Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 832686, 14 pages
doi:10.1155/2009/832686

Research Article

Composition Operator on Bergman-Orlicz Space

Zhijie Jiang[1,2] and Guangfu Cao[1]

[1] *College of Mathematics and Information Science, Guangzhou University, Guangzhou, Guangdong, 510006, China*
[2] *Department of Mathematics, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China*

Correspondence should be addressed to Zhijie Jiang, matjzj@126.com

Received 19 May 2009; Accepted 22 October 2009

Recommended by Shusen Ding

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let $dA(z)$ denote the normalized area measure on $\mathbb{D}$. For $\alpha > -1$ and Φ a twice differentiable, nonconstant, nondecreasing, nonnegative, and convex function on $[0,\infty)$, the Bergman-Orlicz space L_α^Φ is defined as follows $L_\alpha^\Phi = \{f \in H(\mathbb{D}) : \int_{\mathbb{D}} \Phi(\log^+|f(z)|)(1-|z|^2)^\alpha dA(z) < \infty\}$. Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ induced by φ is defined by $C_\varphi f = f \circ \varphi$ for f analytic in $\mathbb{D}$. We prove that the composition operator C_φ is compact on L_α^Φ if and only if C_φ is compact on A_α^2, and C_φ has closed range on L_α^Φ if and only if C_φ has closed range on A_α^2.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ induced by φ is defined by $C_\varphi f = f \circ \varphi$ for f analytic in $\mathbb{D}$. The idea of studying the general properties of composition operators originated from Nordgren [1]. As a sequence of Littlewood's subordinate theorem, each φ induces a bounded composition operator on the Hardy spaces $H^p(\mathbb{D})$ for all p $(0 < p < \infty)$ and the weighted Bergman spaces $A_\alpha^p(\mathbb{D})$ for all p $(0 < p < \infty)$ and for all α $(-1 < \alpha < \infty)$. Thus, boundedness of composition operators on these spaces becomes very clear. Nextly, a natural problem is how to characterize the compactness of composition operators on these spaces, which once was a central problem for mathematicians who were interested in the theory of composition operators. The study of compact composition operators was started by Schwartz, who obtained the first compactness theorem in his thesis [2], showing that the integrability of $(1-|\varphi|)^{-1}$ over $\partial\mathbb{D}$ implied the compactness of C_φ on H^p. The work was continued by Shapiro and Taylor [3], who showed that C_φ was not compact on H^2 whenever φ had a finite angular derivative at some point of

$\partial\mathbb{D}$. Moreover, MacCluer and Shapiro [4] pointed out that nonexistence of the finite angular derivatives of φ was a sufficient condition for the compactness of C_φ on A^p_α but it failed on H^p. So looking for an appropriate tool of characterizing the compactness of C_φ on H^p was difficult at that time. Fortunately, Shapiro [5] developed relations between the essential norm of C_φ on H^2 and the Nevanlinna counting function of φ, and he obtained a nice essential norm formula of C_φ in 1987. As a result, he completely gave a characterization of the compactness of C_φ in terms of the function properties of φ.

Another solution to the compactness of C_φ on H^2 was done by the Aleksandrov measures which was introduced by Cima and Matheson [6]. It is well known that the harmonic function $\Re((\lambda + \varphi(z))/(\lambda - \varphi(z)))$ can be expressed by the Possion integral

$$\Re\frac{\lambda + \varphi(z)}{\lambda - \varphi(z)} = \int_{\partial\mathbb{D}} P(z,\zeta)dm_\lambda(\zeta) \tag{1.1}$$

for each $\lambda \in \partial\mathbb{D}$. Cima and Matheson applied σ_λ the singular part of m_λ to give the following expression:

$$\left\|C_\varphi\right\|_e^2 = \sup_{\lambda\in\partial\mathbb{D}}\|\sigma_\lambda\|. \tag{1.2}$$

They showed that C_φ was compact on H^2 if and only if all the measures m_λ were absolutely continuous.

The study of compactness of composition operators is also an important subject on other analytic function spaces, and we have chosen two typical examples above, and for more related materials one can consult [7, 8]. Another natural interesting subject is the composition operator with closed range. Considering angular derivatives of φ, it is known that C_φ is compact on A^2 if and only if φ fails to have finite angular derivatives on $\partial\mathbb{D}$, in this case, C_φ does not have closed range since C_φ is not a finite rank operator. And if φ has finite angular derivatives on $\partial\mathbb{D}$, then φ is necessarily a finite Blaschke product and hence one can easily verify that C_φ has closed range on A^2. Zorboska has given a necessary and sufficient condition for C_φ with closed range on H^2, and she also has done on A^p_α [9]. Luecking [10] considered the same question on Dirichlet space after Zorboska's work. Recently, Kumar and Partington [11] have studied the weighted composition operators with closed range on Hardy spaces and Bergman spaces.

This paper will study the compactness of composition operator on Bergman-Orlicz space. We are mainly inspired by the following results.

(i) Liu et al. [12] showed that composition operator was bounded on Hardy-Orlicz space. Lu and Cao [13] also showed that composition operator was bounded on Bergman-Orlicz space.

(ii) A composition operator was compact on the Nevanlinna class $\mathcal{N}$ if and only if it was compact on H^2 [14].

(iii) If a composition operator was compact on H^p for some $p > 0$, then it was compact on H^p for all $p > 0$ [3]. Moreover, paper [15] compared the compactness of composition operators on Hardy-Orlicz spaces and on Hardy spaces. All these results lead us to wonder whether there is a equivalence for the compactness of

C_φ on A_α^2 and on the Bergman-Orlicz space, and whether there is a equivalence for the closed range of C_φ on A_α^2 and on the Bergman-Orlicz space. In this paper, we are going to give affirmative answers for the proceeding questions.

2. Preliminaries

Let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. Let $dA(z)$ denote the normalized area measure on $\mathbb{D}$, that is, $A(\mathbb{D}) = 1$. Let $\mathcal{S}$ denote the class of strongly convex functions $\Phi : [0,+\infty) \to [0,+\infty)$, which satisfies

(i) $\Phi(0) = \Phi'(0) = 0$, $\Phi(t)/t \to \infty$ as $t \to \infty$,

(ii) Φ'' exists on $[0,+\infty)$,

(iii) $\Phi(2t) \leq C\Phi(t)$ for some positive constant C and for all $t > 0$.

