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Research Article

On Some Geometrical Properties of Generalized Modular Spaces of Cesáro Type Defined by Weighted Means

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The main purpose of this paper is to introduce modular structure of the sequence space defined by Altay and Başar (2007), and to study Kadec-Klee (H) and uniform Opial properties of this sequence space on Köthe sequence spaces.

1. Introduction

In [1], Malkowsky and Savaş defined a new sequence space by using generalized weighted means and they studied β-dual and matrix transformations of this space.

Recently, Altay and Başar [2] constructed a new paranormed sequence space inspired by the sequence space defined in [1].

On the other hand, Shue [3] first defined the Cesáro sequence spaces with a norm. Many authors studied the Cesáro sequence spaces with several properties. In [4], it is shown that the Cesáro sequence spaces ces_p $(1 \leq p < \infty)$ have Kadec-Klee and local uniform rotundity properties.Cui et al. [5] showed that Banach-Saks of type p property holds in these spaces.

Quite recently, Sanhan and Suantai [6] generalized normed Cesáro sequence spaces to paranormed sequence spaces by making use of Köthe sequence spaces. They also defined and investigated modular structure and some geometrical properties of these generalized sequence spaces. Besides, Petrot and Suantai [7] studied the uniform Opial property of these spaces.

Our goal is first to introduce modular sequence space $l_\rho(u,v;p)$ obtained from paranormed ones by generalized weighted means on Köthe sequence spaces.

In special cases, the sequence space $l_\rho(u,v;p)$ includes the well-known Cesàro and Nörlund sequence spaces that are normed and paranormed spaces having modular structure (see [8]). We also show that the modular space $l_\rho(u,v;p)$ is a Banach space when it is equipped with Luxemburg norm.

The main purpose of this study is to show that the Kadec-Klee and Opial properties hold in the $l_\rho(u,v;p)$ space.

The organization of this paper is as follows. In the first section, we introduce some definitions and the concepts that are used throughout the paper. In the second section, we construct the modular space $l_\rho(u,v;p)$ which was obtained by paranormed space $l(u,v;p)$ and we investigate the Kadec-Klee property of this space. We also show that the modular space $l_\rho(u,v;p)$ is a Banach space under the Luxemburg norm. Finally, in the third section, uniform Opial property of the space $l_\rho(u,v;p)$ is investigated by using some topological structures.

We denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{F}$ the set of natural numbers, the set of real numbers and the scalar field, respectively. Let $(X,\|\cdot\|)$ be a real Banach space and let $B(X)(S(X))$ be the closed unit ball (the unit sphere) of X. The space of all real sequences $x=(x(i))_{i=1}^{\infty}$ is denoted by ℓ^0.

A Banach space $X=(X,\|\cdot\|)$ is said to be a Köthe sequence space if X is a subspace of ℓ^0 such that (see [9]): (i) If $x\in\ell^0$, $y\in X$, and $|x(i)|\le|y(i)|$ for all $i\in\mathbb{N}$, then $x\in X$ and $\|x\|\le\|y\|$. (ii) There is an element $x\in X$ such that $x(i)>0$ for all $i\in\mathbb{N}$.

We say that $x\in X$ is order continuous if for any sequence (x_n) in X such that $x_n(i)\le|x(i)|$ for each $i\in\mathbb{N}$ and $x_n(i)\to 0(n\to\infty)$, we have $\|x_n\|\to 0$ holds.

A Köthe sequence space X is said to be order continuous if all sequences in X are order continuous. It is easy to see that $x\in X$ is order continuous if and only if $\|(0,0,\dots,0,x(n+1),x(n+2),\dots)\|\to 0$ as $n\to\infty$.

A Banach space X is said to have the *Kadec-Klee property* (or property (H)) if every weakly convergent sequence on the unit sphere with the weak limit in the sphere is convergent in norm.

Let X be a real Banach space. We say that X has the *Opial property* if for any weakly null sequence $\{x_n\}$ in X and any x in $X\setminus\{0\}$, the inequality

$$\liminf_{n\to\infty}\|x_n\|<\liminf_{n\to\infty}\|x_n+x\| \tag{1.1}$$

holds (see [10]). Opial [10] has proved that ℓ_p space $(1<p<\infty)$ has this property.

