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Research Article

Univalence of Certain Linear Operators Defined by Hypergeometric Function

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The main object of the present paper is to investigate univalence and starlikeness of certain integral operators, which are defined here by means of hypergeometric functions. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

1. Introduction and Preliminaries

Let $\mathcal{H}$ denote the class of all analytic functions f in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. For $n \geq 0$, a positive integer, let

$$\mathcal{A}_n = \left\{ f \in \mathcal{H} : f(z) = z + \sum_{k=1}^{\infty} a_{n+k} z^{n+k} \right\}, \tag{1.1}$$

with $\mathcal{A}_1 := \mathcal{A}$, where $\mathcal{A}$ is referred to as the normalized analytic functions in the unit disc. A function $f \in \mathcal{A}$ is called starlike in D if $f(D)$ is starlike with respect to the origin. The class of all starlike functions is denoted by $S^* := S^*(0)$. For $\alpha < 1$, we define

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha,\ z \in D \right\}, \tag{1.2}$$

and it is called the class of all starlike functions of order α. Clearly, $S^*(\alpha) \subseteq S^*$ for $0 < \alpha < 1$. For functions $f_j(z)$, given by

$$f_j(z) = \sum_{k=0}^{\infty} a_{k,j} z^k, \quad (j = 1, 2), \tag{1.3}$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) := \sum_{k=0}^{\infty} a_{k,1} a_{k,2} z^k =: (f_2 * f_1)(z). \tag{1.4}$$

An interesting subclass of S (the class of all analytic univalent functions) is denoted by $U(\alpha, \mu, \lambda)$ and is defined by

$$U(\alpha, \mu, \ \lambda) = \left\{ f \in \mathcal{A} : \left| (1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - 1 \right| < \lambda, \ z \in D \right\}, \tag{1.5}$$

where $0 < \alpha \leq 1, \ 0 \leq \mu < \alpha n$, and $\lambda > 0$.

The special case of this class has been studied by Ponnusamy and Vasundhra [1] and Obradović et al. [2].

For a,b,c $\in$ C and c $\neq$ 0,-1,-2,. . ., the Gussian hypergeometric series F(a,b;c;z) is defined as

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad z \in D, \tag{1.6}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ and $(a)_0 = 1$. It is well-known that $F(a,b;c;z)$ is analytic in D. As a special case of the Euler integral representation for the hypergeometric function, we have

$$F(1,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{1}{1-tz} t^{b-1} (1-t)^{c-b-1} dt, \qquad z \in D, \ \operatorname{Re} c > \operatorname{Re} b > 0. \tag{1.7}$$

Now by letting

$$\phi(a;c;z) := F(1,a;c;z), \tag{1.8}$$

it is easily seen that

$$z\phi(a;c+1;z)' = c\phi(a;c;z) - c\phi(a;c+1;z). \tag{1.9}$$

For $f \in \mathcal{A}$, Owa and Srivastava [3] introduced the operator $\Omega^{\lambda} : \mathcal{A} \mapsto \mathcal{A}$ defined by

$$\Omega^{\lambda} f(z) = \frac{\Gamma(2-\lambda)}{\Gamma(1-\lambda)} z^{\lambda} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt, \qquad (\lambda \neq 2,3,4\ldots), \tag{1.10}$$

which is extensions involving fractional derivatives and fractional integrals. Using definition of $\phi(a;c;z) := F(1,a;c;z)$ we may write

$$\Omega^{\lambda} f(z) = z\phi(2;2-\lambda;z) * f(z). \tag{1.11}$$

This operator has been studied by Srivastava et al. [4] and Srivastava and Mishra [5].

Also for $\lambda < 1$, $\operatorname{Re}\alpha > 0$, and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, let us define the function F by

$$\begin{aligned} F(z) &:= \lambda z + \frac{1-\lambda}{\alpha}\int_0^1 t^{(1/\alpha)-2} f(tz)\,dt \\ &= z + (1-\lambda)\sum_{k=2}^{\infty} \frac{a_k}{(k-1)\alpha+1} z^k. \end{aligned} \tag{1.12}$$

This operator has been investigated by many authors such as Trimble [6], and Obradović et al. [7].