For $\Phi \in \mathcal{S}$ and $\alpha > -1$ the Bergman-Orlicz space L_α^Φ is defined as follows:

$$L_\alpha^\Phi = \left\{ f \in H(\mathbb{D}) : \|f\|_\Phi = \int_{\mathbb{D}} \Phi(\log^+|f(z)|)\left(1-|z|^2\right)^\alpha dA(z) < \infty \right\}, \tag{2.1}$$

where $\log^+ x = \max\{0, \log x\}$. Although $\|\cdot\|_\Phi$ does not define a norm in L_α^Φ, it holds that the $d(f,g) = \|f-g\|_\Phi$ defines a metric on L_α^Φ, and makes L_α^Φ into a complete metric space. Obviously, the inequalities

$$\begin{aligned} \log^+ x \leq \log(1+x) \leq 1 + \log^+ x, \quad x \geq 0, \\ 2\log^+ x \leq \log\left(1+x^2\right) \leq 1 + 2\log^+ x, \quad x \geq 0, \end{aligned} \tag{2.2}$$

and the fact that Φ is nondecreasing convex function imply that

$$\begin{aligned} \Phi(\log^+ x) &\leq \Phi(\log(1+x)) \leq \Phi(1+\log^+ x) \\ &\leq \frac{1}{2}\Phi(2) + \frac{1}{2}\Phi(2\log^+ x) \\ &\leq \frac{1}{2}\Phi(2) + \frac{1}{2}C\Phi(\log^+ x), \\ \Phi(\log^+ x) &\leq \Phi(2\log^+ x) \leq \Phi\left(\log\left(1+x^2\right)\right) \\ &\leq \Phi(1+2\log^+(x)) \\ &\leq \frac{1}{2}\Phi(2) + \frac{1}{2}\Phi(4\log^+ x) \\ &\leq \frac{1}{2}\Phi(2) + \frac{1}{2}C\Phi(\log^+ x). \end{aligned} \tag{2.3}$$

Then $f \in L_\alpha^\Phi$ if and only if

$$\int_{\mathbb{D}} \Phi(\log(1+|f(z)|))\left(1-|z|^2\right)^\alpha dA(z) < \infty \tag{2.4}$$

or if and only if

$$\int_{\mathbb{D}} \Phi\left(\log\left(1+|f(z)|^2\right)\right)\left(1-|z|^2\right)^\alpha dA(z) < \infty. \tag{2.5}$$

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preccurlyeq b$ means that there is a positive constant C such that $a \leq Cb$. Moreover, if both $a \preccurlyeq b$ and $b \preccurlyeq a$ hold, we write $a \asymp b$ and say that a is asymptotically equivalent to b.

In this section we will prove several auxiliary results which will be used in the proofs of the main results in this paper.

Lemma 2.1. *If $f \in L_\alpha^\Phi$, then*

$$\|f\|_\Phi \asymp \Phi\left(\log\left(1+f(0)|^2\right)\right) + \int_{\mathbb{D}} f^\Phi(z)\left(1-|z|^2\right)^\alpha dA(z), \tag{2.6}$$

where Δ is Laplacian and $f^\Phi(z) = \Delta\Phi(\log(1+|f(z)|^2))$.

Proof. By the Green Theorem, if $u, v \in C^2(\overline{\Omega})$, where Ω is a domain in the plane with smooth boundary, then

$$\int_\Omega (u\Delta v - v\Delta u)dx\,dy = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds. \tag{2.7}$$

Let $0 < \varepsilon < r < 1$, $u(z) = \log(r/|z|)$, $v(z) = \Phi(\log(1+|f(z)|^2))$, and $\Omega = \{z \in \mathbb{D} : \varepsilon < |z| < r\}$. Since $\Delta u(z) = 0$, by (2.7) we have

$$\begin{aligned}\int_\Omega \Delta\Phi\left(\log\left(1+|f(z)|^2\right)\right)\log\frac{r}{|z|}dx\,dy + \log\frac{r}{\varepsilon}\int_{|z|=\varepsilon}\frac{\partial}{\partial n}\Phi\left(\log\left(1+|f(z)|^2\right)\right)ds \\ = \int_{|z|=r}\frac{\Phi\left(\log\left(1+|f(z)|^2\right)\right)}{r}ds - \int_{|z|=\varepsilon}\frac{\Phi\left(\log\left(1+|f(z)|^2\right)\right)}{\varepsilon}ds.\end{aligned} \tag{2.8}$$

Since $(\partial/\partial n)(\Phi(\log(1+|f(z)|^2)))$ is bounded near to 0, we get

$$\lim_{\varepsilon\to 0}\log\frac{r}{\varepsilon}\int_{|z|=\varepsilon}\frac{\partial}{\partial n}\Phi\left(\log\left(1+|f(z)|^2\right)\right)ds = 0. \tag{2.9}$$

Let $\varepsilon \to 0$ in (2.8), we have

$$\begin{aligned}\int_{|z|<r} \Delta\Phi\left(\log\left(1+|f(z)|^2\right)\right)\log\frac{r}{|z|}dx\,dy \\ = \int_0^{2\pi}\Phi\left(\log\left(1+\left|f\left(re^{i\theta}\right)\right|^2\right)\right)d\theta - 2\pi\Phi\left(\log\left(1+|f(0)|^2\right)\right).\end{aligned} \tag{2.10}$$

Integrating equality (2.10) with respect to r from 0 to 1, we obtain

$$\int_0^1\int_{|z|<r}\Delta\Phi\left(\log\left(1+|f(z)|^2\right)\right)\log\frac{r}{|z|}\left(1-r^2\right)^{\alpha} r dr\, dA(z) = \|f\|_{\Phi} - \frac{2\pi}{\alpha+1}\Phi\left(\log\left(1+|f(0)|^2\right)\right). \tag{2.11}$$

