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Research Article

Conditions for Carathéodory Functions

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The purpose of the present paper is to derive some sufficient conditions for Carathéodory functions in the open unit disk. Our results include several interesting corollaries as special cases.

1. Introduction

Let $\mathcal{P}$ be the class of functions p of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If p in $\mathcal{P}$ satisfies

$$\operatorname{Re}\{p(z)\} > 0 \quad (z \in \mathbb{U}), \tag{1.2}$$

then we say that p is the Catathéodory function.

Let $\mathcal{A}$ denote the class of all functions f analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ with the usual normalization $f(0) = f'(0) - 1 = 0$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For $0 < \alpha \leq 1$, let $\mathcal{STC}(\alpha)$ and $\mathcal{STS}(\alpha)$ denote the classes of functions $f \in \mathcal{A}$ which are strongly convex and starlike of order α; that is, which satisfy

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U}), \tag{1.3}$$

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U}), \tag{1.4}$$

respectively. We note that (1.3) and (1.4) can be expressed, equivalently, by the argument functions. The classes $\mathcal{STC}(\alpha)$ and $\mathcal{STS}(\alpha)$ were introduced by Brannan and Kirwan [1] and studied by Mocanu [2] and Nunokawa [3, 4]. Also, we note that if $\alpha = 1$, then $\mathcal{STS}(\alpha)$ coincides with $\mathcal{S}^*$, the well-known class of starlike(univalent) functions with respect to origin, and if $0 < \alpha < 1$, then $\mathcal{STS}(\alpha)$ consists only of bounded starlike functions [1], and hence the inclusion relation $\mathcal{STS}(\alpha) \subset \mathcal{S}^*$ is proper. Furthermore, Nunokawa and Thomas [4] (see also [5]) found the value $\beta(\alpha)$ such that $\mathcal{STC}(\beta(\alpha)) \subset \mathcal{STS}(\alpha)$.

In the present paper, we consider general forms which cover the results by Mocanu [6] and Nunokawa and Thomas [4]. An application of a certain integral operator is also considered. Moreover, we give some sufficient conditions for univalent (close-to-convex) and (strongly) starlike functions (of order β) as special cases of main results.

2. Main Results

To prove our results, we need the following lemma due to Nunokawa [3].

Lemma 2.1. *Let p be analytic in $\mathbb{U}, p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\begin{gathered} \left|\arg p(z)\right| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|, \\ \left|\arg p(z_0)\right| = \frac{\pi}{2}\alpha \quad (0 < \alpha \le 1). \end{gathered} \tag{2.1}$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k, \tag{2.2}$$

where

$$\begin{gathered} k \ge \frac{1}{2}\left(x + \frac{1}{x}\right) \quad \text{when } \arg p(z_0) = \frac{\pi}{2}\alpha, \\ k \le -\frac{1}{2}\left(x + \frac{1}{x}\right) \quad \text{when } \arg p(z_0) = -\frac{\pi}{2}\alpha, \\ \left\{p(z_0)\right\}^{1/\alpha} = \pm ix \quad (x > 0). \end{gathered} \tag{2.3}$$

With the help of Lemma 2.1, we now derive the following theorem.

Theorem 2.2. *Let p be nonzero analytic in $\mathbb{U}$ with $p(0) = 1$ and let p satisfy the differential equation*

$$\eta z p'(z) + B(z)p(z) = a + ibA(z), \tag{2.4}$$

where $\eta > 0$, $a \in \mathbb{R}^+$, $0 \le b \le a\tan(\pi/2)\alpha$, $0 < \alpha < 1$, $A(z) = \operatorname{sign}(\operatorname{Im} p(z))$ and $B(z)$ is analytic in $\mathbb{U}$ with $B(0) = a$. If

$$\left|\arg B(z)\right| < \frac{\pi}{2}\beta(\eta, \alpha, a, b) \quad (z \in \mathbb{U}), \tag{2.5}$$