If we take

$$\psi(m,\gamma,z) = 1 + (1-m)\sum_{k=2}^{\infty} \frac{1}{(k-1)\gamma+1} z^k, \tag{1.13}$$

then we can rewrite operator F defined by (1.11) as

$$F(z) = z\left(\psi(\lambda,\alpha,z) * \frac{f(z)}{z}\right). \tag{1.14}$$

From the definition of $\psi(m,\gamma,z)$ it is easy to check that

$$z\psi'(m,\gamma,z) + \frac{1}{\gamma}\psi(m,\gamma,z) = \frac{1}{\gamma}\left[1 + (1-m)\frac{z}{1-z}\right]. \tag{1.15}$$

For $f \in U(\alpha,\mu,\lambda)$ with $(z/f(z))^{\mu} * \phi(a;c+1;z) \neq 0$ for all $z \in D$ we define the transform G by

$$G(z) = z\left(\frac{1}{(z/f(z))^{\mu} * \phi(a;c+1;z)}\right)^{1/\mu}, \tag{1.16}$$

where $a, c \in \mathbb{C}$ and $c \neq 0, -1, -2, \ldots$.

Also for $f \in U(\alpha,\mu,\lambda)$ with $(z/f(z))^{\mu} * \psi(m,\gamma,z) \neq 0$ for all $z \in D$ we define the transform H by

$$H(z) = z\left(\frac{1}{(z/f(z))^{\mu} * \psi(m,\gamma,z)}\right)^{1/\mu}, \tag{1.17}$$

where $m < 1$ and $\gamma \neq 0; \operatorname{Re}\gamma \geq 0$.

In this investigation we aim to find conditions on α, μ, λ such that $f \in U(\alpha, \mu, \lambda)$ implies that the function f to be starlike. Also we find conditions on $\alpha, \mu, \lambda, m, \gamma, a, c$ for each $f \in U(\alpha, \mu, \lambda)$; the transforms G and H belong to $U(\alpha, \mu, \lambda)$ and S^*.

For proving our results we need the following lemmas.

Lemma 1.1 (cf. Hallenbeck and Ruscheweyh [8]). *Let $h(z)$ be analytic and convex univalent in the unit disk D with $h(0) = 1$. Also let*

$$g(z) = 1 + b_1 z + b_2 z^2 + \cdots \tag{1.18}$$

be analytic in D. If

$$g(z) + \frac{zg'(z)}{c} \prec h(z) \quad (z \in \mathbb{U};\ c \neq 0), \tag{1.19}$$

then

$$g(z) \prec \psi(z) = \frac{c}{z^c}\int_0^z t^{c-1} h(t)\,dt \prec h(z) \quad (z \in D;\ Re\, c \geq 0;\ c \neq 0). \tag{1.20}$$

and $\psi(z)$ is the best dominant of (1.20).

Lemma 1.2 (cf. Ruscheweyh and Stankiewicz [8]). *If f andg are analytic and F and G are convex functions such that $f \prec F$, $g \prec G$, then $f * g \prec F * G$.*

Lemma 1.3 (cf. Ruscheweyh and Sheil-Small [9]). *Let F and G be univalent convex functions in D. Then the Hadamard product $F * G$ is also univalent convex in D.*

2. Main Results

We follow the method of proof adopted in [1, 10].

Theorem 2.1. *Let n be positive integer with $n \geq 2$. Also let $(n+1)/2n < \alpha \leq 1$ and $n(1-\alpha) < \mu < \alpha n$. If $f(z) = z + a_{n+1}z^{n+1} + \cdots$ belongs to $U(\alpha, \mu, \lambda)$, Then $f \in S^*(\gamma)$ whenever $0 < \lambda \leq \lambda(\alpha, \mu, n, \gamma)$, where*

$$\lambda(\alpha, \mu, n, \gamma) := \begin{cases} \dfrac{(\alpha n - \mu)\sqrt{2\alpha(1-\gamma) - 1}}{\sqrt{(\alpha n - \mu)^2 + \mu^2[2\alpha(1-\gamma) - 1]}}, & 0 \leq \gamma \leq \dfrac{\mu - n(1-\alpha)}{\mu(1+n)}, \\[2ex] \dfrac{(\alpha n - \mu)(1-\gamma)}{n + \mu\gamma - \mu}, & \dfrac{\mu - n(1-\alpha)}{\mu(1+n)} < \gamma < 1. \end{cases} \tag{2.1}$$