Thus

$$\begin{aligned}\|f\|_{\Phi} &= \frac{2\pi}{\alpha+1}\Phi\left(\log\left(1+|f(0)|^2\right)\right)+\int_0^1\int_{|z|<r}\Delta\Phi\left(\log\left(1+|f(z)|^2\right)\right)\log\frac{r}{|z|}\left(1-r^2\right)^{\alpha} rdr\, dA(z) \\ &= \frac{2\pi}{\alpha+1}\Phi\left(\log\left(1+|f(0)|^2\right)\right)+\int_{\mathbb{D}}\Delta\Phi\left(\log\left(1+|f(z)|^2\right)\right)dA(z)\int_{|z|}^1\log\frac{r}{|z|}\left(1-r^2\right)^{\alpha} rdr.\end{aligned} \tag{2.12}$$

Since $\int_{|z|}^1 \log(r/|z|)(1-r^2)^{\alpha} rdr \asymp (1-z|^2)^{2+\alpha}$,

$$\begin{aligned}\|f\|_{\Phi} &\asymp \frac{2\pi}{\alpha+1}\Phi\left(\log\left(1+|f(0)|^2\right)\right)+\int_{\mathbb{D}}\Delta\Phi\left(\log\left(1+|f(z)|^2\right)\right)\left(1-|z|^2\right)^{\alpha}dA(z) \\ &\asymp \Phi\left(\log\left(1+|f(0)|^2\right)\right)+\int_{\mathbb{D}} f^{\Phi}\left(1-|z|^2\right)^{\alpha}dA(z),\end{aligned} \tag{2.13}$$

the proof is complete. □

Let φ be an analytic self-map of $\mathbb{D}$. The generalized Nevanlinna counting function of φ is defined by

$$N_{\varphi,\alpha+2}(w) = \sum_{z\in\varphi^{-1}(w)}\left(\log\frac{1}{|z|}\right)^{\alpha+2}. \tag{2.14}$$

Lemma 2.2 (see [9]). *If φ is an analytic self-map of $\mathbb{D}$ and g is a nonnegative measurable function in $\mathbb{D}$, then*

$$\int_{\mathbb{D}} g \circ \varphi(z) \left|\varphi'(z)\right|^2 \left(\log \frac{1}{|z|}\right)^{\alpha+2} dA(z) = \int_{\mathbb{D}} g(z) N_{\varphi,\alpha+2}(z) dA(z). \tag{2.15}$$

Lemmas 2.1 and 2.2 (see [9]) can lead to the following corollary.

Corollary 2.3. *Let φ be an analytic self-map of $\mathbb{D}$ and $f \in H(\mathbb{D})$, then*

$$\|f \circ \varphi\| \asymp \Phi\left(1 + \log \left|f \circ \varphi(0)\right|^2\right) + \int_{\mathbb{D}} f^{\Phi}(w) N_{\alpha+2}(w) dA(w). \tag{2.16}$$

We will end this section with the following lemma, which illustrates that the counting functional $\delta_z : f \mapsto f(z)$ is continuous on L_α^Φ.

Lemma 2.4. *Let $f \in L_\alpha^\Phi$, then*

$$\left|f(z)\right| \leq \exp\left(\Phi^{-1}\left(\frac{C\|f\|_\Phi}{\left(1-|z|^2\right)^{\alpha+2}}\right)\right) \quad \forall\, z \text{ in } \mathbb{D}. \tag{2.17}$$

Proof. By the subharmonicity of map $z \mapsto \log(1 + |f(z)|)$, we get

$$\log\left(1 + \left|f(z)\right|\right) \leq \frac{1}{A_\alpha(\mathbb{D}(z,(1-|z|)/2))} \int_{\mathbb{D}(z,(1-|z|)/2)} \log\left(1 + \left|f(z)\right|\right)\left(1 - z|^2\right)^\alpha dA(z). \tag{2.18}$$

Since $\Phi(\log(1 + |f(z)|))$ is convex and increasing, we have

$$\begin{aligned}
\Phi\left(\log\left(1 + \left|f(z)\right|\right)\right) &\leq \frac{1}{A_\alpha(\mathbb{D}(z,(1-|z|)/2))} \int_{\mathbb{D}(z,(1-|z|)/2)} \Phi\left(\log\left(1 + \left|f(z)\right|\right)\right) dA_\alpha(z) \\
&\leq \frac{C}{(1-|z|^2)^{\alpha+2}} \int_{\mathbb{D}(z,(1-|z|)/2)} \Phi\left(\log\left(1 + \left|f(z)\right|\right)\right) dA_\alpha(z) \\
&\leq \frac{C}{(1-|z|^2)^{\alpha+2}} \int_{\mathbb{D}} \Phi\left(\log\left(1 + \left|f(z)\right|\right)\right) dA_\alpha(z) \\
&\leq \frac{C\|f\|_\Phi}{\left(1-|z|^2\right)^{\alpha+2}}.
\end{aligned} \tag{2.19}$$

Since $\Phi(\log^+ x) \le \Phi(\log(1+x))$, we get

$$\Phi\left(\log^+ \left|f(z)\right|\right) \le \frac{C\|f\|_\Phi}{\left(1-|z|^2\right)^{\alpha+2}}, \tag{2.20}$$

that is, $\log^+|f(z)| \le \Phi^{-1}(C\|f\|_\Phi/(1-|z|^2)^{\alpha+2})$. Thus $|f(z)| \le \exp(\Phi^{-1}(C\|f\|_\Phi/(1-|z|^2)^{\alpha+2}))$. □

3. Compactness

In this section, we are going to investigate the equivalence between compactness of composition operator on the Bergman-Orlicz space L_α^Φ and on the weighted Bergman space A_α^2. The following lemma characterizes the compactness of C_φ on L_α^Φ in terms of sequential convergence, whose proof is similar to that in [7, Proposition 3.11].

