(-1)^a q^{ha}\xi^a \zeta_{E,q^d,\xi^d}^{(h,1)}\left(s,\frac{a+x}{d}\right)[d]_q^{-s}. \tag{3.7}$$

By substituting $s=-n$ with $n>0$, into (3.7), we obtain

$$\begin{aligned} L_{E,q,\xi}^{(h,1)}(-n,x:\chi) &= \frac{1+q}{1+q^d}\sum_{a=1}^{d}\chi(a)(-1)^a q^{ha}\xi^a \zeta_{E,q^d,\xi^d}^{(h,1)}\left(-n,\frac{a+x}{d}\right)[d]_q^{n} \\ &= \frac{1+q}{1+q^d}\sum_{a=1}^{d}\chi(a)(-1)^a q^{ha}\xi^a E_{n,q^d,\xi^d}^{(h,1)}\left(\frac{a+x}{d}\right)[d]_q^{n} \\ &= E_{n,q,\xi,\chi}^{(h,1)}(x), \end{aligned} \tag{3.8}$$

where $d \equiv 1 \pmod 2$, $d \in \mathbb{N}$.

Thus, we have the following theorem.

Theorem 3.3. *For $n \in \mathbb{N}$, let χ be the Dirichlet's character with conductor d with $d \equiv 1 \pmod 2$. Then one has*

$$L_{E,q,\xi}^{(h,1)}(-n,x:\chi) = E_{n,q,\xi,\chi}^{(h,1)}(x). \tag{3.9}$$

Remark 3.4. If we take $x=1$ in (3.5), then we have (cf. [28, 55])

$$L_{E,q,\xi}^{(h,1)}(s,\chi) = (1+q)\sum_{n=1}^{\infty}\frac{\chi(n)(-1)^n q^{hn}\xi^n}{[n]_q^s}, \quad \text{for } s \in \mathbb{C}. \tag{3.10}$$

From (3.9) and (3.10), we have the following corollary.

Corollary 3.5. *Let χ be the Dirichlet's character with conductor d with $d \equiv 1 \pmod 2$. Then one has*

$$E_{n,q,\xi,\chi}^{(h,1)}(x) = \frac{1+q}{1+q^d}\sum_{a=1}^{d}\chi(a)(-1)^a q^{ha}\xi^a E_{n,q^d,\xi^d}^{(h,1)}\left(\frac{a+x}{d}\right)[d]_q^{n}. \tag{3.11}$$

Secondly, we will define two-variable twisted Euler (h,q)-L-function as follows.

Definition 3.6. Let χ be the Dirichlet's character with conductor d with $d \equiv 1 \pmod 2$, $d \in \mathbb{N}$. For $s \in \mathbb{C}$, $h \in \mathbb{Z}$, $x \in \mathbb{R}$, $0 < x \le 1$ and $\xi^r = 1$ with $\xi \neq 1$, we define

$$L_{E,q,\xi}^{(h,1)}(s,x:\chi) = (1+q)\sum_{k=0}^{\infty}\frac{\chi(k)(-1)^k q^{hk}\xi^k}{[k+x]_q^s}. \tag{3.12}$$

We consider the well-known identity (cf. [44, 65])

$$\frac{1}{(1-x)^s} = \sum_{j=0}^{\infty} \binom{s+j-1}{j} x^j. \tag{3.13}$$

By using (3.12), we define two-variable twisted Euler (h,q)-L-function as follows:

$$L_{E,q,\xi}^{(h,1)}(s,x:\chi) = (1+q)(1-q)^s \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{s+j-1}{j} \chi(k)(-1)^k \xi^k q^{hk+j(k+x)}. \tag{3.14}$$

We will investigate the relations between $L_{E,q,\xi}^{(h,1)}(s,x:\chi)$ and $L_{E,q,\xi}^{(h,1)}(s,\chi)$ as follows.