Proof. Let us define

$$p(z) = \left(\frac{z}{f(z)}\right)^{\mu}. \tag{2.2}$$

Since $f \in U(\alpha, \mu, \lambda)$, we have

$$\begin{aligned}(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) &= p(z) - \frac{\alpha}{\mu} z f'(z) \\ &= 1 + (\alpha n - \mu) a_{n+1} z^n + \cdots \\ &= 1 + \lambda\omega(z),\end{aligned} \tag{2.3}$$

where $\omega(z)$ is an analytic function with $|\omega(z)| < 1$ and $\omega(0) = \omega'(0) = \cdots = \omega^{(n-1)}(0) = 0$. By Schwarz lemma, we have $|\omega(z)| \le |z|^n$. By (2.3), it is easy to check that

$$\begin{gathered} p(z) = 1 - \frac{\mu\lambda}{\alpha}\int_0^1 \frac{\omega(tz)}{t^{\mu/\alpha+1}} dt, \\ (1-\alpha) + \alpha\frac{zf'(z)}{f(z)} = \frac{1 + \lambda\omega(z)}{1 - \mu\lambda/\alpha\int_0^1 \omega(tz)/(t^{\mu/\alpha+1})dt}. \end{gathered} \tag{2.4}$$

Therefore

$$\begin{aligned} &\frac{1}{1-\gamma}\left(\frac{zf'(z)}{f(z)} - \gamma\right) \\ &= \frac{[((\alpha-1)-\alpha\gamma)/(1-\gamma)]\left(\alpha - \mu\lambda\int_0^1(\omega(tz)/t^{\mu/\alpha+1})dt\right) + (\alpha/(1-\gamma))(1+\lambda\omega(z))}{\alpha\left(\alpha - \mu\lambda\int_0^1(\omega(tz)/t^{\mu/\alpha+1})dt\right)}. \end{aligned} \tag{2.5}$$

We need to show that $f \in S^*(\gamma)$. To do this, according to a well-known result [9] and (2.5) it suffices to show that

$$\frac{[((\alpha-1)-\alpha\gamma)/(1-\gamma)]\left(\alpha - \mu\lambda\int_0^1(\omega(tz)/t^{(\mu/\alpha)+1})dt\right) + (\alpha/(1-\gamma))(1+\lambda\omega(z))}{\alpha\left(\alpha - \mu\lambda\int_0^1(\omega(tz)/t^{(\mu/\alpha)+1}dt)\right)} \ne -iT, \quad T \in \mathbb{R}, \tag{2.6}$$

which is equivalent to

$$\lambda\left[\frac{\omega(z) + \mu((\alpha\gamma+1-\alpha)/\alpha - i(1-\gamma)T)\int_0^1(\omega(tz)/t^{\mu/\alpha+1})dt}{\alpha(1-\gamma)(1+iT)}\right] \ne -1, \quad T \in \mathbb{R}. \tag{2.7}$$

Suppose that B_n denote the class of all Schwarz functions ω such that $\omega(0) = \omega'(0) = \cdots = \omega^{(n-1)}(0) = 0$, and let

$$M = \sup_{z\in D, \omega\in B_n, T\in\mathbb{R}}\left|\frac{\omega(z) + \mu((\alpha\gamma+1-\alpha)/\alpha - i(1-\gamma)T)\int_0^1(\omega(tz)/t^{\mu/\alpha+1})dt}{\alpha(1-\gamma)(1+iT)}\right|, \tag{2.8}$$

then, $f \in S^*(\gamma)$ if $\lambda M \leq 1$. This observation shows that it suffices to find M. First we notice that

$$M \leq \sup_{T\in\mathbb{R}} \left\{ \frac{1 + (\mu/(n-\mu)/\alpha)\sqrt{(\alpha\gamma + 1 - \alpha)^2/\alpha^2 + (1-\gamma)^2 T^2}}{\alpha(1-\gamma)(1+T^2)} \right\}. \tag{2.9}$$

