Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 759251, 12 pages
doi:10.1155/2009/759251

Research Article

Certain Classes of Harmonic Multivalent Functions Based on Hadamard Product

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Received 25 February 2009; Accepted 12 September 2009

Recommended by Yong Zhou

We define and investigate two special subclasses of the class of complex-valued harmonic multivalent functions based on Hadamard product.

1. Introduction

A continuous function $f = u+iv$ is a complex-valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subseteq \mathbb{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. Note that $f = h + \overline{g}$ reduces to h if the coanalytic part g is zero.

For $p \geq 1$, denote by $H(p)$ the set of all multivalent harmonic functions $f = h + \overline{g}$ defined in the open unit disc U, where h and g defined by

$$h(z) = z^p + \sum_{n=p+t}^{\infty} a_n z^n, \quad g(z) = \sum_{n=p+t-1}^{\infty} b_n z^n, \quad \left|b_{p+t-1}\right| < 1, \quad t \in \mathbb{N} := \{1, 2, \dots\} \tag{1.1}$$

are analytic functions in U.

Let F be a fixed multivalent harmonic function given by

$$F(z) = H(z) + \overline{G(z)} = z^p + \sum_{n=p+t}^{\infty} |A_n| z^n + \overline{\sum_{n=p+t-1}^{\infty} |B_n| z^n}, \quad \left|B_{p+t-1}\right| < 1, \quad t \in \mathbb{N}. \tag{1.2}$$

A function $f \in H(p)$ is said to be in the class $H_F(p,t,\alpha,k)$ if

$$\operatorname{Re}\left(\frac{z(f*F)'(z)}{z'(f*F)(z)}\right) \geq k\left|\frac{z(f*F)'(z)}{z'(f*F)(z)} - p\right| + p\alpha\ , \tag{1.3}$$

where $f * F$ is a harmonic convolution of f and F. Note that $z' = (\partial/\partial\theta)(r\ e^{i\theta})$, $f'(z) = (\partial/\partial\theta)f(r\ e^{i\theta})$. Using the fact

$$\operatorname{Re}\ w > k|w - p| + p\alpha \Longleftrightarrow \operatorname{Re}\left(\left(1 + ke^{i\theta}\right)w - kpe^{i\theta}\right) \geq p\alpha, \tag{1.4}$$

it follows that $f \in H_F(p,t,\alpha,k)$ if and only if

$$\operatorname{Re}\left(\frac{(1+ke^{i\theta})z(f*F)'(z)}{z'(f*F)(z)} - kpe^{i\theta}\right) \geq p\alpha, \quad 0 \leq \alpha < 1. \tag{1.5}$$

A function f in $H_F(p,t,\alpha,k)$ is called k-uniformly multivalent harmonic starlike function associated with a fixed multivalent harmonic function F. The set $H_F(p,t,\alpha,k)$ is a comprehensive family that contains several previously studied subclasses of $H(p)$; for example, if we let

$$F(z) = I(z) := \left(\frac{z^p}{1-z}\right) + \overline{\left(\frac{z^p}{1-z}\right)} = z^p + \sum_{n=p+1}^{\infty} z^n + \overline{\sum_{n=p}^{\infty} z^n}, \tag{1.6}$$

then

$$H_I(p,\alpha,0) \equiv S_H^*(p,\alpha) := \left\{f \in H(p) : \operatorname{Re}\left(\frac{z\ f'(z)}{z' f(z)}\right) \geq p\alpha\right\}, \tag{1.7}$$

(see [1, 2]);

$$H_I(p,1,0,0) \equiv S_H^*(p,0) \equiv S_H^*(p), \tag{1.8}$$

(see [3]);

$$H_I(1,1,\alpha,0) \equiv S_H^*(1,\alpha) \equiv S_H^*(\alpha), \tag{1.9}$$

(see [4]);

$$H_I(1,1,0,0) \equiv S_H^*(0) \equiv S_H^*, \tag{1.10}$$

(see [5, 6]);

$$H_I(1,1,\alpha,k) \equiv G_H(k,\alpha) := \left\{ f \in H(1) : \operatorname{Re}\left(\frac{(1+ke^{i\theta})zf'(z)}{z'f(z)} - ke^{i\theta} \right) \ge \alpha \right\}, \tag{1.11}$$

(see [7]);

$$H_I(p,1,\alpha,1) \equiv M_H(p,\alpha) := \left\{ f \in H(p) : \operatorname{Re}\left(\left(1+e^{i\theta}\right) \frac{zf'(z)}{z'f(z)} - pe^{i\theta} \right) \ge p\alpha \right\}, \tag{1.12}$$

(see [8]).