Substituting $k = a+jd$, $a = 1,2,\ldots,d$ with $d \equiv 1 \pmod 2$, $j = 0,1,2,\ldots,$ into (3.12), we have

$$L_{E,q,\xi}^{(h,1)}(s,x:\chi) = (1+q)\sum_{a=1}^{d}\sum_{j=0}^{\infty} \frac{\chi(a+jd)(-1)^{a+jd} q^{h(a+jd)} \xi^{a+jd}}{[a+jd+x]_q^s}, \tag{3.15}$$

Thus we obtain the following theorem.

Theorem 3.7. *For $s \in \mathbb{C}$ with $h \in \mathbb{Z}$, let χ be the Dirichlet character with conductor d with $d \equiv 1 \pmod 2$ and $x \in \mathbb{R}$, $0 < x \le 1$, $\xi^r = 1$ with $\xi \neq 1$. Then one has*

$$L_{E,q,\xi}^{(h,1)}(s,x:\chi) = \frac{1+q}{1+q^d}\sum_{a=1}^{d}\chi(a)(-1)^a q^{ha}\xi^a \zeta_{E,q^d,\xi^d}^{(h,1)}\left(s, \frac{a+x}{d}\right)[d]_q^{-s}. \tag{3.16}$$

By substituting $s = -n$ with $n \in \mathbb{N}$ into (3.16) and using (3.4), we can obtain

$$\begin{aligned} L_{E,q,\xi}^{(h,1)}(-n,x:\chi) &= \frac{1+q}{1+q^d}\sum_{a=1}^{d}\chi(a)(-1)^a q^{ha}\xi^a \zeta_{E,q^d,\xi^d}^{(h,1)}\left(-n, \frac{a+x}{d}\right)[d]_q^{n} \\ &= \frac{1+q}{1+q^d}\sum_{a=1}^{d}\chi(a)(-1)^a q^{ha}\xi^a E_{n,q^d,\xi^d}^{(h,1)}\left(\frac{a+x}{d}\right)[d]_q^{n} \\ &= E_{n,q,\xi,\chi}^{(h,1)}(x). \end{aligned} \tag{3.17}$$

Thus, we see that the function $L_{E,q,w}^{(h,1)}(s,x:\chi)$ interpolates generalized (h,q)-Euler polynomials attached to χ at negative integer values of s as followings.

Theorem 3.8. *For $n \in \mathbb{N}$, let χ be the Dirichlet's character with odd conductor d. Then one has*

$$L_{E,q,\xi}^{(h,1)}(-n,x:\chi) = E_{n,q,\xi,\chi}^{(h,1)}(x). \tag{3.18}$$

Note that if we take $x = 1$, then Theorem 3.8 reduces to Theorem 3.3.

Let a and F be integers with $F \equiv 1 \pmod 2$ and $0 < a < F$. For $s \in \mathbb{C}$, we define partial (h,q)-Hurwitz type zeta function $H^{(h,1)}_{E,q,\xi}(s,a,x \mid F)$ as follows:

$$H^{(h,1)}_{E,q,\xi}(s,a,x \mid F) = \sum_{\substack{m \equiv a \pmod F, \\ m>0}} \frac{(-1)^m q^{hm} \xi^m}{[m+x]_q^s}. \tag{3.19}$$

By substituting $m = a + jF$, we have

$$\begin{aligned} H^{(h,1)}_{E,q,\xi}(s,a,x \mid F) &= \sum_{j=0}^{\infty} \frac{(-1)^{a+jF} q^{h(a+jF)} \xi^{a+jF}}{[a+jF+x]_q^s} \\ &= (-1)^a q^{ha} \xi^a [F]_q^{-s} \sum_{j=0}^{\infty} \frac{(-1)^{jF} (q^F)^{hj} (\xi^F)^j}{[((a+x)/F)+j]_{q^F}^s} \\ &= [F]_q^{-s} (-1)^a (q)^{ha} \xi^a \frac{1}{1+q^F} \sum_{j=0}^{\infty} \frac{(-1)^{jF} (q^F)^{hj} (\xi^F)^j}{[((a+x)/F)+j]_{q^F}^s} \\ &= [F]_q^{-s} \frac{(-1)^a (q)^{ha} \xi^a}{1+q^F} \zeta^{(h,1)}_{E,q^F,\xi^F}\left(s, \frac{a+x}{F}\right). \end{aligned} \tag{3.20}$$