Lemma 3.1. *Let φ be an analytic self-map of $\mathbb{D}$, bounded operator C_φ is compact on L_α^Φ if and only if whenever $\{f_n\}$ is bounded in L_α^Φ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_\varphi f_n\|_\Phi \to 0$ as $n \to \infty$.*

In order to characterize the compactness of C_φ, we need to introduce the notion of Carleson measure. For $|\xi| = 1$ and $\delta > 0$ we define $Q_\delta(\xi) = \{z \in \mathbb{D} : |z-\xi| < \delta\}$. A positive Borel measure μ on $\mathbb{D}$ is called a Carleson measure if $\sup_{|\xi|=1}\mu(Q_\delta(\xi)) = O(\delta^{\alpha+2})$. Moreover, if μ satisfies the additional condition $\lim_{\delta\to 0}\mu(Q_\delta(\xi))/\delta^{\alpha+2} = 0$, μ is called a vanishing Carleson measure (see [16] for the further information of Carleson measure). The following result for the compactness of C_φ on A_α^2 is useful in the proof of Theorem 3.3.

Lemma 3.2 (see [14, 17]). *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent:*

(i)*C_φ is compact on A_α^2,* (ii)$\lim_{|z|\to 1}(N_{\varphi,\alpha+2}(z)/(1-|z|^2)^{\alpha+2}) = 0$, *and* (iii) *the pull measure μ_φ is a vanshing Carleson measure on $\mathbb{D}$.*

Theorem 3.3. *Let φ be an analytic self-map of $\mathbb{D}$, then C_φ is compact on A_α^2 if and only if C_φ is compact on L_α^Φ.*

Proof. First we assume that C_φ is compact on A_α^2. Choose a sequence $\{f_n\}$ that is bounded by a positive constant M in L_α^Φ and converges to zero uniformly on compact subsets of $\mathbb{D}$. By Lemma 3.1, it is enough to show that $\|f_n \circ \varphi\|_\Phi \to 0$ as $n \to \infty$. Let $\varepsilon > 0$, we can find $0 < r < 1$ such that $N_{\varphi,\alpha+2}(z) < \varepsilon(1-|z|^2)^{\alpha+2}$ for all $|z| > r$. Since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, so is f_n'. Thus we can choose $N > 0$ such that $|f_n| < \varepsilon$ and $|f_n'| < \varepsilon$ on $\overline{r\mathbb{D}}$, whenever $n > N$. Hence for such n we have

$$\|C_\varphi f_n\|_\Phi \le \Phi\left(\log\left(1+\left|f_n\circ\varphi(0)\right|^2\right)\right) + \int_{\mathbb{D}} f_n^\Phi N_{\varphi,\alpha+2}\,dA(w). \tag{3.1}$$

As $|f_n \circ \varphi(0)| \to 0$ as $n \to \infty$ and $\Phi(\log(1+|f_n \circ \varphi(0)|^2)) \to 0$ as $n \to \infty$, we only need to verify that $\int_{\mathbb{D}} f_n^{\Phi} N_{\varphi,\alpha+2} dA(w) \to 0$ as $n \to \infty$. Now

$$\int_{\mathbb{D}} f_n^{\Phi} N_{\varphi,\alpha+2} dA(w) = \int_{\overline{r\mathbb{D}}} f_n^{\Phi} N_{\varphi,\alpha+2} dA(w) + \int_{\mathbb{D}\setminus\overline{r\mathbb{D}}} f_n^{\Phi} N_{\varphi,\alpha+2} dA(w) = \mathrm{I} + \mathrm{II}. \tag{3.2}$$

We first prove that the first term in previous equality is bounded by a constant multiple of ε

$$\begin{aligned}
\mathrm{I} &= \int_{\overline{r\mathbb{D}}} f_n^{\Phi} N_{\varphi,\alpha+2} dA(w) \\
&= \int_{\overline{r\mathbb{D}}} \left[\Phi''\left(\log\left(1+|f_n(w)|^2\right)\right)|f_n(w)|^2 + \Phi'\left(\log\left(1+|f_n(w)|^2\right)\right)\right] \\
&\quad \times \frac{|f_n(w)'|^2}{\left(1+|f_n(w)|^2\right)^2} N_{\varphi,\alpha+2}(w) dA(w) \\
&\leq \left[\Phi''(\log(1+\varepsilon))\varepsilon + \Phi'(\log(1+\varepsilon))\right]\varepsilon \int_{\overline{r\mathbb{D}}} N_{\varphi,\alpha+2}(w) dA(w) \\
&\leq \left[\Phi''(\log(1+\varepsilon))\varepsilon + \Phi'(\log(1+\varepsilon))\right]\varepsilon.
\end{aligned} \tag{3.3}$$

Now, we show that the previous second term above is also bounded by a constant multiple of ε

$$\mathrm{II} = \int_{\mathbb{D}\setminus\overline{r\mathbb{D}}} f_n^{\Phi} N_{\varphi,\alpha+2} dA(w) \leq \varepsilon f_n^{\Phi}(w)\left(1-|w|^2\right)^{\alpha} \leq \varepsilon \left\| f_n^{\Phi} \right\| \leq M\varepsilon. \tag{3.4}$$

Conversely, we assume that C_φ is compact on L_α^Φ. By Lemma 3.2, we need to verify that μ_φ is a vanishing Carleson measure. For $0 < \delta < 1$ and $\xi \in \partial\mathbb{D}$ we write $a = (1-\delta)\xi$ and $g_a(z) = (1-|a|^2)^{\alpha+2}/(1-\overline{a}z)^{2\alpha+4}$. Then $|g_a| \in L^1(\mathbb{D}, dA_\alpha)$. Put $G(z) = \Phi(|g_a(z)|)$. G is well defined, beacuse G is nondecreasing on range of $|g_a|$. Since Φ^{-1} is concave, there is a constant $C > 0$ such that $\Phi^{-1}(t) \leq Ct$ for enough big t. Thus we get $G(z) \in L^1(\mathbb{D}, dA_\alpha)$. Set $h(z) = \exp\int_0^{2\pi}((e^{it}+z)/(e^{it}-z))G(e^{it})dt$. Since $\Phi(\log^+|h(z)|) = \Phi(G(z)) = |g_a(z)| \in L^1(\mathbb{D}, dA_\alpha)$, it means that $h \in L_\alpha^\Phi$. Let $f_a(z) = (1-|a|)h(z)$. Then clearly $f_a \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Moreover,

$$\begin{aligned}
\|f_a\|_\Phi &= \int_{\mathbb{D}} \Phi\left(\log^+|f_a(z)|\right) dA_\alpha(z) \leq \int_{\mathbb{D}} \Phi\left(\log^+|h(z)|\right) dA_\alpha(z) \\
&= \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^{\alpha+2}}{|1-\overline{a}z|^{2\alpha+4}} dA_\alpha(z) = 1.
\end{aligned} \tag{3.5}$$