Finally, denote by $TH(p)$ the subclass of functions $f(z) = h(z) + \overline{g(z)}$ in $H(p)$ where

$$h(z) = z^p - \sum_{n=p+t}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=p+t-1}^{\infty} |b_n| z^n. \tag{1.13}$$

Let $H_{\overline{F}}(p,t,\alpha,k) := TH(p) \cap H_F(p,t,\alpha,k)$.

In this paper, we investigate coefficient conditions, extreme points, and distortion bounds for functions in the family $H_{\overline{F}}(p,t,\alpha,k)$. We observe that the results so obtained for this main family can be viewed as extensions and generalizations for various subclasses of $H(p)$ and $H(1)$.

2. Main Results

Theorem 2.1. *Let $f = h + \overline{g}$ be such that h and g are given by* (1.1). *Then $f \in H_F(p,t,\alpha,k)$ if the inequality*

$$\sum_{n=p+t}^{\infty} \left[\frac{n(1+k) - p(k+\alpha)}{(p(1-\alpha)+1) - \left|p(1-\alpha)-1\right|} \right] |a_n A_n| + \sum_{n=p+t-1}^{\infty} \left[\frac{n(1+k) + p(k+\alpha)}{(p(1-\alpha)+1) - \left|p(1-\alpha)-1\right|} \right] |b_n B_n| \le \frac{1}{2} \tag{2.1}$$

is satisfied for some k $(k \ge 0)$, p $(p \ge 1)$, α $(0 \le \alpha < 1)$, and t $(t \ge 1)$.

Proof. In view of (1.5), we need to prove that $\operatorname{Re}(w) > 0$, where

$$w = \frac{(ke^{i\theta}+1)\left[z(h*H)'(z) - \overline{z(g*G)'(z)} \right] - p(ke^{i\theta}+\alpha)\left[(h*H)(z) + \overline{(g*G)(z)} \right]}{(h*H)(z) + \overline{(g*G)(z)}} := \frac{A(z)}{B(z)}. \tag{2.2}$$

Using the fact that $\operatorname{Re}(w) > 0 \Leftrightarrow |1+w| \ge |1-w|$, it sufficies to show that

$$|A(z) + B(z)| - |A(z) - B(z)| \ge 0. \tag{2.3}$$

Therefore, we obtain

$$
\begin{aligned}
&|A(z)+B(z)|-|A(z)-B(z)| \\
&\quad \ge \left(\left[p(1-\alpha)+1\right]-\left|p(1-\alpha)-1\right|\right)|z|^p-\sum_{n=p+t}^{\infty}\left[n(1+k)-p(k+\alpha)+1\right]|a_nA_n||z|^n \\
&\qquad -\sum_{n=p+t-1}^{\infty}\left[n(1+k)+p(k+\alpha)-1\right]|b_nB_n||z|^n-\sum_{n=p+t}^{\infty}\left[n(1+k)-p(k+\alpha)-1\right]|a_nA_n||z|^n \\
&\qquad -\sum_{n=p+t-1}^{\infty}\left[n(1+k)+p(k+\alpha)+1\right]|b_nB_n||z|^n \\
&\quad =\left(\left[p(1-\alpha)+1\right]-\left|p(1-\alpha)-1\right|\right)|z|^p \\
&\qquad -\sum_{n=p+t}^{\infty}2\left[n(1+k)-p(k+\alpha)\right]|a_nA_n||z|^n-\sum_{n=p+t-1}^{\infty}2\left[n(1+k)+p(k+\alpha)\right]|b_nB_n||z|^n \\
&\quad =\left(\left[p(1-\alpha)+1\right]-\left|p(1-\alpha)-1\right|\right)|z|^p \\
&\qquad \times\left\{1-\sum_{n=p+t}^{\infty}\frac{2\left[n(1+k)-p(k+\alpha)\right]}{\left[p(1-\alpha)+1\right]-\left|p(1-\alpha)-1\right|}|a_nA_n|\right. \\
&\qquad\qquad \left.-\sum_{n=p+t-1}^{\infty}\frac{2\left[n(1+k)+p(k+\alpha)\right]}{\left[p(1-\alpha)+1\right]-\left|p(1-\alpha)-1\right|}|b_nB_n|\right\}\ge 0.
\end{aligned}
\tag{2.4}
$$