Franchetti [11] has shown that any infinite dimensional Banach space has an equivalent norm that satisfies the Opial property.

We say that X has the *uniform Opial property* (see [10]) if for any $\varepsilon>0$ there exists $r>0$ such that for any x from X with $\|x\|\ge\varepsilon$ and any weakly null sequence $\{x_n\}$ in the unit sphere $S(X)$ of X, the following inequality

$$1+r\le\liminf_{n\to\infty}\|x_n+x\| \tag{1.2}$$

holds. It is well known that the Opial property of a Banach space X plays an important role in the fixed point theory and in the theory of differential and integral equations (see e.g [12–15]). Also the geometrical properties of some modular sequence spaces have been studied in ([16, 17]).

For a real vector space X, a function $\rho : X \to [0,\infty]$ is called a *modular* if it satisfies the following conditions:

(i) $\rho(x) = 0 \Leftrightarrow x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for all $\alpha \in \mathbb{F}$ with $|\alpha| = 1$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Further, the modular ρ is called *convex* if

(iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ holds for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For any modular ρ on X, the space

$$X_\rho = \left\{x \in X : \rho(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0^+\right\} \tag{1.3}$$

is called the *modular space*.

A sequence (x_n) of elements of X_ρ is called *modular convergent* to $x \in X_\rho$ if there exists a $\lambda > 0$ such that $\rho(\lambda(x_n - x)) \to 0$ as $n \to \infty$. If ρ is a convex modular, then the following formulas:

$$\begin{aligned} \|x\|_L &= \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}, \\ \|x\|_A &= \inf_{\lambda>0} \frac{1}{\lambda}\left(1 + \rho(\lambda x)\right) \end{aligned} \tag{1.4}$$

define two norms on X_ρ which are called the *Luxemburg norm* and the *Amemiya norm*, respectively. In addition

$$\|x\|_L \leq \|x\|_A \leq 2\|x\|_L \tag{1.5}$$

for all $x \in X_\rho$ holds (see [18]).

Proposition 1.1. *Let $(x_n) \subset X_\rho$. Then $\|x_n\|_L \to 0$ (or equivalently $\|x\|_A \to 0$) if and only if $\rho(\lambda(x_n)) \to 0$ as $n \to \infty$, for every $\lambda > 0$.*

Proof. See [19, page 15, Theorem]. □

Throughout this paper, the sequence $p = (p_r)$ is a bounded sequence of positive real numbers with $p_r > 1$, also $H = \sup p_r$ and $M = \max(1, H)$.

For $x \in \ell^0$, $i \in \mathbb{N}$, we denote

$$\begin{aligned} e_i &= (0,0,\ldots 0,1,0,0,\ldots)\left(1 \text{ seats-in } i\text{th place of } e_i\right), \\ x_{|i} &= (x(1), x(2), \ldots, x(i), 0, 0, \ldots), \\ x_{|\mathbb{N}-i} &= (0,0,\ldots,0,0,x(i+1),x(i+2),\ldots), \end{aligned} \tag{1.6}$$

and $\mathrm{supp}(x) = \{i \in \mathbb{N} : x(i) \neq 0\}$. Also let E be the set of all sequences with finite number of coordinates different from zero. Besides we will need the following inequality in the sequel:

$$|a_r + b_r|^{p_r} \leq K\left(|a_r|^{p_r} + |b_r|^{p_r}\right), \tag{1.7}$$

where $K = \max\{1, 2^{H-1}\}$, with $H = \sup_r p_r$.

By using the sequence space defined in [1], Altay and Başar [2] defined the sequence space $l(u,v;p)$ as

$$l(u,v;p) = \left\{ x = (x_k) \in \ell^0 : \sum_{k=0}^{\infty} \left| \sum_{j=0}^{k} u_k v_j x_j \right|^{p_k} < \infty \right\}. \tag{1.8}$$

We write U for the set of all sequences u such that $u_k \neq 0$ for all $k \in \mathbb{N}$. Let $u, v \in U$ and define the matrix $G(u,v) = (g_{nk})$ by

$$g_{nk} = \begin{cases} u_n v_k, & 0 \leq k \leq n, \\ 0, & k > n \end{cases} \tag{1.9}$$

for all $k, n \in \mathbb{N}$ where u_n depends only on n and v_k depends only on k.