By substituting (3.2), for $s = -n$, we get

$$H^{(h,1)}_{E,q,\xi}(s,a,x \mid F) = [F]_q^n \frac{(-1)^a q^{ha} \xi^a}{1+q^F} E^{(h,1)}_{n,q^F,\xi^F}\left(\frac{a+x}{F}\right). \tag{3.21}$$

Equation (3.20) means that the function $H^{(h,1)}_{E,q,\xi}(s,a,x \mid F)$ interpolates $E^{(h,1)}_{n,q,\xi}(s,a,x \mid F)$ polynomials at negative integers.

From (3.16) and (3.20), we have the following theorem.

Theorem 3.9. *For $s \in \mathbb{C}$, $\xi^r = 1$ with $\xi \neq 1$, let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and $x \in \mathbb{R}$, $0 < x \leq 1$, F is any multiple of d. Then one has*

$$L^{(h,1)}_{E,q,\xi}(s,x:\chi) = (1+q) \sum_{a=1}^{F} \chi(a)(-1)^a H^{(h,1)}_{E,q,\xi}(s,a,x \mid F). \tag{3.22}$$

Remark 3.10. If we take $s = 0$ in (3.22), then we have

$$\begin{aligned} L^{(h,1)}_{E,q,\xi}(0,x:\chi) &= (1+q) \sum_{a=1}^{F} \chi(a) H^{(h,1)}_{E,q,\xi}(0,a,x \mid F) \\ &= \frac{1+q}{1+q^F} \sum_{a=1}^{F} \chi(a)(-1)^a q^{ha} \xi^a E^{(h,1)}_{0,q^F,\xi^F}\left(\frac{a+x}{F}\right). \end{aligned} \tag{3.23}$$

From (2.12), if we take $s = 0$, then we have the following corollary.

Corollary 3.11. *For $s \in \mathbb{C}$, $\xi^r = 1$ with $\xi \neq 1$, let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and $x \in \mathbb{R}$, $0 < x \leq 1$, F is any multiple of d. Then one has*

$$L_{E,q,\xi}^{(h,1)}(0, x : \chi) = \frac{(1+q)^2}{(1+q^F)(1+\xi q^h)} \sum_{a=1}^{F} \chi(a)(-1)^a q^{ha} \xi^a. \tag{3.24}$$

4. p-Adic Twisted Two-Variable Euler (h,q)-L-Functions

In [62], Washington constructed one-variable p-adic-L-function which interpolates generalized classical Bernoulli numbers negative integers. Kim [22] investigated the p-adic analogues of two-variables Euler q-L-function. In this section, we will construct p-adic twisted two-variable Euler-(h,q)-L-functions, which interpolate generalized twisted (h,q)-Euler polynomials at negative integers. Our notations and methods are essentially due to Kim and Washington (cf. [22, 62]).

We assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-(1/(p-1))}$, so that $q^x = \exp(x \log q)$. Let p be an odd prime number. Let ω denote the Teichmüller character having conductor p. For an arbitrary character χ, we define $\chi_n = \chi\omega^{-n}$, where $n \in \mathbb{Z}$, in the sense of the product of characters. Let $\langle a \rangle = \langle a : q \rangle = \omega^{-1}(a)[a]_q = [a]_q/\omega(a)$. Then $\langle a \rangle \equiv 1 \pmod{p^{1+(1/(p-1))}}$. Hence we see that

$$\begin{aligned} \langle\ a+pt\ \rangle &= \omega^{-1}(a+pt)[a+pt]_q \\ &= \omega^{-1}(a)[a]_q + \omega^{-1}(a)q^a[pt]_q \\ &\equiv 1 \left(\bmod\ p^{1+(1/(p-1))}\right), \end{aligned} \tag{4.1}$$

where $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, $(a,p) = 1$.