where

$$\beta(\eta,\alpha,a,b) = \frac{2}{\pi}\tan^{-1}\left\{\frac{S(\alpha)T(\alpha)(a\sin(\pi/2)\alpha - b\cos(\pi/2)\alpha) + \eta\alpha}{S(\alpha)T(\alpha)(a\cos(\pi/2)\alpha + b\sin(\pi/2)\alpha)}\right\}, \tag{2.6}$$

$$S(\alpha) = (1+\alpha)^{(1+\alpha)/2}, \qquad T(\alpha) = (1-\alpha)^{(1-\alpha)/2}, \tag{2.7}$$

then

$$\left|\arg p(z)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{2.8}$$

Proof. If there exists a point $z_0 \in \mathbb{U}$ such that the conditions (2.1) are satisfied, then (by Lemma 2.1) we obtain (2.2) under the restrictions (2.3). Then we obtain

$$\begin{aligned}
A(z_0) &= \begin{cases} 1, & \text{if } p(z_0) = (ix)^{\alpha}, \\ -1, & \text{if } p(z_0) = (-ix)^{\alpha}, \end{cases} \\
B(z_0) &= \frac{a + ibA(z_0)}{p(z_0)} - \eta\frac{z_0 p'(z_0)}{p(z_0)} \\
&= (a + ibA(z_0))(\pm ix)^{-\alpha} - i\eta\alpha k \\
&= \left(\frac{a}{x^{\alpha}}\cos\frac{\pi}{2}\alpha + \frac{b}{x^{\alpha}}A(z_0)\sin\left(\pm\frac{\pi}{2}\alpha\right)\right) \\
&\quad + i\left(\frac{b}{x^{\alpha}}A(z_0)\cos\frac{\pi}{2}\alpha - \frac{a}{x^{\alpha}}\sin\left(\pm\frac{\pi}{2}\alpha\right) - \eta\alpha k\right).
\end{aligned} \tag{2.9}$$

Now we suppose that

$$\{p(z_0)\}^{1/\alpha} = ix \quad (x > 0). \tag{2.10}$$

Then we have

$$\arg B(z_0) = -\tan^{-1}\left\{\frac{a\sin(\pi/2)\alpha - b\cos(\pi/2)\alpha + \eta\alpha x^{\alpha}k}{a\cos(\pi/2)\alpha + b\sin(\pi/2)\alpha}\right\}, \tag{2.11}$$

where

$$kx^{\alpha} \geq \frac{1}{2}\left(x^{\alpha+1} + x^{\alpha-1}\right) \equiv g(x) \quad (x > 0). \tag{2.12}$$

Then, by a simple calculation, we see that the function $g(x)$ takes the minimum value at $x = \sqrt{(1-\alpha)/(1+\alpha)}$. Hence, we have

$$\begin{aligned}\arg B(z_0) &\le -\tan^{-1}\left\{\frac{(1+\alpha)^{(1+\alpha)/2}(1-\alpha)^{(1-\alpha)/2}(a\sin(\pi/2)\alpha - b\cos(\pi/2)\alpha) + \eta\alpha}{(1+\alpha)^{(1+\alpha)/2}(1-\alpha)^{(1-\alpha)/2}(a\cos(\pi/2)\alpha + b\sin(\pi/2)\alpha)}\right\} \\ &= -\frac{\pi}{2}\beta(\eta,\alpha,a,b),\end{aligned} \tag{2.13}$$

where $\beta(\eta,\alpha,a,b)$ is given by (2.6). This evidently contradicts the assumption of Theorem 2.2.