They also showed that the space $l(u,v;p)$ is a complete linear metric space paranormed by

$$h(x) = \left(\sum_{k=0}^{\infty} \left| \sum_{j=0}^{k} u_k v_j x_j \right|^{p_k} \right)^{1/M}. \tag{1.10}$$

We now introduce a generalized modular sequence space $l_\rho(u,v;p)$ by

$$l_\rho(u,v;p) = \left\{ x \in \ell^0 : \rho(\lambda x) < \infty, \text{ for some } \lambda > 0 \right\}, \tag{1.11}$$

where

$$\rho(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} u_k v_j |x(j)| \right)^{p_k}. \tag{1.12}$$

It can be seen that $\rho : l_\rho(u,v;p) \to [0,\infty]$ is a modular on $l_\rho(u,v;p)$.

Note that the Luxemburg norm on the sequence space $l_\rho(u,v;p)$ is defined as follows:

$$\|x\|_L = \inf\left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}, \quad \forall x \in l_\rho(u,v;p) \tag{1.13}$$

or equally

$$\|x\|_L = \inf\left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|\frac{x(j)}{\lambda}\right|\right)^{p_k} \le 1 \right\}. \tag{1.14}$$

In the same way we can introduce the Amemiya norm on the sequence space $l_\rho(u,v;p)$ as follows:

$$\|x\|_A = \inf_{\lambda>0} \frac{1}{\lambda}(1+\rho(\lambda x)), \qquad \forall x \in l_\rho(u,v;p). \tag{1.15}$$

By combining special case of u_k and v_j, we get the following modular spaces: if $u_k = 1/(k+1)$ and $v_j = 1$ for all $k, j \in \mathbb{N}$, then the space $l_\rho(u,v;p)$ reduces to the modular space $\text{ces}(p)$ (see [7]) normed by

$$\|x\|_{L,\text{ces}(p)} = \inf\left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) = \sum_{k=0}^{\infty}\left(\frac{1}{k+1}\sum_{j=0}^{k} \left|\frac{x(j)}{\lambda}\right|\right)^{p_k} \le 1 \right\}. \tag{1.16}$$

If $u_k = 1/P_k$, $v_j = p_j$, and $P_k = \sum_{j=0}^{k} p_j$ for all $k, j \in \mathbb{N}$, then the space $l_\rho(u,v;p)$ reduces to the modular space $N_\rho(p)$ (see [8]), normed by

$$\|x\|_{L,N_\rho(p)} = \inf\left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) = \sum_{k=0}^{\infty}\left(\frac{1}{P_k}\sum_{j=0}^{k} p_j \left|\frac{x(j)}{\lambda}\right|\right)^{p_k} \le 1 \right\}. \tag{1.17}$$

2. Kadec-Klee Property and Modular Structure of $l_\rho(u,v;p)$

In this section we will give some basic properties of the modular ρ on the space $l_\rho(u,v;p)$. Also we will investigate some relationships between the modular ρ and the Luxemburg norm on $l_\rho(u,v;p)$.

Proposition 2.1. *The functional ρ is a convex modular on $l_\rho(u,v;p)$.*

Proof. The proof is obvious. Hence we omitted it. □

Proposition 2.2. *For any $x \in l_\rho(u,v;p)$, the following assertions are satisfied:*

(i) *If $0 < a < 1$ and $\|x\|_L > a$, then $\rho(x) > a^H$,*

(ii) *if $a \ge 1$ and $\|x\|_L < a$, then $\rho(x) < a^H$,*

(iii) *if $\|x\|_L \le 1$, then $\rho(x) \le \|x\|_L$,*

(iv) *$\|x\|_L = 1$ if and only if $\rho(x) = 1$.*

Proof. It can be proved with standard techniques in a similar way as in [16, 20]. □

Proposition 2.3. *Let (x_n) be a sequence in $l_\rho(u,v;p)$. Then*

(i) *if* $\lim_{n\to\infty}\|x_n\|_L = 1$, *then* $\lim_{n\to\infty}\rho(x_n) = 1$,

(ii) *if* $\lim_{n\to\infty}\rho(x_n) = 0$, *then* $\lim_{n\to\infty}\|x_n\|_L = 0$.