We denote the subset D of $\mathbb{C}_p^*$ by (cf. [62])

$$D = \{s \in \mathbb{C}_p : |s|_p \leq p^{1-(1/(p-1))}\}. \tag{4.2}$$

Let

$$A_j(x) = \sum_{j=0}^{\infty} a_{n,j} x^n, \quad a_{n,j} \in \mathbb{C}_p,\ j = 0, 1, 2, \ldots, \tag{4.3}$$

be a sequence of power series, each of which converges in a fixed subset D such that

(1) $a_{n,j} \to a_{n,0}$ as $j \to \infty$ for all n, j and

(2) for each $s \in D$ and $\varepsilon > 0$, there exists $n_0 = n_0(s, \varepsilon)$ such that

$$\left| \sum_{n \geq n_0} a_{n,j} s^n \right|_p < \varepsilon, \quad \text{for } \forall j. \tag{4.4}$$

Then $\lim_{j\to\infty} A_j(s) = A_0(s)$ for all $s \in D$ (cf. [2, 22, 50, 51, 60, 62]).

Let χ be the Dirichlet's character with conductor d with $d \equiv 1 \pmod 2$ and let F be a positive multiple of p and d.

Now we set

$$L_{E,p,q,\xi}^{(h,1)}(s,x:\chi) = \frac{1+q}{1+q^F}\sum_{\substack{a=1,\\ p\nmid a}}^{F}\chi(a)(-1)^a\xi^a\langle a+pt\rangle^{-s} \cdot \sum_{j=0}^{\infty}\binom{-s}{j}E_{j,q^F,\xi^F}^{(h,1)}q^{j(a+pt)}\left[\frac{F}{a+pt}\right]_{q^{a+pt}}^{j}. \tag{4.5}$$

Then $L_{E,p,q,\xi}^{(h,1)}(s,x:\chi)$ is analytic for $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, when $s \in D$. For $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, we have

$$\sum_{j=0}^{\infty}\binom{-s}{j}E_{j,q^F,\xi^F}^{(h,1)}q^{j(a+pt)}\left[\frac{F}{a+pt}\right]_{q^{a+pt}}^{j} \tag{4.6}$$

is analytic for $s \in D$. It readily follows that

$$\langle a+pt\rangle^{s} = \omega^{-s}(a)[a+pt]_q^s = \langle a\rangle^s\sum_{m=0}^{\infty}\binom{s}{m}\left(q^a[a]_q^{-1}[pt]_q\right)^m \tag{4.7}$$

is analytic for $s \in \mathbb{C}_p$ with $|t|_p \leq 1$ when $s \in D$. Thus we see that

$$L_{E,p,q,\xi}^{(h,1)}(0,x:\chi) = \frac{1+q}{2}\sum_{a=1}^{F}(-1)^a\chi_n(a)\xi^a. \tag{4.8}$$

Let $n \in \mathbb{Z}_+$ and fixed $t \in \mathbb{C}_p$ with $|t|_p \leq 1$. Then we have that

$$E_{n,q,\xi,\chi_n}^{(h,1)}(pt) = [F]_q^n\frac{1+q}{1+q^F}\sum_{a=0}^{F}\chi_n(a)(-1)^a\xi^a E_{n,q^F,\xi^F}^{(h,1)}\left(\frac{a+pt}{F}\right). \tag{4.9}$$