By hypothesis, last expression is nonnegative. Thus the proof is complete. □

The coeficient bounds (2.1) is sharp for the function

$$
\begin{aligned}
f(z)=z^p&+\sum_{n=p+t}^{\infty}\frac{\left[p(1-\alpha)+1\right]-\left|p(1-\alpha)-1\right|}{2\left[n(1+k)-p(k+\alpha)\right]}X_nz^n \\
&+\sum_{n=p+t-1}^{\infty}\frac{\left[p(1-\alpha)+1\right]-\left|p(1-\alpha)-1\right|}{2\left[n(1+k)+p(k+\alpha)\right]B_n}\overline{Y_n}\overline{z^n},
\end{aligned}
\tag{2.5}
$$

where $\sum_{n=p+t}^{\infty}|X_n|+\sum_{n=p+t-1}^{\infty}|Y_n|=1$.

Corollary 2.2. *For $p\ge(1/1-\alpha)$, $0\le\alpha<1$, if the inequality*

$$
\sum_{n=p+t}^{\infty}\left[n(1+k)-p(k+\alpha)\right]|a_nA_n|+\sum_{n=p+t-1}^{\infty}\left[n(1+k)+p(k+\alpha)\right]|b_nB_n|\le 1 \tag{2.6}
$$

holds, then $f\in H_F(p,t,\alpha,k)$.

Corollary 2.3. *For $0 \leq \alpha < 1$ and $1 \leq p \leq (1/1-\alpha)$, if the inequality*

$$\sum_{n=p+t}^{\infty} \left[n(1+k) - p(k+\alpha)\right] |a_n A_n| + \sum_{n=p+t-1}^{\infty} \left[n(1+k) + p(k+\alpha)\right] |b_n B_n| \leq p(1-\alpha) \tag{2.7}$$

holds, then $f \in H_F(p,t,\alpha,k)$.

Theorem 2.4. *Let $f = h + \overline{g}$ be such that h and g are given by* (1.13)*. Also, suppose that $k \geq 0$ and $0 \leq \alpha < 1$. Then*

(i) *for $1 \leq p \leq (1/1-\alpha)$, $f \in H_{\overline{F}}\ (p,t,\alpha,k)$ if and only if*

$$\sum_{n=p+t}^{\infty} \left[n(1+k) - p(k+\alpha)\right] |a_n A_n| + \sum_{n=p+t-1}^{\infty} \left[n(1+k) + p(k+\alpha)\right] |b_n B_n| \leq p(1-\alpha); \tag{2.8}$$

(ii) *for $p(1-\alpha) \geq 1$, $f \in H_{\overline{F}}\ (p,t,\alpha,k)$ if and only if*

$$\sum_{n=p+t}^{\infty} \left[n(1+k) - p(k+\alpha)\right] |a_n A_n| + \sum_{n=p+t-1}^{\infty} \left[n(1+k) + p(k+\alpha)\right] |b_n B_n| \leq 1. \tag{2.9}$$

Proof. According to Corollaries 2.2 and 2.3, we must show that if the condition (2.9) does not hold, then $f \notin H_{\overline{F}}(p,t,\alpha,k)$, that is, we must have

$$\operatorname{Re}\left(\frac{\left(1+ke^{i\theta}\right)\left[z(h*H)'(z) - \overline{z(g*G)'(z)}\right] - p\left(ke^{i\theta}+\alpha\right)\left[(h*H)(z) + \overline{(g*G)(z)}\right]}{\left[(h*H)(z) + \overline{(g*G)(z)}\right]} \right) \geq 0. \tag{2.10}$$

Choosing the values of $z = r$ on positive real axis where $0 \leq z = r < 1$, and using $\operatorname{Re}(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the inequality (2.10) reduces to