Proof. (i) Suppose that $\lim_{n\to\infty}\|x_n\|_L = 1$ and let $\varepsilon \in (0,1)$. Then there exists n_0 such that $1-\varepsilon < \|x_n\|_L < 1+\varepsilon$ for all $n \geq n_0$. By Proposition 2.2(i) and (ii), for all $n \geq n_0$, the inequality $(1-\varepsilon)^H < \|x_n\|_L < (1+\varepsilon)^H$ implies that $\lim_{n\to\infty}\rho(x_n) = 1$.

(ii) Suppose that $\|x_n\|_L \nrightarrow 0$. Then there is an $\varepsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\|_L > \varepsilon$ for all $k \in \mathbb{N}$. By Proposition 2.2(i), we obtain that $\rho(x_{n_k}) > \varepsilon^H$ for all $k \in \mathbb{N}$, which means that $\rho(x_{n_k}) \nrightarrow 0$ as $n \to \infty$. Hence $\rho(x_n) \nrightarrow 0$. □

Now we have the following.

Theorem 2.4. *The space $l_\rho(u,v;p)$ is a Banach space with respect to the Luxemburg norm defined by*

$$\|x\|_L = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}. \tag{2.1}$$

Proof. Let $(x_n(j))$ be a Cauchy sequence in $l_\rho(u,v;p)$ and $\varepsilon \in (0,1)$. Thus, there exists n_0 such that

$$\|x_n - x_m\|_L < \varepsilon, \tag{2.2}$$

for all $m, n \geq n_0$. By Proposition 2.2(iii) we have

$$\rho(x_n - x_m) < \|x_n - x_m\|_L < \varepsilon, \tag{2.3}$$

for all $n, m \geq n_0$, which means that

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|x_n(j) - x_m(j)\right|\right)^{p_k} < \varepsilon, \tag{2.4}$$

for $m, n \geq n_0$. For fixed k, the last inequality gives that

$$\left|x_n(j) - x_m(j)\right| < \varepsilon, \tag{2.5}$$

Hence we obtain that the sequence $(x_n(j))$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $x_m(j) \to x(j)$ as $m \to \infty$. Therefore, for fixed k

$$\sum_{j=0}^{k} u_k v_j \left|x_n(j) - x(j)\right| < \varepsilon \tag{2.6}$$

as $m \to \infty$ and for all $n \geq n_0$.

It remains to show that the sequence $(x(j))$ is an element of $l_\rho(u,v;p)$. From the inequality (2.3), we can write

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_k v_j\left|x_n(j)-x_m(j)\right|\right)^{p_k}<\varepsilon \tag{2.7}$$

for all $m,n\geq n_0$. So we obtain

$$\rho(x_n-x_m)\longrightarrow\rho(x_n-x), \tag{2.8}$$

as $m\to\infty$ for all $n\geq n_0$. Since for all $n\geq n_0$,

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_k v_j\left|x_n(j)-x_m(j)\right|\right)^{p_k}\longrightarrow\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_k v_j\left|x_n(j)-x(j)\right|\right)^{p_k}, \tag{2.9}$$

as $m\to\infty$, then by (2.3) we have $\rho(x_n-x)<\|x_n-x\|_L<\varepsilon$ for all $n\geq n_0$. This means that $x_n\to x$ as $n\to\infty$. So we have $(x_{n_0}-x)\in l_\rho(u,v;p)$. Since $l_\rho(u,v;p)$ is a linear space, we have $x=x_{n_0}-(x_{n_0}-x)\in l_\rho(u,v;p)$. Therefore the sequence space $l_\rho(u,v;p)$ is a Banach space with respect to Luxemburg norm. This completes the proof. □

Now, we give a proposition concerning Kadec-Klee property of the space $l_\rho(u,v;p)$.

Proposition 2.5. *Let $x\in l_\rho(u,v;p)$ and $(x_n)\subseteq l_\rho(u,v;p)$. If $\rho(x_n)\to\rho(x)$ as $n\to\infty$ and $x_n(j)\to x(j)$ as $n\to\infty$ for all $j\in\mathbb{N}$, then $x_n\to x$ as $n\to\infty$.*

Proof. Let $\varepsilon>0$. Since $\rho(x)=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}u_k v_j|x(j)|\right)^{p_k}<\infty$, we have

$$\sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k}u_k v_j\left|x(j)\right|\right)^{p_k}<\frac{\varepsilon}{6K}, \tag{2.10}$$

where $H=\sup p_k$, $K=\max\{1,2^{H-1}\}$.