Next, we suppose that

$$\{p(z_0)\}^{1/\alpha} = -ix \quad (x > 0). \tag{2.14}$$

Applying the same method as the above, we have

$$\begin{aligned}\arg B(z_0) &\ge \tan^{-1}\left\{\frac{(1+\alpha)^{(1+\alpha)/2}(1-\alpha)^{(1-\alpha)/2}(a\sin(\pi/2)\alpha - b\cos(\pi/2)\alpha) + \eta\alpha}{(1+\alpha)^{(1+\alpha)/2}(1-\alpha)^{(1-\alpha)/2}(a\cos(\pi/2)\alpha + b\sin(\pi/2)\alpha)}\right\} \\ &= \frac{\pi}{2}\beta(\eta,\alpha,a,b),\end{aligned} \tag{2.15}$$

where $\beta(\eta,\alpha,a,b)$ is given by (2.6), which is a contradiction to the assumption of Theorem 2.2. Therefore, we complete the proof of Theorem 2.2. □

Corollary 2.3. *Let $f \in \mathcal{A}$ and $\eta > 0$, $0 < \alpha < 1$. If*

$$\left|\arg\left\{(1-\eta)\frac{zf'(z)}{f(z)} + \eta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}\right| < \frac{\pi}{2}\beta(\eta,\alpha) \quad (z \in U), \tag{2.16}$$

where $\beta(\eta,\alpha)$ is given by (2.6) *with $a = 1$ and $b = 0$, then $f \in \mathcal{STS}(\alpha)$.*

Proof. Taking

$$p(z) = \frac{f(z)}{zf'(z)}, \qquad B(z) = (1-\eta)\frac{zf'(z)}{f(z)} + \eta\left(1 + \frac{zf''(z)}{f'(z)}\right) \tag{2.17}$$

in Theorem 2.2, we can see that (2.4) is satisfied. Therefore, the result follows from Theorem 2.2. □

Corollary 2.4. *Let $f \in \mathcal{A}$ and $0 < \alpha < 1$. Then $\mathcal{STC}(\beta(\alpha)) \subset \mathcal{STS}(\alpha)$, where $\beta(\alpha)$ is given by* (2.6) *with $\eta = a = 1$ and $b = 0$.*

By a similar method of the proof in Theorem 2.2, we have the following theorem.

Theorem 2.5. *Let p be nonzero analytic in $\mathbb{U}$ with $p(0) = 1$ and let p satisfy the differential equation*

$$\frac{zp'(z)}{p(z)} + B(z) = a + ibA(z), \tag{2.18}$$

where $a \in \mathbb{R}^+$, $b \in \mathbb{R}^- \cup \{0\}$, $A(z) = \operatorname{sign}(\operatorname{Im} p(z))$, and $B(z)$ is analytic in $\mathbb{U}$ with $B(0) = a$. If

$$\left|\arg B(z)\right| < \frac{\pi}{2}\alpha(\delta, a, b) \quad (z \in \mathbb{U}), \tag{2.19}$$

where

$$\alpha(\delta) := \alpha(\delta, a, b) = \frac{2}{\pi}\tan^{-1}\frac{\delta - b}{a} \quad (\delta > 0), \tag{2.20}$$

then

$$\left|\arg p(z)\right| < \frac{\pi}{2}\delta \quad (z \in \mathbb{U}). \tag{2.21}$$

Corollary 2.6. *Let $f \in \mathcal{STS}(\alpha(\delta))$, where $\alpha(\delta)$ is given by* (2.20) *with $a = 1$ and $b = 0$. Then*

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2}\delta \quad (z \in \mathbb{U}). \tag{2.22}$$

Proof. Letting

$$p(z) = \frac{z}{f(z)}, \qquad B(z) = \frac{zf'(z)}{f(z)} \tag{2.23}$$

in Theorem 2.5, we have Corollary 2.6 immediately. □

If we combine Corollaries 2.4 and 2.6, then we obtain the following result obtained by Nunokawa and Thomas [4].