If $\chi_n(p) \neq 0$, then $(p, d_{\chi_n}) = 1$, so F/p is a multiple of d_{χ_n}. Therefore, we have

$$\begin{aligned}
&\chi_n(p)[p]_q^n E_{n,q^F,\xi^F,\chi_n}^{(h,1)}(t) \\
&\quad = \chi_n(p)[p]_q^n\left\{\left[\frac{F}{p}\right]_{q^p}^n\frac{1+q^p}{1+q^{pF/p}}\sum_{a=0}^{F/p-1}\chi_n(a)(-1)^a\xi^a E_{n,(q^p)^{F/p},(\xi^p)^{F/p}}^{(h,1)}\left(\frac{a+t}{F/p}\right)\right\} \\
&\quad = [F]_q^n\frac{1+q^p}{1+q^F}\sum_{\substack{a=0\\ p\nmid a}}^{F}\chi_n(a)(-1)^a\xi^a E_{n,q^F,\xi^F}^{(h,1)}\left(\frac{a+pt}{F}\right).
\end{aligned} \tag{4.10}$$

Then we note that

$$\frac{1+q}{1+q^p}\chi_n(p)[p]_q^n E^{(h,1)}_{n,q^F,\xi^F,\chi_n}(t)=\frac{1+q}{1+q^F}[F]_q^n\sum_{\substack{a=0\\p\mid a}}^{F}\chi_n(a)(-1)^a\xi^a E^{(h,1)}_{n,q^F,\xi^F}\left(\frac{a+pt}{F}\right). \tag{4.11}$$

The difference of these equations yields

$$E^{(h,1)}_{n,q,\xi,\chi_n}(pt)-\frac{1+q}{1+q^p}\chi_n(p)[p]_q^n E^{(h,1)}_{n,q^F,\xi^F,\chi_n}(t)=\frac{1+q}{1+q^F}[F]_q^n\sum_{\substack{a=0\\p\nmid a}}^{F}\chi_n(a)(-1)^a\xi^a E^{(h,1)}_{n,q^F,\xi^F}\left(\frac{a+pt}{F}\right). \tag{4.12}$$

Using distribution for (h,q)-Euler polynomials, we easily see that

$$E^{(h,1)}_{n,q^F,\xi^F}\left(\frac{a+pt}{F}\right)=[F]_q^{-n}[a+pt]_q^n\sum_{k=0}^{n}\binom{n}{k}q^{(a+pt)k}\xi^a\left[\frac{F}{a+pt}\right]^k_{q^{a+pt}}E^{(h,1)}_{k,q^F,\xi^F}. \tag{4.13}$$

Since $\chi_n(a)=\chi(a)\omega^{-n}(a)$, for $(a,p)=1$, and $t\in\mathbb{C}_p$, with $|t|_p\le 1$, we have

$$\begin{aligned}
&E^{(h,1)}_{n,q,\xi,\chi_n}(pt)-\frac{1+q}{1+q^p}\chi_n(p)[p]_q^n E^{(h,1)}_{n,q^F,\xi^F,\chi_n}(t)\\
&\quad=\frac{1+q}{1+q^F}\sum_{a=0}^{F-1}\chi_n(a)(-1)^a\xi^a E^{(h,1)}_{n,q^F,\xi^F}\left(\frac{a+pt}{F}\right)\\
&\quad=\frac{1+q}{1+q^p}\sum_{\substack{a=0,\\p\nmid a}}^{F-1}\chi_n(a)(-1)^a\xi^a\langle a+pt\rangle^n\sum_{k=0}^{n}\binom{n}{k}q^{(a+pt)k}\left[\frac{F}{a+pt}\right]^k_{q^{a+pt}}E^{(h,1)}_{k,q^F,\xi^F}.
\end{aligned} \tag{4.14}$$

From (4.5)–(4.14), we can derive that

$$E^{(h,1)}_{n,q,\xi,\chi_n}(pt)-\frac{1+q}{1+q^p}\chi_n(p)[p]_q^n E^{(h,1)}_{n,q^p,\xi^p,\chi_n}(t)=L^{(h,1)}_{E,p,q,\xi}(-n,t:\chi). \tag{4.15}$$

Therefore we obtain the following theorem.