On the other hand, if $|1 - z\overline{\zeta}|/(1 - |a|) < \gamma$ for some fixed $0 < \gamma < 1/4$, where $\zeta = a/|a|$, that is, $z \in Q_{\gamma\delta}(\zeta)$, we have

$$\begin{aligned}\frac{1-|a|^2}{|1-\overline{a}z|^2} &= \frac{1-|a|^2}{(1-|a|)^2}\frac{(1-|a|)^2}{|1-\overline{a}z|^2} = \frac{1-|a|^2}{(1-|a|)^2}\left(1+\frac{|a|\left(1-z\overline{\zeta}\right)}{1-|a|}\right)^{-2} \\ &\geq \frac{1}{(1+\gamma)^2}\frac{1-|a|^2}{(1-|a|)^2} > \frac{1-|a|^2}{4(1-|a|)^2} \geq \frac{1}{4\delta}.\end{aligned} \tag{3.6}$$

Hence, for $z \in Q_{\gamma\delta}(\zeta)$ we have

$$\Phi^{-1}\left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{\alpha+2} \geq \Phi^{-1}\left(\frac{1}{4\delta}\right)^{\alpha+2}. \tag{3.7}$$

Thus, for $z \in Q_{\gamma\delta}(\zeta)$ we obtain

$$\begin{aligned}\Phi\left(\log^+\left|f_a(z)\right|\right) &= \Phi\left(\log^+(1-|a|)h(z)\right) = \Phi\left(\log^+(1-|a|) + \log^+|h(z)|\right) \\ &\geq \Phi\left(\log^+|h(z)|\right) = \left|g_a(z)\right| = \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{\alpha+2} \geq \left(\frac{1}{4\delta}\right)^{\alpha+2}.\end{aligned} \tag{3.8}$$

So, for all $\xi \in \partial\mathbb{D}$ and $0 < \delta < 1$ we get

$$\begin{aligned}\left(\frac{1}{4\delta}\right)^{\alpha+2}\mu_\varphi(Q_{\gamma\delta}(\xi)) &\leq \int_{Q_{\gamma\delta}(\xi)}\Phi\left(\log^+\left|f_a(z)\right|\right)d\mu_\varphi \leq \int_{\mathbb{D}}\Phi\left(\log^+\left|f_a(z)\right|\right)d\mu_\varphi \\ &= \int_{\mathbb{D}}\Phi\left(\log^+\left|f_a\circ\varphi(z)\right|\right)dA_\alpha(z) = \left\|C_\varphi f_a\right\|_\Phi.\end{aligned} \tag{3.9}$$

For the compactness of C_φ, we know that $\|C_\varphi f_a\|_\Phi \to 0$ as $|a| \to 1$, which means that $\lim_{\delta\to 1}(\mu_\varphi(Q_{\gamma\delta}(\xi))/\delta^{\alpha+2}) = 0$ uniformly for $\xi \in \partial\mathbb{D}$. This means that μ_φ is a vanishing Carleson measure. By Lemma 3.2, C_φ is compact on A^2_α. □

For special case $\Phi(t) = t^p (p > 1)$, the Bergman-Orlicz space L^Φ_α is called the area-type Nevanlinna class and we write $\mathcal{N}^p_\alpha$.

Corollary 3.4. *Let φ be an analytic self-map of $\mathbb{D}$, then C_φ is compact on A^2_α if and only if C_φ is compact on $\mathcal{N}^p_\alpha$.*

Remark 3.5. Theorem 3.3 may be not true if Φ does not satisfy the given conditions in this paper. For example, if Φ is a nonnegative function on $\mathbb{R}$ such that $\Phi \to 0$ as $x \to -\infty$, and

Φ is nondecreasing but $\Phi(x) > 0$ for some $x \neq 0$. Then the compactness of C_φ on the Bergman space A^2 (i.e., $\alpha = 0$) is different from that on L^Φ_α. Here L^Φ_α is defined as follows:

$$L^\Phi_\alpha = \left\{ f \in H(\mathbb{D}) : \exists t > 0, \text{ s.t.} \int_{\mathbb{D}} \Phi\left(\log\left|tf(z)\right|\right) dA(z) < \infty \right\}. \tag{3.10}$$

If we take $\Phi(x) = 0$ for $x \leq 1$, and $\Phi(x) = \infty$ for $x > 1$, then L^Φ_α is $H^\infty(\mathbb{D})$. We know that C_φ is compact on $H^\infty(\mathbb{D})$ if and only if $\|\varphi\|_\infty < 1$ (consult [2]). But MacCluer and Shapiro constructed an inner function φ in [4] such that C_φ was compact on A^2.

4. Closed Range

In this section we will develop a relatively tractable if and only if condition for the composition operator on L^Φ_α with closed range. Considering that any analytic automorphism of $\mathbb{D}$ has the form $\varphi_a(z) = c(z-a)/(1-\overline{a}z)$, where $|c| = 1$ and $a \in \mathbb{D}$. By [13], we have the following lemma.

Lemma 4.1. *If one of C_φ, $C_{\varphi\circ\varphi_a}$, $C_{\varphi_a\circ\varphi}$ has closed range on L^Φ_α, so have the other two.*

Now that $L^\Phi_{\alpha,0} = \{f \in L^\Phi_\alpha : f(0) = 0\}$ is a closed subspace of L^Φ_α and $\dim(L^\Phi_\alpha / L^\Phi_{\alpha,0}) = 1$, the following lemma is easily proved.