Define $\phi : [0, \infty) \mapsto \mathbb{R}$ by

$$\phi(x) = \frac{(\alpha n - \mu) + \mu\sqrt{(\alpha\gamma + 1 - \alpha)^2 + (1-\gamma)^2\alpha^2 x}}{(\alpha n - \mu)\alpha(1-\gamma)\sqrt{1+x}}. \tag{2.10}$$

Differentiating ϕ with respect to x, we get

$$\begin{aligned} \phi'(x) = {} & \frac{\mu(\alpha n - \mu)\alpha^3(1-\gamma)^3\sqrt{1+x}/2\sqrt{(\alpha\gamma + 1 - \alpha)^2 + (1-\gamma)^2\alpha^2 x}}{(\alpha n - \mu)^2\alpha^2(1-\gamma)^2(1+x)} \\ & - \frac{[(\alpha n - \mu)\alpha(1-\gamma)]\left[(\alpha n - \mu) + \mu\sqrt{(\alpha\gamma + 1 - \alpha)^2 + (1-\gamma)^2\alpha^2 x}\right]/2\sqrt{1+x}}{(\alpha n - \mu)^2\alpha^2(1-\gamma)^2(1+x)}. \end{aligned} \tag{2.11}$$

Case 1. Let $0 < \gamma < (\mu - n(1-\alpha))/\mu(1+n)$. Then we see that ϕ has its only critical point in the positive real line at

$$x_0 = \frac{1}{(1-\gamma)^2\alpha^2}\left[\frac{\mu^2(2\alpha(1-\gamma) - 1)^2}{(\alpha n - \mu)^2} - (\alpha\gamma + 1 - \gamma)^2\right]. \tag{2.12}$$

Furthermore, we can see that $\phi'(x) > 0$ for $0 \leq x < x_0$ and $\phi'(x) < 0$ for $x > x_0$. Hence $\phi(x)$ attains its maximum value at x_0 and

$$\phi(x) \leq \phi(x_0) = \frac{(\alpha n - \mu)^2 + \mu^2[2\alpha(1-\gamma) - 1]}{(\alpha n - \mu)\sqrt{[2\alpha(1-\gamma) - 1](\alpha n - \mu)^2 + \mu^2[2\alpha(1-\gamma) - 1]^2}}. \tag{2.13}$$

Case 2. Let $\gamma > (\mu - n(1-\alpha))/\mu(1+n)$, then it is easy to see that $\phi'(x) < 0$, and so $\phi(x)$ attains its maximum value at 0 and

$$\phi(x) \leq \phi(0) = \frac{n + \mu\gamma - \mu}{(\alpha n - \mu)(1-\gamma)}, \quad \forall x \geq 0. \tag{2.14}$$

Now the required conclusion follows from (2.13) and (2.14). □

By putting $\gamma = 0$ in Theorem 2.1 we obtain the following result.

Corollary 2.2. *Let n be the positive integer with $n \geq 2$. Also let $(n+1)/2n < \alpha \leq 1$ and $n(1-\alpha) < \mu < \alpha n$. If $f(z) = z + a_{n+1}z^{n+1} + \cdots$ belongs to $U(\alpha,\mu,\lambda)$, then $f \in S^*$ whenever $0 < \lambda \leq (\alpha n - \mu)\sqrt{2\alpha - 1}/\sqrt{(\alpha n - \mu)^2 + \mu^2(2\alpha - 1)}$.*

Remark 2.3. Taking $\alpha = 1$, $\mu = 1$ in Theorem 2.1 and Corollary 2.2 we get results of [10].

We follow the method ofproof adopted in [11].