Since $\rho(x_n)-\sum_{k=0}^{k_0}\left(\sum_{j=0}^{k}u_k v_j|x_n(j)|\right)^{p_k}\to\rho(x)-\sum_{k=0}^{k_0}\left(\sum_{j=0}^{k}u_k v_j|x(j))|\right)^{p_k}$ as $n\to\infty$ and $x_n(j)\to x(j)$ as $n\to\infty$ for all $j\in\mathbb{N}$, there exists $n_0\in\mathbb{N}$ such that

$$\left|\sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k}u_k v_j\left|x_n(j)\right|\right)^{p_k}-\sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k}u_k v_j\left|x(j)\right|\right)^{p_k}\right|<\frac{\varepsilon}{3K} \tag{2.11}$$

for all $n\geq n_0$. Also since $x_n(j)\to x(j)$ for all $j\in\mathbb{N}$, we have

$$\sum_{k=0}^{k_0}\left(\sum_{j=0}^{k}u_k v_j\left|x_n(j)-x(j)\right|\right)^{p_k}<\frac{\varepsilon}{3} \tag{2.12}$$

for all $n \geq n_0$. It follows from (2.3), (2.10), and (2.11) that for $n \geq n_0$,

$$\begin{aligned}
\rho(x_n - x) &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|x_n(j) - x(j)\right|\right)^{p_k} \\
&= \sum_{k=0}^{k_0}\left(\sum_{j=0}^{k} u_k v_j \left|x_n(j) - x(j)\right|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|x_n(j) - x(j)\right|\right)^{p_k} \\
&< \frac{\varepsilon}{3} + K\left[\sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|x_n(j)\right|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|x(j)\right|\right)^{p_k}\right] \\
&= \frac{\varepsilon}{3} + K\left[\rho(x_n) - \sum_{k=0}^{k_0}\left(\sum_{j=0}^{k} u_k v_j \left|x_n(j)\right|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|x(j)\right|\right)^{p_k}\right] \\
&< \frac{\varepsilon}{3} + K\left[\rho(x) - \sum_{k=0}^{k_0}\left(\sum_{j=0}^{k} u_k v_j \left|x(j)\right|\right)^{p_k} + \frac{\varepsilon}{3K} + \sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|x(j)\right|\right)^{p_k}\right] \\
&= \frac{\varepsilon}{3} + K\left[\sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|x(j)\right|\right)^{p_k} + \frac{\varepsilon}{3K} + \sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|x(j)\right|\right)^{p_k}\right] \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned} \tag{2.13}$$

This shows that $\rho(x_n - x) \to 0$ as $n \to \infty$. Hence by Proposition 2.3(ii), we have $\|x_n - x\|_L \to 0$ as $n \to \infty$, that is, $x_n \to x$. This completes the proof. □

Now, we give the main results of this paper involving geometric properties of the space $l_\rho(u, v; p)$.

Theorem 2.6. *The space $l_\rho(u, v; p)$ has the Kadec-Klee property.*

Proof. Let $x \in S(l_\rho(u, v; p))$ and $(x_n) \subseteq B(l_\rho(u, v; p))$ such that $\|x_n\|_L \to 1$ and $x_n \xrightarrow{w} x$ as $n \to \infty$. From Proposition 2.2(iv), we have $\rho(x) = 1$, so it follows from Proposition 2.3(i) that $\rho(x_n) \to \rho(x)$ as $n \to \infty$. Since $x_n \xrightarrow{w} x$ and the ith-coordinate mapping $\pi_j : l_\rho(u, v; p) \to \mathbb{R}$ defined by $\pi_j(x) = x(j)$ is continuous linear function on $l_\rho(u, v; p)$, it follows that $x_n(j) \to x(j)$ as $n \to \infty$ for all $j \in \mathbb{N}$. Thus, by Proposition 2.5, $x_n \to x$ as $n \to \infty$. □

3. Uniform Opial Property of $l_\rho(u, v; p)$

In this section, we give some topological properties of $l_\rho(u, v; p)$ and investigate its uniform opial property.