Corollary 2.7. *Let $f \in \mathcal{STC}(\beta(\delta))$, where*

$$\beta(\delta) = \frac{2}{\pi}\tan^{-1}\left\{\tan\frac{\pi}{2}\alpha(\delta) + \frac{\alpha(\delta)}{(1+\alpha(\delta))^{(1+\alpha(\delta))/2}(1-\alpha(\delta))^{(1-\alpha(\delta))/2}\cos(\pi/2)\alpha(\delta)}\right\} \tag{2.24}$$

and $\alpha(\delta)$ is given by (2.20). *Then*

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2}\delta \quad (z \in \mathbb{U}). \tag{2.25}$$

Corollary 2.8. *Let* $f \in \mathcal{A}$, $0 < \alpha < 1$ *and* β, γ *be real numbers with* $\beta \neq 0$ *and* $\beta + \gamma > 0$. *If*

$$\left|\arg\left(\beta \frac{zf'(z)}{f(z)} + \gamma\right)\right| < \frac{\pi}{2}\delta(\alpha,\beta,\gamma) \quad (z \in \mathbb{U}), \tag{2.26}$$

where

$$\delta(\alpha,\beta,\gamma) = \frac{2}{\pi}\tan^{-1}\left\{\tan\frac{\pi}{2}\alpha + \frac{\alpha}{(\beta+\gamma)(1+\alpha)^{(1+\alpha)/2}(1-\alpha)^{(1-\alpha)/2}\cos(\pi/2)\alpha}\right\}, \tag{2.27}$$

then

$$\left|\arg\left(\beta \frac{zF'(z)}{F(z)} + \gamma\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}), \tag{2.28}$$

where F is the integral operator defined by

$$F(z) = \left(\frac{\beta+\gamma}{z^{\gamma}}\int_0^z f^{\beta}(t)t^{\gamma-1}dt\right)^{1/\beta} \quad (z \in \mathbb{U}). \tag{2.29}$$

Proof. Let

$$B(z) = \frac{1}{\beta+\gamma}\left(\beta\frac{zf'(z)}{f(z)} + \gamma\right), \tag{2.30}$$

$$p(z) = \frac{\beta+\gamma}{z^{\gamma}f^{\beta}(z)}\int_0^z f^{\beta}(t)t^{\gamma-1}dt. \tag{2.31}$$

Then $B(z)$ and $p(z)$ are analytic in $\mathbb{U}$ with $B(0) = p(0) = 1$. By a simple calculation, we have

$$\frac{1}{\beta+\gamma}zp'(z) + B(z)p(z) = 1. \tag{2.32}$$

Using a similar method of the proof in Theorem 2.2, we can obtain that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{2.33}$$

From (2.29) and (2.31), we easily see that

$$F(z) = f(z)\{p(z)\}^{1/\beta}. \tag{2.34}$$

Since

$$\beta\frac{zF'(z)}{F(z)}+\gamma=\frac{\beta+\gamma}{p(z)}, \tag{2.35}$$

the conclusion of Corollary 2.8 immediately follows. □

Remark 2.9. Letting $\alpha \to 1$ in Corollary 2.8, we have the result obtained by Miller and Mocanu [7].

The proof of the following theorem below is much akin to that of Theorem 2.2 and so we omit for details involved.

Theorem 2.10. *Let p be nonzero analytic in $\mathbb{U}$ with $p(0)=1$ and let p satisfy the differential equation*

$$\frac{zp'(z)}{p(z)}+B(z)p(z)=a+ibA(z), \tag{2.36}$$

where $a\in\mathbb{R}^{+}$, $b\in\mathbb{R}^{-}\cup\{0\}$, $A(z)=\operatorname{sign}(\operatorname{Im}p(z))$ and $B(z)$ is analytic in $\mathbb{U}$ with $B(0)=a$. If

$$\left|\arg B(z)\right|<\frac{\pi}{2}\beta(\alpha,a,b)\quad(z\in\mathbb{U}), \tag{2.37}$$

where

$$\beta(\alpha,a,b)=\alpha+\frac{2}{\pi}\tan^{-1}\frac{\alpha-b}{a}\quad(0<\alpha\leq 1), \tag{2.38}$$

then

$$\left|\arg p(z)\right|<\frac{\pi}{2}\alpha\quad(z\in\mathbb{U}). \tag{2.39}$$

Corollary 2.11. *Let $f\in\mathcal{A}$ with $f'(z)\neq 0$ in $\mathbb{U}$ and $0<\alpha\leq 1$. If*