Theorem 2.4. *Let $n \geq 2$, $a \neq 0$, $c \in \mathbb{C}$ with $\operatorname{Re} c \geq 0 \neq c$ and the function $\varphi(z) = 1 + b_1 z + b_2 z^2 + \cdots$ with $b_n \neq 0$ be univalent convex in D. If $f(z) = z + a_{n+1}z^{n+1} + \cdots \in U(\alpha,\mu,\lambda)$ and $\phi(a;c;z)$ defined by* (1.8) *satisfy the conditions*

$$\begin{gathered}\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c+1;z) \neq 0 \quad \forall z \in D,\\ \phi(a;c;z) \prec \varphi(z),\end{gathered} \tag{2.15}$$

then the transform G defined by (1.16) *has the following:*

(1) $G \in U(\alpha,\mu,\lambda|b_n||c|/|c+n|)$,

(2) $G \in S^*$*whenever*

$$0 < \lambda \leq \frac{|c+n|(\alpha n - \mu)\sqrt{2\alpha - 1}}{|b_n||c|\sqrt{(\alpha n - \mu)^2 + \mu^2(2\alpha - 1)}}. \tag{2.16}$$

Proof. From the definition of G we obtain

$$\left(\frac{z}{G(z)}\right)^{\mu} = \left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c+1;z). \tag{2.17}$$

Differentiating $(z/G(z))^{\mu}$ shows that

$$z\left(\left(\frac{z}{G(z)}\right)^{\mu}\right)' = \mu\left(\frac{z}{G(z)}\right)^{\mu} - \mu\left(\frac{z}{G(z)}\right)^{\mu+1} G'(z). \tag{2.18}$$

It is easy to see that

$$z\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)' * \phi(a;c+1;z) = z\left(\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c+1;z)\right)'. \tag{2.19}$$

From (1.9) and (2.19) we deduce that

$$z\left(\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c+1;z)\right)' = c\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c;z) - c\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c+1;z), \tag{2.20}$$

or

$$z\left(\left(\frac{z}{G(z)}\right)^{\mu}\right)' + c\left(\frac{z}{G(z)}\right)^{\mu} = c\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c;z). \tag{2.21}$$

Let us define

$$p(z) = (1-\alpha)\left(\frac{z}{G(z)}\right)^{\mu} + \alpha\left(\frac{z}{G(z)}\right)^{\mu+1} G'(z) := 1 + d_n z^n + \cdots, \tag{2.22}$$

then $p(z)$ is analytic in D, with $p(0) = 1$ and $p'(0) = \cdots = p^{(n-1)}(0) = 0$. Combining (2.18) with (2.21), one can obtain

$$p(z) = \left(1 + \frac{\alpha c}{\mu}\right)\left(\frac{z}{G(z)}\right)^{\mu} - \frac{\alpha c}{\mu}\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c;z). \tag{2.23}$$

Differentiating $p(z)$ yields

$$zp'(z) = \left(1 + \frac{\alpha c}{\mu}\right) z\left(\left(\frac{z}{G(z)}\right)^{\mu}\right)' - \frac{\alpha c}{\mu} z\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)' * \phi(a;c;z). \tag{2.24}$$

In view of (2.21), (2.23), and (2.24), we obtain

$$\begin{aligned} cp(z) + zp'(z) &= c\left(1 + \frac{\alpha c}{\mu}\right)\left(\frac{z}{G(z)}\right)^{\mu} - \frac{\alpha c^2}{\mu}\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c;z) \\ &\quad + \left(1 + \frac{\alpha c}{\mu}\right) z\left[\left(\frac{z}{G(z)}\right)^{\mu}\right]' - \frac{\alpha c}{\mu} z\left[\left(\frac{z}{f(z)}\right)^{\mu}\right]' * \phi(a;c;z) \\ &= c\left(1 + \frac{\alpha c}{\mu}\right)\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c;z) \\ &\quad - \frac{\alpha c^2}{\mu}\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c;z) - \frac{\alpha c}{\mu}\left[\left(\frac{z}{f(z)}\right)^{\mu}\right]' * \phi(a;c;z) \\ &= c\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a;c;z) - c\alpha\left[\left(\frac{z}{f(z)}\right)^{\mu} - \left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right] * \phi(a;c;z) \\ &= c\left[(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right] * \phi(a;c;z). \end{aligned} \tag{2.25}$$