Theorem 4.1. *Let F be a positive integral multiple of p and $d(=d_\chi)$ with $F\equiv 1 \pmod 2$, and let*

$$L^{(h,1)}_{E,p,q,\xi}(s,t:\chi)=\frac{1+q}{1+q^d}\sum_{\substack{a=1,\\p\nmid a}}^{F}\chi(a)(-1)^a\xi^a\langle a+pt\rangle^{-s}\sum_{m=0}^{\infty}\binom{-s}{m}q^{(a+pt)m}\left[\frac{F}{a+pt}\right]^m_{q^{a+pt}}E^{(h,1)}_{m,q^F,\xi^F}. \tag{4.16}$$

Then $L^{(h,1)}_{E,p,q,\xi}(s,t : \chi)$ is analytic for $t \in \mathbb{C}_p$, $|t|_p \le 1$, provides $s \in D$ when $\chi = 1$. Furthermore, for each $n \in \mathbb{Z}_+$, we have

$$L^{(h,1)}_{E,p,q,\xi}(-n,t : \chi) = E^{(h,1)}_{n,q,\xi,\chi_n}(pt) - \frac{1+q}{1+q^p}\chi_n(p)[p]^n_q E^{(h,1)}_{n,q^p,\xi^p,\chi_n}(t). \tag{4.17}$$

Thus we note that $L^{(h,1)}_{E,p,q,\xi}(s,0 : \chi) = L^{(h,1)}_{E,p,q,\xi}(s,\chi)$ for all $s \in D$, where $L^{(h,1)}_{E,p,q,\xi}(s,\chi)$ is twisted p-adic Euler (h,q)-L-function, (cf. [15, 22]).

We now generalized to two-variable p-adic Euler (h,q)-L-function, $L^{(h,1)}_{E,p,q,\xi}(s,t : \chi)$ which is first defined by the interpolation function

$$\begin{aligned} H^{(h,1)}_{E,p,q,\xi}(s,a,x \mid F) &= \frac{(-1)^a}{1+q^F} q^{ha}\xi^a \langle a+pt\rangle^{-s} \\ &\quad \cdot \sum_{j=0}^{\infty}\binom{-s}{j} q^{j(a+pt)}\left(\frac{[F]_q}{[a+pt]_q}\right)^j E^{(h,1)}_{j,q^F,\xi^F}, \end{aligned} \tag{4.18}$$

for $s \in \mathbb{Z}_p$.

From (4.18), we have that

$$\begin{aligned} H^{(h,1)}_{E,p,q,\xi}(-n,a,x \mid F) &= \frac{(-1)^a}{1+q^F}\xi^a q^{ha}\langle a+pt\rangle^n \sum_{j=0}^{a}\binom{n}{j} q^{(a+pt)j}\left(\frac{[F]_q}{[a]_q}\right)^j E^{(h,1)}_{j,q^F,\xi^F} \\ &= \frac{(-1)^a}{1+q^F} q^{ha}\xi^a \omega^{-n}(a)[F]^n_q E_{n,q^F,\xi^F}\left(\frac{a}{F}\right) \\ &= \omega^{-n}(a) H^{(h,1)}_{E,q,\xi}(-n,a,x \mid F). \end{aligned} \tag{4.19}$$

By using the definition of $H^{(h,1)}_{E,p,q,\xi}(s,a,x \mid F)$, we can express $L^{(h,1)}_{E,p,q,\xi}(s,t : \chi)$ for all $a \in \mathbb{Z}$, $(a,p)=1$ and $t \in \mathbb{C}_p$ with $|t| \le 1$ as follows:

$$L^{(h,1)}_{E,p,q,\xi}(s,t : \chi) = \sum_{\substack{a=1,\\ p\nmid a}}^{F} \chi(a) H^{(h,1)}_{E,p,q,\xi}(s,a+pt \mid F). \tag{4.20}$$