Lemma 4.2. *Let φ be an analytic self-map of $\mathbb{D}$, then C_φ has closed range on L^Φ_α if and only if C_φ has closed range on $L^\Phi_{\alpha,0}$.*

Recall that the pseudohyperbolic metric $\rho(z,w)$, $z,w \in \mathbb{D}$ is given by

$$\rho(z,w) = \left|\frac{w-z}{1-\overline{w}z}\right|. \tag{4.1}$$

For $z \in \mathbb{D}$ and $0 < r < 1$ we define $D(z,r) = \{w \in \mathbb{D} : \rho(z,w) < r\}$. For $\varepsilon > 0$ we put $\Omega_\varepsilon = \{z \in \mathbb{D} : (1-|z|^2)/(1-|\varphi(z)|^2) \geq \varepsilon\}$ and $G_\varepsilon = \varphi(\Omega_\varepsilon)$. We say that G_ε satisfies the Φ-reverse Carleson measure condition if there exists a positive constant η such that

$$\int_{G_\varepsilon} \Phi\left(1+\log|f(z)|^2\right)\left(1-|z|^2\right)^{\alpha+2} dA(z) \geq \eta \int_{\mathbb{D}} \Phi\left(1+\log|f(z)|^2\right)\left(1-|z|^2\right)^{\alpha+2} dA(z), \tag{4.2}$$

where f is analytic in $\mathbb{D}$ and $\int_{\mathbb{D}} \Phi(1+\log|f(z)|^2)(1-|z|^2)^{\alpha+2} dA(z) < \infty$.

Theorem 4.3. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ has closed range on L^Φ_α if and only if there exists $\varepsilon > 0$ such that G_ε satisfies the Φ-reverse Carleson measure condition.*

Proof. We first assume that there exists $\varepsilon > 0$ such that G_ε satisfies the Φ-reverse Carleson measure condition. If $f \in L^\Phi_\alpha$, then

$$\begin{aligned}
\|f\circ\varphi\|_\Phi &\asymp \Phi\Big(\log\Big(1+\left|f\circ\varphi(0)\right|^2\Big)\Big)+\int_{\mathbb{D}}(f\circ\varphi)^\Phi(z)\Big(1-|z|^2\Big)^{\alpha+2}dA(z)\\
&\geq \Phi\Big(\log\Big(1+\left|f\circ\varphi(0)\right|^2\Big)\Big)+\int_{\Omega_\varepsilon}(f\circ\varphi)^\Phi(z)\Big(1-|z|^2\Big)^{\alpha+2}dA(z)\\
&\geq \varepsilon^{\alpha+2}\left(\Phi\Big(\log\Big(1+\left|f\circ\varphi(0)\right|^2\Big)\Big)+\int_{\Omega_\varepsilon}(f\circ\varphi)^\Phi(z)\Big(1-\left|\varphi(z)\right|^2\Big)^{\alpha+2}dA(z)\right)\\
&= \varepsilon^{\alpha+2}\sum_n\int_{\Omega_\varepsilon\cap R_n}(f\circ\varphi)^\Phi(z)\Big(1-\left|\varphi(z)\right|^2\Big)^{\alpha+2}dA(z)+\varepsilon^{\alpha+2}\Phi\Big(\log\Big(1+\left|f\circ\varphi(0)\right|^2\Big)\Big),
\end{aligned} \tag{4.3}$$

where $\mathcal{Z}$ is the zero point set of φ and $\{R_n\}$ is a partition of $\mathbb{D}\setminus\mathcal{Z}$ into at most countably many semiclosed polar rectangles such that φ is univalent on each R_n. Let $S_n = \varphi(R_n\cap\Omega_\varepsilon)$. Then by the change of variables involving φ, the last line above becomes

$$\begin{aligned}
&\varepsilon^{\alpha+2}\sum_n\int_{G_\varepsilon}f^\Phi\Big(1-|w|^2\Big)^{\alpha+2}\chi_{S_n}(w)dA(z)+\varepsilon^{\alpha+2}\Phi\Big(\log\Big(1+\left|f\circ\varphi(0)\right|^2\Big)\Big)\\
&\quad\geq \varepsilon^{\alpha+2}\int_{G_\varepsilon}f^\Phi\Big(1-|w|^2\Big)^{\alpha+2}dA(z)+\varepsilon^{\alpha+2}\Phi\Big(\log\Big(1+\left|f\circ\varphi(0)\right|^2\Big)\Big)\\
&\quad\geq \eta\varepsilon^{\alpha+2}\int_{\mathbb{D}}f^\Phi\Big(1-|w|^2\Big)^{\alpha+2}dA(z)+\eta\varepsilon^{\alpha+2}\Phi\Big(\log\Big(1+\left|f\circ\varphi(0)\right|^2\Big)\Big).
\end{aligned} \tag{4.4}$$

So we show that C_φ has closed range on L^Φ_α.

Conversely, by Lemma 4.2, we need to prove that C_φ has closed range on $L^\Phi_{\alpha,0}$. Suppose that there does not exist $\varepsilon > 0$ such that G_ε satisfies Φ-reverse Carleson measure condition. We can choose a sequence $\{f_n\}$ in $L^\Phi_{\alpha,0}$ such that $\int_{\mathbb{D}}\Phi(1+\log|f_n(z)|^2)dA_{\alpha+2}(z) = 1$ for all n and yet $\int_{G_n}f^\Phi(1-w|^2)^{\alpha+2}dA \to 0$ as $n\to\infty$, where $G_n = \varphi(\Omega_n)$ and $\Omega_n = \{z\in\mathbb{D} : 1-|\varphi(z)|^2 \leq n(1-|z|^2)\}$. Now

$$\begin{aligned}
\|f_n\circ\varphi\|_\Phi &\asymp \Phi\Big(\log\Big(1+\left|f_n\circ\varphi(0)\right|^2\Big)\Big)+\int_{\mathbb{D}}(f_n\circ\varphi)^\Phi(z)dA_{\alpha+2}(z)\\
&= \Phi\Big(\log\Big(1+\left|f_n\circ\varphi(0)\right|^2\Big)\Big)+\int_{\Omega_n}(f_n\circ\varphi)^\Phi(z)dA_{\alpha+2}(z)\\
&\quad+\int_{\mathbb{D}\setminus\Omega_n}(f_n\circ\varphi)^\Phi(z)dA_{\alpha+2}(z).
\end{aligned} \tag{4.5}$$