We introduce the notation $l_\rho^A(u, v; p) = (l_\rho(u, v; p), \|\cdot\|_A)$ for brevity.

Theorem 3.1. *$S(l_\rho^A(u,v;p))$ is a closed subspace of $l_\rho^A(u,v;p)$.*

Proof. Let us recall the definitions of $S(l_\rho^A(u,v;p))$ and $l_\rho^A(u,v;p)$, that is,

$$\begin{aligned} S\left(l_\rho^A(u,v;p)\right) &= \left\{x \in \ell^0 : \rho(\lambda x) < \infty \ \forall \lambda > 0\right\}, \\ l_\rho^A(u,v;p) &= \left\{x \in \ell^0 : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\right\}. \end{aligned} \tag{3.1}$$

It is easy to see that $S(l_\rho^A(u,v;p))$ is a subspace of $l_\rho^A(u,v;p)$. Next we must prove that $S(l_\rho(u,v;p))$ is closed in $l_\rho^A(u,v;p)$. In order to establish this fact, we show that if $x_n \subset S(l_\rho^A(u,v;p))$ for each $n \in \mathbb{N}$ and $x_n \to x \in l_\rho^A(u,v;p)$, then $x \in S(l_\rho^A(u,v;p))$.

Take any $c > 0$. Since $x_n \to x \in l_\rho^A(u,v;p)$, by Proposition 1.1, $\|x_n - x\|_A \to 0$ and $\rho(2c(x_n - x)) < \infty$ for some $c > 0$. Besides, since $x_n \in S(l_\rho^A(u,v;p))$, $\rho(2cx_n) < \infty$ for every $c > 0$. We must show that $\rho(cx) < \infty$ for every $c > 0$. We put

$$\rho(x) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j |x(j)|\right)^{p_k}, \tag{3.2}$$

and take the sequence $cx(j)$ such that

$$cx(j) = \frac{2c(x(j) - x_n(j))}{2} + \frac{2cx_n(j)}{2}. \tag{3.3}$$

Thus

$$\begin{aligned} \rho(cx) &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j |cx(j)|\right)^{p_k} \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{2}\sum_{j=0}^{k} u_k v_j |2c(x_n(j) - x(j)) - 2c(x_n(j))|\right)^{p_k} \\ &\le \sum_{k=0}^{\infty}\left(\frac{1}{2}\sum_{j=0}^{k} u_k v_j |2c(x_n(j) - x(j))| + \frac{1}{2}\sum_{j=0}^{k} u_k v_j |2c(x_n(j))|\right)^{p_k} \\ &\le \frac{K}{2}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j |2c(x_n(j) - x(j))|\right)^{p_k} + \frac{K}{2}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} u_k v_j |2c(x_n(j))|\right)^{p_k} \\ &\le \frac{K}{2}\rho(2c(x_n - x)) + \frac{K}{2}\rho(2cx_n), \end{aligned} \tag{3.4}$$

where $K = \max(1, 2^{H-1})$. Since $\rho(2c(x_n - x)) < \infty$ and $\rho(2cx_n) < \infty$ for every $c > 0$, we obtain $\rho(cx) < \infty$ for every $c > 0$. So $x \in S(l_\rho^A(u,v;p))$. $\square$

Lemma 3.2. *If $\rho(x) < \infty$, then the distance from x to E, $d(x, E)$, is less than or equal to* 1.

Proof. See [7, Lemma 2.2] □

Theorem 3.3. *If $\lim_{j\to\infty}\inf p_j > 1$, and $u_k \sum_{j=0}^{m} v_j = 1$, then the following assertions are true:*

(i) $S(l_\rho^A(u, v; p)) = cl(E)$, *the closure of the set E,*

(ii) $S(l_\rho^A(u, v; p))$ *is the subspace of all order continuous elements of $l_\rho^A(u, v; p)$,*

(iii) $S(l_\rho^A(u, v; p))$ *is a separable space.*

Proof. (i) Suppose that $S(l_\rho^A(u, v; p)) \subseteq cl(E)$. Then for any $x \in S(l_\rho^A(u, v; p))$ and $r \geq 1$, we have $rx \in S(l_\rho^A(u, v; p))$. Therefore by Lemma 3.2 we get $d(rx, E) \leq 1$ or $d(x, E) \leq 1/r$. Since r was arbitrary, we have that $x \in cl(E)$.