$$\begin{aligned}
&= \operatorname{Re}\left\{ \frac{\left(1+ke^{i\theta}\right)\left[pz^p - \sum_{n=p+t}^{\infty} n|a_n A_n| z^n - \sum_{n=p+t-1}^{\infty} n|b_n B_n| z^n\right] - \mathfrak{A}}{p\left[z^p - \sum_{n=p+t}^{\infty} |a_n A_n| z^n + \overline{\sum_{n=p+t-1}^{\infty} |b_n B_n| z^n}\right]} \right\} \\
&\geq \left\{ \frac{(1+k)\left[pr^p - \sum_{n=p+t}^{\infty} n|a_n A_n| r^n - \sum_{n=p+t-1}^{\infty} n|b_n B_n| r^n\right] - \mathfrak{B}}{p\left[r^p - \sum_{n=p+t}^{\infty} |a_n A_n| r^n + \sum_{n=p+t-1}^{\infty} |b_n B_n| r^n\right]} \right\} \\
&= \left\{ \frac{p(1-\alpha)r^p - \sum_{n=p+t}^{\infty}\left[n(1+k) - p(k+\alpha)\right]|a_n A_n| r^n - \mathfrak{C}}{p\left[r^p - \sum_{n=p+t}^{\infty} |a_n A_n| r^n + \sum_{n=p+t-1}^{\infty} |b_n B_n| r^n\right]} \right\},
\end{aligned} \tag{2.11}$$

where $\mathfrak{A}$ denotes $(k\,e^{i\theta}+\alpha)p[z^p - \sum_{n=p+t}^{\infty}|a_nA_n|z^n + \overline{\sum_{n=p+t-1}^{\infty}|b_nB_n|z^n}]$, $\mathfrak{B}$ denotes $(k+\alpha)p\,[r^p - \sum_{n=p+t}^{\infty}|a_nA_n|r^n + \sum_{n=p+t-1}^{\infty}|b_nB_n|r^n]$, and $\mathfrak{C}$ denotes $\sum_{n=p+t-1}^{\infty}[n(1+k)+p(k+\alpha)]|b_nB_n|r^n$.

Letting $r \to 1^-$, we obtain

$$\frac{p(1-\alpha) - \left\{\sum_{n=p+t}^{\infty}\left[n(1+k)-p(k+\alpha)\right]|a_nA_n| + \sum_{n=p+t-1}^{\infty}\left[n(1+k)+p(k+\alpha)\right]|b_nB_n|\right\}}{1-\sum_{n=p+t}^{\infty}|a_nA_n| + \sum_{n=p+t-1}^{\infty}|b_nB_n|} \geq 0. \tag{2.12}$$

If the condition (2.10) does not hold, then the numerator in (2.12) is negative for r sufficiently close to 1. Hence there exists $z_0 = r_0$ in $(0,1)$ for which (2.12) is negative. Therefore, it follows that $f \notin H_{\overline{F}}(p,t,\alpha,k)$ and so the proof is complete. □

Theorem 2.5. *If $f \in H_{\overline{F}}(p,t;\alpha,k)$, then for $|z| = r < 1$, $|A_{p+t}| \leq |A_n| \leq |B_n|$, and $A_{p+t} \neq 0$,*

$$\left|f(z)\right| \leq \begin{cases} \left(1+\left|b_{p+t-1}\right|\right)r^{p+t-1} \\ +\left(\dfrac{p(1-\alpha)}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}\right. \\ \qquad \left. -\dfrac{\left[(p+t-1)(1+k)+p(k+\alpha)\right]}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}\left|b_{p+t-1}B_{p+t-1}\right|\right)r^{p+1}; \quad p(1-\alpha)\leq 1 \\ \left(1+\left|b_{p+t-1}\right|\right)r^{p+t-1} \\ +\left(\dfrac{1}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}\right. \\ \qquad \left. -\dfrac{\left[(p+t-1)(1+k)+p(k+\alpha)\right]}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}\left|b_{p+t-1}B_{p+t-1}\right|\right)r^{p+1}; \quad p(1-\alpha)\geq 1, \end{cases}$$

$$\left|f(z)\right| \geq \begin{cases} \left(1-\left|b_{p+t-1}\right|\right)r^{p+t-1} \\ -\left(\dfrac{p(1-\alpha)}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}\right. \\ \qquad \left. -\dfrac{\left[(p+t-1)(1+k)+p(k+\alpha)\right]}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}\left|b_{p+t-1}B_{p+t-1}\right|\right)r^{p+1}; \quad p(1-\alpha)\leq 1 \\ \left(1-\left|b_{p+t-1}\right|\right)r^{p+t-1} \\ -\left(\dfrac{1}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}\right. \\ \qquad \left. -\dfrac{\left[(p+t-1)(1+k)+p(k+\alpha)\right]}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}\left|b_{p+t-1}B_{p+t-1}\right|\right)r^{p+1}; \quad p(1-\alpha)\geq 1. \end{cases} \tag{2.13}$$

These bounds are sharp.