Since φ is an analytic self-map of $\mathbb{D}$. The Nevanlinna counting function $N_{\varphi,\alpha+2}$ satisfies

$$N_{\varphi,\alpha+2}(z) = O\left(\left(\log\frac{1}{|z|}\right)^{\alpha+2}\right) \tag{4.6}$$

as $|z| \longrightarrow 1$. Using (4.6) and decompositions of the disk into polar rectangles [8], one can find a positive constant c_1 such that

$$\int_{\Omega_n} (f_n \circ \varphi)^{\Phi} dA_{\alpha+2}(z) = \int_{\Omega_n} f_n^{\Phi} N_{\varphi,\alpha+2} dA(z) \le c_1 \int_{G_n} f_n^{\Phi}(z)\left(1-|z|^2\right)^{\alpha+2} dA(z) \longrightarrow 0 \tag{4.7}$$

as $n \longrightarrow \infty$, and

$$\begin{aligned}\int_{\mathbb{D}\setminus\Omega_n} (f_n \circ \varphi)^{\Phi} dA_{\alpha+2}(z) &\le \frac{1}{n^{\alpha+2}} \int_{\mathbb{D}\setminus\Omega_n} (f_n \circ \varphi)^{\Phi}\left(1-|\varphi(z)|^2\right)^{\alpha+2} dA(z) \\ &\le \frac{1}{n^{\alpha+2}} \int_{\mathbb{D}} f_n^{\Phi} N_{\varphi,\alpha+2} dA(z) \\ &\le \frac{1}{n^{\alpha+2}} \|f_n\|_{\Phi} \longrightarrow 0\end{aligned} \tag{4.8}$$

as $n \longrightarrow \infty$. Evidently, $\|f_n \circ \varphi\|_{\Phi} \longrightarrow 0$ as $n \longrightarrow \infty$, though $\|f_n\|_{\Phi} = 1$ for all n. It follows that C_φ does not have closed range on L^{Φ}_{α}. □

We have offered a criterion for the composition operator with closed range on L^{Φ}_{α}, but it seems that it is difficult to check whether or not G_ε satisfies the Φ-reverse Carleson measure condition.

Theorem 4.4. *The composition operator C_φ has closed range on L^{Φ}_{α} if and only if there are positive constants ε, c, and r such that $A_\alpha(G_\varepsilon \cap D(z,r)) \ge c|D(z,r)|^{\alpha+2}$ for all $z \in \mathbb{D}$.*

Proof. We first assume that C_φ has closed range on L^{Φ}_{α}. Then there is a constant $\varepsilon > 0$ such that G_ε satisfies the Φ-reverse Carleson measure condition. Thus, applying the proceeding constructed function $h(z)$ to the Φ-reverse Carleson condition gives

$$\int_{G_\varepsilon} \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{\alpha+2} dA_\alpha(z) \ge \eta \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{\alpha+2} dA_\alpha(z) = \eta. \tag{4.9}$$

Since $\int_{\mathbb{D}}(1-|z|^2)^{\alpha} dA(z) < \infty$, it allows to choose a fixed constant $r > 0$ such that

$$\int_{\mathbb{D}\setminus D(b,r)} \left(1-|z|^2\right)^{\alpha} dA(z) \le \frac{1}{2C} \int_{\mathbb{D}} \left(1-|z|^2\right)^{\alpha} dA(z). \tag{4.10}$$

Changing $w = (b - a + (1 - \overline{a}b)z)/(1 - \overline{b}a + (\overline{b} - \overline{a})z)$ in (4.10) gives

$$\int_{\mathbb{D}\setminus D(a,r)} \left(\frac{1 - |a|^2}{|1 - \overline{a}z|^2} \right)^{\alpha+2} dA_\alpha(z) \le \frac{1}{2C} \int_{\mathbb{D}} \left(\frac{1 - |a|^2}{|1 - \overline{a}z|^2} \right)^{\alpha+2} dA_\alpha(z). \tag{4.11}$$

Combing (4.11) gives

$$\int_{G_\varepsilon \cap D(a,r)} \left(\frac{1 - |a|^2}{|1 - \overline{a}z|^2} \right)^{\alpha+2} dA_\alpha(z) \ge \frac{1}{2C} \int_{\mathbb{D}} \left(1 - |z|^2\right)^\alpha dA(z). \tag{4.12}$$

The integral in the left of (4.12) is dominated by

$$A_\alpha(G_\varepsilon \cap D(a,r)) \sup \left\{ \left(\frac{1 - |a|^2}{|1 - \overline{a}z|^2} \right)^{\alpha+2} : z \in D(a,r) \right\}. \tag{4.13}$$

Since $(1 - a|^2)/|1 - \overline{a}z|^2 \asymp 1/|D(a,r)|$ for $z \in D(a,r)$, we get

$$\frac{A_\alpha(G_\varepsilon \cap D(a,r))}{|D(a,r)|^{\alpha+2}} \ge \frac{1}{2C} \int_{\mathbb{D}} \left(1 - |z|^2\right)^\alpha dA(z). \tag{4.14}$$

The converse can be derived from modification of [18], so we omit it here. □

Remark 4.5. From [18], we find that the composition operator has closed range on the weighted Bergman space A_α^p if and only if there are positive constants ε, c and r such that $A_\alpha(G_\varepsilon \cap D(z,r)) \ge c|D(z,r)|^{\alpha+2}$ for all $z \in \mathbb{D}$. Thus, we have the following fact.

The composition operator C_φ has closed range on L_α^Φ if and only if C_φ has closed range on A_α^p.

Let us further investigate the Φ-reverse Carleson measure condition, which can be formulated as follows.

The space $\{f|_{G_\varepsilon} : f \in L_\alpha^\Phi\}$ is a closed subspace of L_α^Φ if and only if there exists a constant $\varepsilon > 0$ such that G_ε satisfies Φ-reverse Carleson measure condition.