Conversely, we have to show that $S(l_\rho^A(u, v; p)) \supseteq cl(E)$. By Theorem 3.1, $S(l_\rho^A(u, v; p))$ is a closed linear subspace of $l_\rho^A(u, v; p)$. To complete the proof, it remains to show that $cl(E) \subseteq S(l_\rho^A(u, v; p))$. So it suffices to show that $e_m \in S(l_\rho^A(u, v; p))$ for each $m \in \mathbb{N}$.

Let $\beta = \liminf p_k > 1$. Fix $k \in \mathbb{N}$ and take any $r > 0$. If the sequence (v_m) is monotone, we have $u_k v_m < 1/m$ for fixed $m \neq 0$. Choose k_0 such that $k_0 \geq \max\{m, r\}$ so that $p_k \geq \beta$ for all $k \geq k_0$. Thus we get

$$\rho(re_m) = \sum_{k=m}^{k_0} (u_k v_m r)^{p_k} + \sum_{k=k_0+1}^{\infty} (u_k v_m r)^{p_k} \leq \sum_{k=m}^{k_0} \left(\frac{r}{k}\right)^{p_k} + \sum_{k=k_0+1}^{\infty} \left(\frac{r}{k}\right)^{\beta} < \infty. \tag{3.5}$$

Hence $e_m \in S(l_\rho^A(u, v; p))$.

(ii) From (i) $S(l_\rho^A(u, v; p))$ is a closed subspace of $l_\rho^A(u, v; p)$. We only need to show that each element of $S(l_\rho^A(u, v; p))$ is an order continuous element of $l_\rho^A(u, v; p)$. For $x \in S(l_\rho^A(u, v; p))$ and $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that $\rho((x - x_{|_i})/\varepsilon) < \varepsilon$ for all $i > i_0$. Therefore,

$$\left\| \frac{x - x_{|i}}{\varepsilon} \right\|_A \leq 1 + \rho\left(\frac{x - x_{|i}}{\varepsilon}\right) \leq 1 + \varepsilon \tag{3.6}$$

for all $i > i_0$. $\|x - x_{|i}\|_A \to 0$ as $i \to \infty$ holds since ε is arbitrary.

(iii) From the definition of E it can be seen that E is a countable dense. From (i) $S(l_\rho^A(u, v; p))$ has at least one countable dense subset, that is, E. Hence $S(l_\rho^A(u, v; p))$ is separable. □

Now, we establish some conditions for $l_\rho^A(u, v; p)$ to posses the uniform Opial property.

Theorem 3.4. *If $p_k > 1$ for all $k \in \mathbb{N}$ and $\limsup_{k\to\infty} p_k < \infty$, then $l_\rho^A(u, v; p)$ has the uniform Opial property.*

Proof. Take any $\varepsilon > 0$ and $x \in l_\rho^A(u, v; p)$ such that $\|x\|_A \geq \varepsilon$. Let (x_n) be a weakly null sequence in $S(l_\rho^A(u, v; p))$. Since $\limsup_{k\to\infty} p_k < \infty$, by Theorem 2.6, there exists $\delta \in (0, 2/3)$ independent of x such that $\rho(x/2) > \delta$. Also since $\limsup_{k\to\infty} p_k < \infty$, we have

$S(l_\rho^A(u,v;p)) = l_\rho^A(u,v;p)$. By Theorem 3.3(ii) x is an order continuous element. Hence we can find $k_0 \in \mathbb{N}$ such that

$$\|x_{|\mathbb{N}-k_0}\|_A < \frac{\delta}{4},$$
$$\sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|\frac{x(j)}{2}\right|\right)^{p_k} < \frac{\delta}{8}. \tag{3.7}$$