Proof. Suppose $p(1-\alpha) \le 1$. Let $f \in H_{\overline{F}}(p,t,\alpha,k)$ and $|A_{p+t}| \le |B_n|$. In view of (1.13), we get

$$\begin{aligned}
\left|f(z)\right| &= \left| z^p + \left|b_{p+t-1}\right|(\overline{z})^{p+t-1} - \sum_{n=p+t}^{\infty}\left(|a_n|z^n - |b_n|\overline{z^n}\right)\right| \\
&\le \left(1+\left|b_{p+t-1}\right|\right)r^{p+t-1} + \sum_{n=p+t}^{\infty}(|a_n|+|b_n|)r^{p+1} \\
&\le \left(1+\left|b_{p+t-1}\right|\right)r^{p+t-1} + \frac{p(1-\alpha)}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|} \\
&\quad \times \sum_{n=p+t}^{\infty}\frac{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}{p(1-\alpha)}(|a_n|+|b_n|)r^{p+1} \\
&\le \left(1+\left|b_{p+t-1}\right|\right)r^{p+t-1} + \frac{p(1-\alpha)}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|} \\
&\quad \times \left(\sum_{n=p+t}^{\infty}\frac{\left[n(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}{p(1-\alpha)}|a_n| + \frac{\left[n(1+k)+p(k+\alpha)\right]\left|A_{p+t}\right|}{p(1-\alpha)}|b_n|\right)r^{p+1} \\
&\le \left(1+\left|b_{p+t-1}\right|\right)r^{p+t-1} + \frac{p(1-\alpha)}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|} \\
&\quad \times \left(\sum_{n=p+t}^{\infty}\frac{\left[n(1+k)-p(k+\alpha)\right]}{p(1-\alpha)}|A_n a_n| + \frac{\left[n(1+k)+p(k+\alpha)\right]}{p(1-\alpha)}|B_n b_n|\right)r^{p+1}.
\end{aligned} \tag{2.14}$$

Using Theorem 2.4(i), we obtain

$$\begin{aligned}
\left|f(z)\right| &\le \left(1+\left|b_{p+t-1}\right|\right)r^{p+t-1} + \frac{p(1-\alpha)}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|} \\
&\quad \times \left(1 - \frac{\left[(p+t-1)(1+k)+p(k+\alpha)\right]}{p(1-\alpha)}\left|b_{p+t-1}B_{p+t-1}\right|\right)r^{p+1} \\
&= \left(1+\left|b_{p+t-1}\right|\right)r^{p+t-1} \\
&\quad + \left(\frac{p(1-\alpha)}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}\right. \\
&\qquad \left. - \frac{\left[(p+t-1)(1+k)+p(k+\alpha)\right]}{\left[(p+t)(1+k)-p(k+\alpha)\right]\left|A_{p+t}\right|}\left|b_{p+t-1}B_{p+t-1}\right|\right)r^{p+1}.
\end{aligned} \tag{2.15}$$

The proofs of other cases are similar and so are omitted. □

Corollary 2.6. *If $f \in H_{\overline{F}}(p,t,\alpha,k)$, then*

$$w : |w(z)| < \left\{\begin{array}{l} 1 - \dfrac{p(1-\alpha)}{\left[(p+t)(1+k) - p(k+\alpha)\right]\left|A_{p+t}\right|} \\ -\dfrac{\left[(p+t)(1+k) - p(k+\alpha) - \left[(p+t-1)(1+k) + p(k+\alpha)\right]\left|B_{p+t-1}\right|\right]\left|b_{p+t-1}\right|}{\left[(p+t)(1+k) - p(k+\alpha)\right]\left|A_{p+t}\right|}; \\ \hfill p(1-\alpha) \le 1 \\ 1 - \dfrac{1}{\left[(p+t)(1+k) - p(k+\alpha)\right]\left|A_{p+t}\right|} \\ -\dfrac{\left[(p+t)(1+k) - p(k+\alpha) - \left[(p+t-1)(1+k) + p(k+\alpha)\right]\left|B_{p+t-1}\right|\right]\left|b_{p+t-1}\right|}{\left[(p+t)(1+k) - p(k+\alpha)\right]\left|A_{p+t}\right|}; \\ \hfill p(1-\alpha) \ge 1 \end{array}\right\} \subset f(U). \tag{2.16}$$