From the perspective of closed subspace, we will see the following special setting. Let $\mathfrak{F} = \{z_n : n = 1, 2, \ldots\}$ be a δ-sequence in $\mathbb{D}$. That is, there is k with $\rho(z, z_k) < \delta$ for every $z \in \mathbb{D}$. We also assume that $\mathfrak{F}$ is γ separated for some fixed $\gamma > 0$, that is, $\rho(z_m, z_n) \ge \gamma$ for all $m \ne n$. Using the subharmonicity of $\log^+|f(z)|$ for analytic function $f(z)$, it is easy to see that

$$\log^+\left|f(z_k)\right| \le \frac{C}{A_\alpha(D(z_k, \gamma/2))} \int_{D(z_k,\gamma/2)} \log^+\left(\left|f(z)\right|\right) dA_\alpha(z). \tag{4.15}$$

Since $\Phi(\log^+|f(z)|)$ is convex and increasing, we have

$$\Phi\left(\log^+\left|f(z_k)\right|\right) \le \frac{C}{A_\alpha(D(z_k, \gamma/2))} \int_{D(z_k,\gamma/2)} \Phi\left(\log^+\left|f(z)\right|\right) dA_\alpha(z). \tag{4.16}$$

Moreover, the formula $A_\alpha(D(z_k,\gamma/2)) \asymp (1-|z_k|^2)^{\alpha+2}$ allows us to write

$$\Phi\left(\log^+\left|f(z_k)\right|\right) \leq C\left(1-|z_k|^2\right)^{-\alpha-2}\int_{D(z_k,\gamma/2)}\Phi\left(\log^+\left|f(z)\right|\right)dA_\alpha(z). \tag{4.17}$$

Since $D(z_k,\gamma/2)$ are disjoint, we obtain

$$\sum_k \Phi\left(\log^+\left|f(z_k)\right|\right)\left(1-|z_k|^2\right)^{\alpha+2} \leq \int_{\mathbb{D}}\Phi\left(\log^+\left|f(z)\right|\right)dA_\alpha(z). \tag{4.18}$$

Hence, the map $\sigma: f \mapsto f|_{\mathcal{F}}$ takes L_α^Φ into $L^\Phi(\mathcal{F},\mu)$, where μ is a measure on $\mathcal{F}$ that assigns z_k to the mass $(1-|z_k|^2)^{\alpha+2}$ and space $L^\Phi(\mathcal{F},\mu) = \{f : \mathcal{F} \to \mathbb{C} \mid \int_{\mathcal{F}}\Phi(\log^+|f|)d\mu < \infty\}$. Of course, the map σ may be one to one. If the map σ is one to one, the map σ has closed range if and only if

$$\int_{\mathbb{D}}\Phi\left(\log^+\left|f(z)\right|\right)dA_\alpha(z) \leq \int_{\mathcal{F}}\Phi\left(\log^+\left|f\right|\right)d\mu = \sum_k\Phi\left(\log^+\left|f(z_k)\right|\right)\left(1-|z_k|^2\right)^{\alpha+2}. \tag{4.19}$$

Acknowledgments

The authors are extremely thankful to the editor for pointing out several errors. This work was supported by the Science Foundation of Sichuan Province (no. 20072A04) and the Scientific Research Fund of School of Science SUSE.

References

[1] E. A. Nordgren, "Composition operators," *Canadian Journal of Mathematics*, vol. 20, pp. 442–449, 1968.

[2] H. J. Schwartz, *Composition operators on* H^p, Ph.D. thesis, University of Toledo, Toledo, Ohio, USA, 1968.

[3] J. H. Shapiro and P. D. Taylor, "Compact, nuclear, and Hilbert-Schmidt composition operators on H^2," *Indiana University Mathematics Journal*, vol. 23, pp. 471–496, 1973.

[4] B. D. MacCluer and J. H. Shapiro, "Angular derivatives and compact composition operators on the Hardy and Bergman spaces," *Canadian Journal of Mathematics*, vol. 38, no. 4, pp. 878–906, 1986.

[5] J. H. Shapiro, "The essential norm of a composition operator," *Annals of Mathematics*, vol. 125, no. 2, pp. 375–404, 1987.

[6] J. A. Cima and A. L. Matheson, "Essential norms of composition operators and Aleksandrov measures," *Pacific Journal of Mathematics*, vol. 179, no. 1, pp. 59–64, 1997.

[7] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.

[8] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext: Tracts in Mathematics, Springer, New York, NY, USA, 1993.

[9] K. H. Zhu, *Operator Theory in Function Spaces*, vol. 139 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1990.

[10] D. H. Luecking, "Bounded composition operators with closed range on the Dirichlet space," *Proceedings of the American Mathematical Society*, vol. 128, no. 4, pp. 1109–1116, 2000.

[11] R. Kumar and J. R. Partington, "Weighted composition operators on Hardy and Bergman spaces," in *Recent Advances in Operator Theory, Operator Algebras, and Their Applications*, vol. 153 of *Operator Theory Advance and Application*, pp. 157–167, Birkhäuser, Basel, Switzerland, 2005.

[12] L. Liu, G. Cao, and X. Wang, "Composition operators on Hardy-Orlicz spaces," *Acta Mathematica Scientia. Series B*, vol. 25, no. 1, pp. 105–111, 2005.

[13] Q. Lu and G. F. Cao, "Weighted Orlicz-Bergman spaces and their composition operators," *Acta Analysis Functionalis Applicata*, vol. 7, no. 4, pp. 366–369, 2005 (Chinese).

[14] J. S. Choa and H. O. Kim, "Compact composition operators on the Nevanlinna class," *Proceedings of the American Mathematical Society*, vol. 125, no. 1, pp. 145–151, 1997.

[15] P. Lefèvre, D. Li, H. Queffèlec, and L. Rodríguez-Piazza, "Compact composition operators on H^2 and Hardy-Orlicz spaces," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 360–371, 2009.

[16] J. B. Garnett, *Bounded Analytic Functions*, vol. 96 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1981.

[17] J. S. Choa and S. Ohno, "Products of composition and analytic Toeplitz operators," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 1, pp. 320–331, 2003.

[18] D. H. Luecking, "Inequalities on Bergman spaces," *Illinois Journal of Mathematics*, vol. 25, no. 1, pp. 1–11, 1981.