It follows that

$$\delta \le \sum_{k=0}^{k_0}\left(\sum_{j=0}^{k} u_k v_j \left|\frac{x(j)}{2}\right|\right)^{p_k} + \sum_{k=k_0+1}^{\infty}\left(\sum_{j=0}^{k} u_k v_j \left|\frac{x(j)}{2}\right|\right)^{p_k}$$
$$\delta \le \sum_{k=0}^{k_0}\left(\sum_{j=0}^{k} u_k v_j \left|\frac{x(j)}{2}\right|\right)^{p_k} + \frac{\delta}{8}, \tag{3.8}$$

which implies

$$\frac{7\delta}{8} \le \sum_{k=0}^{k_0}\left(\sum_{j=0}^{k} u_k v_j \left|\frac{x(j)}{2}\right|\right)^{p_k}. \tag{3.9}$$

From $x_n \xrightarrow{w} 0$, it follows that $x_n(j) \to 0$ for all $j \in \mathbb{N}$. So there exists $n_0 \in \mathbb{N}$ such that

$$\left\|x_{n|_{k_0}}\right\|_A < \frac{\delta}{4} \tag{3.10}$$

for all $n > n_0$. We must show that

$$1 + \delta \le \liminf_{n\to\infty}\|x_n + x\|_A. \tag{3.11}$$

We have

$$\begin{aligned}\|x + x_n\|_A &= \left\|(x + x_n)_{|k_0} + (x + x_n)_{|\mathbb{N}-k_0}\right\|_A \\ &\ge \left\|x_{|k_0} + x_{n|\mathbb{N}-k_0}\right\|_A - \left\|x_{|\mathbb{N}-k_0}\right\|_A - \left\|x_{n|_{k_0}}\right\|_A \\ &\ge \left\|x_{|k_0} + x_{n|\mathbb{N}-k_0}\right\|_A - \frac{\delta}{2}.\end{aligned} \tag{3.12}$$

We now consider $\|x_{|k_0} + x_{n|\mathbb{N}-k_0}\|_A$. Since $p_k > 1$ for all $k \in \mathbb{N}$, we have that there exists $c_n > 0$ such that

$$\left\|x_{|k_0} + x_{n|\mathbb{N}-k_0}\right\|_A = \frac{1}{c_n}\left[1 + \rho\left(c_n\left(x_{|k_0} + x_{n|\mathbb{N}-k_0}\right)\right)\right]. \tag{3.13}$$

Combining this fact with (3.12) and considering the fact that $\rho(x+y) \geq \rho(x)+\rho(y)$ if supp(x)$\cap$ supp(y) $= \emptyset$, we get

$$\|x + x_n\|_A \geq \frac{1}{c_n} + \frac{1}{c_n}\rho\left(c_n x_{|k_0}\right) + \frac{1}{c_n}\rho\left(c_n x_{|\mathbb{N}-k_0}\right) - \frac{\delta}{2} = \left\|x_{n|\mathbb{N}-k_0}\right\|_A + \frac{1}{c_n}\rho\left(c_n x_{|k_0}\right) - \frac{\delta}{2}. \tag{3.14}$$

It suffices to consider the case $c_n \geq 1/2$, since in the other case we have $||x + x_n||_A > 2 - \delta/2 > 1 + \delta$. Since $2c_n \geq 1$, by convexity of the function $t \to |t|^{p_k}$, we have $\rho(c_n x_{|k_0}) \geq 2c_n\rho(x_{|k_0})$. Thus, inequalities (3.9) and (3.12) imply that

$$\begin{aligned} \|x + x_n\|_A &\geq \left\|x_{|\mathbb{N}-k_0}\right\|_A + 2\rho\left(\frac{x_{|k_0}}{2}\right) - \frac{\delta}{2} \\ &= \left\|x_{|\mathbb{N}-k_0}\right\|_A + 2\sum_{k=0}^{k_0}\left(\sum_{j=0}^{k} u_k v_j \left|\frac{x(j)}{2}\right|\right)^{p_k} - \frac{\delta}{2} \\ &> 1 - \frac{\delta}{4} + \frac{14\delta}{8} - \frac{\delta}{2} = 1 + \delta, \end{aligned} \tag{3.15}$$

which implies that $\liminf_{n\to\infty}||x_n + x||_A \geq 1 + \delta$. This completes the proof . □

Corollary 3.5. *If $p_k > 1$ for all $k \in \mathbb{N}$ and $\limsup_{k\to\infty} p_k < \infty$, then $N_\rho(p)$ has the uniform Opial property.*

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