Theorem 2.7. *Suppose $A_n \neq 0$ $(n = p+t, n = p+t+1, \dots)$ and $B_n \neq 0$ $(n = p+t-1, n = p+t, \dots)$. Then $f \in clco\, H_{\overline{F}}(p,t,\alpha,k)$ if and only if*

$$f(z) = \sum_{n=p+t-1}^{\infty} \left(X_n h_n(z) + Y_n g_n(z)\right), \quad z \in U, \tag{2.17}$$

where

$$\begin{aligned} h_{p+t-1}(z) &= z^p \\ h_n(z) &= \begin{cases} z^p - \dfrac{p(1-\alpha)}{\left[n(1+k) - p(k+\alpha)\right]|A_n|} z^n; & (n = p+t, p+t+1, \dots),\ p(1-\alpha) \le 1, \\ z^p - \dfrac{1}{\left[n(1+k) - p(k+\alpha)\right]|A_n|} z^n; & (n = p+t, p+t+1, \dots),\ p(1-\alpha) \ge 1, \end{cases} \\ g_n(z) &= \begin{cases} z^p + \dfrac{p(1-\alpha)}{\left[n(1+k) + p(k+\alpha)\right]|B_n|} \overline{z^n}; & (n = p+t-1, p+t, \dots),\ p(1-\alpha) \le 1 \\ z^p + \dfrac{1}{\left[n(1+k) + p(k+\alpha)\right]|B_n|} \overline{z^n}; & (n = p+t-1, p+t, \dots),\ p(1-\alpha) \ge 1 \end{cases} \\ X_{p+t-1} &\equiv X_p = 1 - \left(\sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n\right), \quad X_n \ge 0,\ Y_n \ge 0. \end{aligned} \tag{2.18}$$

In particular, the extreme points of $H_{\overline{F}}(p,t,\alpha,k)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose $p(1-\alpha) \leq 1$. For functions of the form (2.17), we can write

$$f(z) = z^p - \sum_{n=p+t}^{\infty} \frac{p(1-\alpha)}{[n(1+k)-p(k+\alpha)]|A_n|} X_n z^n + \sum_{n=p+t-1}^{\infty} \frac{p(1-\alpha)}{[n(1+k)+p(k+\alpha)]|B_n|} Y_n \overline{z}^n. \quad (2.19)$$

On the other hand, for $0 \leq X_p \leq 1$, we obtain

$$\begin{aligned} &\sum_{n=p+t}^{\infty} \frac{[n(1+k)-p(k+\alpha)]|A_n|}{p(1-\alpha)} \left(\frac{p(1-\alpha)}{[n(1+k)-p(k+\alpha)]|A_n|} X_n \right) \\ &\quad + \sum_{n=p+t-1}^{\infty} \frac{[n(1+k)+p(k+\alpha)]|B_n|}{p(1-\alpha)} \left(\frac{p(1-\alpha)}{[n(1+k)+p(k+\alpha)]|B_n|} Y_n \right) \\ &= \left(\sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n \right) = 1 - X_p \leq 1. \end{aligned} \quad (2.20)$$

Thus $f \in H_{\overline{F}}(p,t,\alpha,k)$, by Theorem 2.4.

Conversely, suppose that $f \in H_{\overline{F}}(p,t,\alpha,k)$. Then, it follows Theorem 2.4 that

$$|a_n| \leq \frac{p(1-\alpha)}{[n(1+k)-p(k+\alpha)]|A_n|}, \qquad |b_n| \leq \frac{p(1-\alpha)}{[n(1+k)+p(k+\alpha)]|B_n|}. \quad (2.21)$$

Setting

$$X_n = \frac{[n(1+k)-p(k+\alpha)]|a_n A_n|}{p(1-\alpha)}, \qquad Y_n = \frac{[n(1+k)+p(k+\alpha)]|b_n B_n|}{p(1-\alpha)}, \quad (2.22)$$

and defining

$$X_p = 1 - \left(\sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n \right), \quad (2.23)$$