We know that $H^{(h,1)}_{E,p,q,\xi}(s,a+pt \mid F)$ is analytic for $t \in \mathbb{C}_p$, $|t| \le 1$, when $s \in D$. The value of $(\partial/\partial s)L^{(h,1)}_{E,p,q,\xi}(s,t : \chi)$ is the coefficients of s in the expansion of $L^{(h,1)}_{E,p,q,\xi}(s,t : \chi)$ at $s=0$. Using the Taylor expansion at $s=0$, we see that

$$\langle a+pt\rangle^{-s} = 1 - s\log\langle a+pt\rangle + \cdots, \qquad \binom{-s}{m} = \frac{(-1)^m}{m}s + \cdots. \tag{4.21}$$

The p-adic logarithmic function, $\log_p$, is the unique function $\mathbb{C}_p^* \to \mathbb{C}_p$ that satisfies

$$\begin{gathered}\log_p(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n, \quad |x|_p < 1, \\ \log_p(xy) = \log_p(x) + \log_p(y), \quad \forall x, y \in \mathbb{C}_p^*, \\ \log_p(p) = 0.\end{gathered} \tag{4.22}$$

By employing these expansion and some algebraic manipulations, we evaluate the derivative $(\partial/\partial s)L_{E,p,q,\xi}^{(h,1)}(0, t : \chi)$. It follows from the definition of $L_{E,p,q,\xi}(s, t : \chi)$ that

$$\begin{aligned}L_{E,p,q,\xi}^{(h,1)}(s, t : \chi) = \frac{1+q}{1+q^F} \sum_{\substack{a=1,\\ p\nmid a}}^{F} \chi(a)(-1)^a \xi^a \langle a + pt\rangle^{-s} \\ \cdot \sum_{m=0}^{\infty} \binom{-s}{m} q^{(a+pt)m} \left[\frac{F}{a+pt}\right]_{q^{a+pt}}^m E_{m,q^F,\xi^F}^{(h,1)}.\end{aligned} \tag{4.23}$$

Thus, we have

$$\begin{aligned}\frac{\partial}{\partial s} L_{E,p,q,\xi}^{(h,1)}(s, t : \chi)|_{s=0} = \frac{1+q}{1+q^F} \sum_{\substack{a=1,\\ p\nmid a}}^{F} \chi(a)(-1)^a \xi^a \\ \cdot \left(-\log(a+pt) E_{0,q^F,\xi^F}^{(h,1)} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} q^{(a+pt)m} \left[\frac{F}{a+pt}\right]_{q^{a+pt}}^m E_{m,q^F,\xi^F}^{(h,1)}\right).\end{aligned} \tag{4.24}$$

Since $\omega(a)$ is a root of unity for $(a, p) = 1$, we have

$$\log_p\langle a + pt\rangle = \log_p(a + pt) + \log_p \omega^{-1}(a) = \log_p(a + pt). \tag{4.25}$$

Thus we have the following theorem.

Theorem 4.2. *Let χ be a primitive Dirichlet's character with odd conductor d, $d \in \mathbb{N}$ and let F be a odd positive integral multiple of p and d. Then for any $t \in \mathbb{C}_p$ with $|t| \le 1$, one has*

$$\begin{aligned}\frac{\partial}{\partial s} L_{E,p,q,\xi}^{(h,1)}(s, t : \chi) = \frac{1+q}{1+q^F} \sum_{\substack{a=1,\\ p\nmid a}}^{F} \chi(a)(-1)^a \xi^a \sum_{m=1}^{\infty} \frac{(-1)^m}{m} q^{(a+pt)m} \left(\frac{[F]_q}{[a+pt]_q}\right)^m E_{m,q^F,\xi^F}^{(h,1)} \\ - \frac{1+q}{2} \sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(a)(-1)^a \xi^a \log(a+pt).\end{aligned} \tag{4.26}$$

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