Hence

$$p(z) + \frac{1}{c} zp'(z) = \left[(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right] * \phi(a;c;z). \tag{2.26}$$

Since $1+\lambda z^n$ and $\varphi(z)=1+b_1z+b_2z^2+\cdots$ are convex and

$$(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu}+\alpha\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z)\prec 1+\lambda z^n, \qquad \phi(a;c;z)\prec\varphi(z), \tag{2.27}$$

by using Lemmas 1.2 and 1.3, from (2.26) we deduce that

$$p(z)+\frac{1}{c}zp'(z)\prec 1+b_n\lambda z^n. \tag{2.28}$$

It now follows from Lemma 1.1 that

$$p(z)\prec\psi(z)=\frac{c}{z^c}\int_0^z t^{c-1}(1+b_n\lambda z^n)dt. \tag{2.29}$$

Therefore

$$p(z)\prec 1+\frac{\lambda b_n c}{c+n}z^n, \tag{2.30}$$

and the result follows from the last subordination and Corollary 2.2. □

It is well-known that (see, [12]) if $c,a>0$ and $c\geq\max\{2,a\}$, then $\phi(a;c;z)$ is univalent convex function in D. So if we take $\varphi(z)=\phi(a;c;z)$ in the Theorem 2.4, we obtain the following.

Corollary 2.5. *For $n\geq 2,c,a>0$, and $c\geq\max\{2,a\}$, let the function $f(z)=z+a_nz^{n+1}+\cdots\in U(\alpha,\mu,\lambda)$ and $\phi(a;c;z)$ defined by* (1.8) *satisfy the condition*

$$\left(\frac{z}{f(z)}\right)^{\mu}*\phi(a;c+1;z)\neq 0 \quad \forall z\in D. \tag{2.31}$$

Then the transform G defined by (1.16) *has the following:*

(1) $G\in U(\alpha,\mu,\lambda|(a)_n|c/|(c)_n|(c+n))$;

(2) $G\in S^*$ *whenever*

$$0<\lambda\leq\frac{(c+n)|(c)_n|(\alpha n-\mu)\sqrt{2\alpha-1}}{|(a)_n|c\sqrt{(\alpha n-\mu)^2+\mu^2(2\alpha-1)}}. \tag{2.32}$$

By putting $a=c$ on the (1.8), we get $\phi(c;c;z)=1/(1-z)$ which is evidently convex. So by taking $\varphi(z)=1/(1-z)$ on Theorem 2.4 we have the following.

Corollary 2.6. *For $n \geq 2, c \in \mathbb{C}$ with $\mathrm{Re}\, c \geq 0 \neq c$, let the function $f(z) = z + a_n z^{n+1} + \cdots \in U(\alpha, \mu, \lambda)$ and $\phi(a; c; z)$ defined by* (1.8) *satisfy the condition*

$$\left(\frac{z}{f(z)}\right)^{\mu} * \phi(a; c+1; z) \neq \quad \forall z \in D. \tag{2.33}$$

Then the transform G defined by (1.16) *has the following:*

(1) $G \in U(\alpha, \mu, \lambda |c| / |c+n|)$;

(2) $G \in S^*$ *whenever*

$$0 < \lambda \leq \frac{|c+n|(\alpha n - \mu)\sqrt{2\alpha - 1}}{|c|\sqrt{(\alpha n - \mu)^2 + \mu^2(2\alpha - 1)}}. \tag{2.34}$$

Remark 2.7. Taking $\alpha = 1$ and $\mu = 1$ on Corollary 2.6, we get a result of [11].

By putting $c = 1 - M$ and $a = 2$ on Theorem 2.10 we obtain the following.