where $X_p \geq 0$, we obtain

$$\begin{aligned}
f(z) &= z^p - \sum_{n=p+t}^{\infty} |a_n| z^n + \sum_{n=p+t-1}^{\infty} |b_n| (\overline{z})^n \\
&= z^p - \sum_{n=p+t}^{\infty} \frac{p(1-\alpha)X_n}{[n(1+k) - p(k+\alpha)]|A_n|} z^n + \sum_{n=p+t-1}^{\infty} \frac{p(1-\alpha)Y_n}{[n(1+k) + p(k+\alpha)]|B_n|} (\overline{z})^n \\
&= z^p - \sum_{n=p+t}^{\infty} (z^p - h_n(z))X_n - \sum_{n=p+t-1}^{\infty} (z^p - g_n(z))Y_n \\
&= \left(1 - \left(\sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n\right)\right) z^p + \sum_{n=p+t}^{\infty} h_n(z) X_n + \sum_{n=p+t-1}^{\infty} g_n(z) Y_n \\
&= X_p z^p + \sum_{n=p+t}^{\infty} X_n h_n(z) + \sum_{n=p+t-1}^{\infty} Y_n g_n(z).
\end{aligned} \tag{2.24}$$

Thus f can be expressed as (2.17). The proof for the case $p(1-\alpha) \geq 1$ is similar and hence is omitted. □

Theorem 2.8. *The class $H_{\overline{F}}(p, t, \alpha, k)$ is closed under convex combinations.*

Proof. For $j = 1, 2, \ldots,$ let the functions f_j given by

$$f_j(z) = z^p - \sum_{n=p+t}^{\infty} |a_{j_n}| z^n + \sum_{n=p+t-1}^{\infty} |b_{j_n}| (\overline{z})^n \tag{2.25}$$

are in $H_{\overline{F}}(p, t, \alpha, k)$. Also suppose the given fixed harmonic functions are given by

$$F_j(z) = z^p + \sum_{n=p+t}^{\infty} |A_{j_n}| z^n + \sum_{n=p+t-1}^{\infty} |B_{j_n}| (\overline{z})^n. \tag{2.26}$$

For $\sum_{j=1}^{\infty} \mu_j = 1,\ 0 \leq \mu_j \leq 1$ the convex combinations of f_j can be expressed as

$$\sum_{j=1}^{\infty} \mu_j f_j(z) = z^p - \sum_{n=p+t}^{\infty} \left(\sum_{j=1}^{\infty} \mu_j |a_{j_n}|\right) z^n + \sum_{n=p+t-1}^{\infty} \left(\sum_{j=1}^{\infty} \mu_j |b_{j_n}|\right) (\overline{z})^n. \tag{2.27}$$

Since

$$\sum_{n=p+t}^{\infty}\left[n(1+k)-p(k+\alpha)\right]\left|a_{j_n}A_{j_n}\right|+\sum_{n=p+t-1}^{\infty}\left[n(1+k)+p(k+\alpha)\right]\left|b_{j_n}B_{j_n}\right| \le \begin{cases} p(1-\alpha) & \text{if } p(1-\alpha)\ge 1 \\ 1 & \text{if } p(1-\alpha)\le 1, \end{cases} \tag{2.28}$$

(2.27) yields

$$\begin{aligned} &\sum_{n=p+t}^{\infty}\left[n(1+k)-p(k+\alpha)\right]\sum_{j=1}^{\infty}\mu_j\left|a_{j_n}A_{j_n}\right|+\sum_{n=p+t-1}^{\infty}\left[n(1+k)+p(k+\alpha)\right]\sum_{j=1}^{\infty}\mu_j\left|b_{j_n}B_{j_n}\right| \\ &\quad=\sum_{j=1}^{\infty}\mu_j\left\{\sum_{n=p+t}^{\infty}\left[n(1+k)-p(k+\alpha)\right]\left|a_{j_n}A_{j_n}\right|+\sum_{n=p+t-1}^{\infty}\left[n(1+k)+p(k+\alpha)\right]\left|b_{j_n}B_{j_n}\right|\right\} \\ &\quad\le\begin{cases} p(1-\alpha)\displaystyle\sum_{j=1}^{\infty}\mu_j=p(1-\alpha) & \text{if } \ p(1-\alpha)\le 1 \\ \displaystyle\sum_{j=1}^{\infty}\mu_j=1 & \text{if } \ p(1-\alpha)\ge 1. \end{cases} \end{aligned} \tag{2.29}$$

Thus the coefficient estimate given by Theorem 2.4 holds. Therefore, we obtain

$$\sum_{j=1}^{\infty}\mu_j f_j(z)\in H_{\overline{F}}(p,t,\alpha,k). \tag{2.30}$$

□

Acknowledgment

This present investigation is supported with the Project no. DÜBAP-07-02-21 by Dicle University, The committee of the Scientific Research Projects.

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