Corollary 2.8. *Let $n \geq 2$ and $\varphi(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ with $b_n \neq 0$ be univalent convex function in D. Also let $M \in \mathbb{C}$ with $\mathrm{Re}\, M < 1$ and $f(z) = z + a_{n+1} z^{n+1} + \cdots \in U(\alpha, \mu, \lambda)$, satisfy*

$$\Omega^M \left[\left(\frac{z}{f(z)}\right)^{\mu}\right] \neq 0 \quad z \in D, \tag{2.35}$$

and let G be the function which is defined by

$$G(z) = z\left(\frac{1}{\Omega^M\left[(z/f(z))^{\mu}\right]}\right)^{1/\mu}. \tag{2.36}$$

If

$$\phi(2; 1-M; z) \prec \varphi(z), \tag{2.37}$$

then we have the following:

(1) $G \in U(\alpha, \mu, \lambda |b_n| |1-M| / |n+1-M|)$;

(2) $G \in S^*$ *whenever*

$$0 < \lambda \leq \frac{|1 - M + n|(\alpha n - \mu)\sqrt{2\alpha - 1}}{|b_n||1-M|\sqrt{(\alpha n - \mu)^2 + \mu^2(2\alpha - 1)}}. \tag{2.38}$$

Remark 2.9. We note that if $M < -1$, then $\phi(2; 1-M; z)$ is convex function, and so we can replace $\varphi(z)$ with $\phi(2; 1-M; z)$ in Corollary 2.8 to get other new results.

In [13], Pannusamy and Sahoo have also considered the class $U(\alpha,\ \mu,\ \lambda)$ for the case $\alpha = 1$ with $\mu = n$.

Theorem 2.10. *For $m < 1$, $\gamma \neq 0$; $\operatorname{Re}\gamma > 0$, $n \geq 2$, let $f(z) = z + a_{n+1}z^{n+1} + \cdots \in U(\alpha, \mu, \lambda)$ and $\psi(m, \gamma, z)$ defined by (1.13) satisfy the condition*

$$\left(\frac{z}{f(z)}\right)^{\mu} * \psi(m, \gamma, z) \neq 0 \quad \forall z \in D. \tag{2.39}$$

Then the transform H defined by (1.17) has the following:

(1) $H \in U(\alpha, \mu, \lambda(1 - m)/|1 + n\gamma|)$;

(2) $H \in S^*$ *whenever*

$$0 < \lambda \leq \frac{|1 + n\gamma|(\alpha n - \mu)\sqrt{2\alpha - 1}}{(1 - m)\sqrt{(\alpha n - \mu)^2 + \mu^2(2\alpha - 1)}}. \tag{2.40}$$

Proof. Let us define

$$p(z) = (1 - \alpha)\left(\frac{z}{H(z)}\right)^{\mu} + \alpha\left(\frac{z}{H(z)}\right)^{\mu+1} H'(z), \tag{2.41}$$

then $p(z)$ is analytic in D, with $p(0) = 1$ and $p'(0) = \cdots = p^{(n-1)}(0) = 0$. Using the same method as on Theorem 2.4 we get

$$p(z) + \gamma z p'(z) = \left[(1 - \alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right] * \left(1 + (1 - m)\frac{z}{1 - z}\right). \tag{2.42}$$

Since $1 + \lambda z^n$ and $h(z) = (1 + (1 - m)(z/(1 - z)))$ are convex,

$$(1 - \alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) \prec 1 + \lambda z^n. \tag{2.43}$$

Using Lemmas 1.2 and 1.3, from (2.42) it yields

$$p(z) + \gamma z p'(z) \prec (1 - m)\lambda z^n. \tag{2.44}$$

It now follows from Lemma 1.1 that

$$p(z) \prec \frac{1}{\gamma z^{1/\gamma}} \int_0^z t^{(1/\gamma)-1}(1 + (1 - m)\lambda t^n)\,dt. \tag{2.45}$$

Therefore

$$\left|p(z)-1\right| \leq \frac{\lambda(1-m)}{\left|1+n\gamma\right|}|z|^{n}, \tag{2.46}$$

and the result follows from (2.46) and Corollary 2.2. □

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