$$\left|\arg\left(f'(z)+zf''(z)\right)\right|<\frac{\pi}{2}\beta(\alpha)\quad(z\in\mathbb{U}), \tag{2.40}$$

where $\beta(\alpha)$ is given by (2.38) *with $a=1$ and $b=0$, then*

$$\left|\arg f'(z)\right|<\frac{\pi}{2}\alpha\quad(z\in\mathbb{U}), \tag{2.41}$$

that is, f is univalent (close-to-convex) in $\mathbb{U}$.

Proof. Let

$$p(z) = \frac{1}{f'(z)}, \qquad B(z) = f'(z) + zf''(z) \tag{2.42}$$

in Theorem 2.10. Then (2.36) is satisfied and so the result follows. □

By applying Theorem 2.10, we have the following result obtained by Mocanu [6].

Corollary 2.12. *Let $f \in \mathcal{A}$ with $f(z)/z \neq 0$ and α_0 be the solution of the equation given by*

$$2\alpha + \frac{2}{\pi}\tan^{-1}\alpha = 1 \quad (0 < \alpha < 1). \tag{2.43}$$

If

$$\left|\arg f'(z)\right| < \frac{\pi}{2}(1-\alpha_0) \quad (z \in \mathbb{U}), \tag{2.44}$$

then $f \in \mathcal{S}^*$.

Proof. Let

$$p(z) = \frac{z}{f(z)}, \qquad B(z) = f'(z). \tag{2.45}$$

Then, by Theorem 2.10, condition (2.44) implies that

$$\left|\arg \frac{z}{f(z)}\right| < \frac{\pi}{2}\alpha_0. \tag{2.46}$$

Therefore, we have

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leq \left|\arg f'(z)\right| + \left|\arg \frac{z}{f(z)}\right| < \frac{\pi}{2}, \tag{2.47}$$

which completes the proof of Corollary 2.12. □

Corollary 2.13. *Let $f \in \mathcal{A}$ with $f(z)f'(z)/z \neq 0$ in $\mathbb{U}$ and $0 < \alpha \leq 1$. If*

$$\left|\arg \frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta(\alpha) \quad (z \in \mathbb{U}), \tag{2.48}$$

where $\beta(\alpha)$ is given by (2.38), *then $f \in \mathcal{STS}(\alpha)$.*

Finally, we have the following result.

Theorem 2.14. *Let p be nonzero analytic in $\mathbb{U}$ with $p(0) = 1$. If*

$$\left|\arg\left((1-\lambda)p(z) + \lambda z p'(z)\right)\right| < \frac{\pi}{2}\beta(\lambda, \alpha), \tag{2.49}$$

$$\beta(\lambda, \alpha) = \alpha + \frac{2}{\pi}\tan^{-1}\frac{\lambda\alpha}{1-\lambda} \quad (0 \le \lambda < 1;\ 0 < \alpha < 1), \tag{2.50}$$

then

$$\left|\arg p(z)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{2.51}$$

Proof. If there exists a point $z_0 \in \mathbb{U}$ satisfying the conditions of Lemma 2.1, then we have

$$(1-\lambda)p(z_0) + \lambda z_0 p'(z_0) = (\pm ix)^{\alpha}(1 - \lambda + i\lambda\alpha k). \tag{2.52}$$

Now we suppose that

$$\left\{p(z_0)\right\}^{1/\alpha} = ix \quad (x > 0). \tag{2.53}$$

Then we have

$$\begin{aligned} \arg\left((1-\lambda)p(z_0) + \lambda z_0 p'(z_0)\right) &= \frac{\pi}{2}\alpha + \tan^{-1}\frac{\lambda\alpha k}{1-\lambda} \\ &\ge \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\frac{\lambda\alpha}{1-\lambda}\right) \\ &= \frac{\pi}{2}\beta(\lambda, \alpha), \end{aligned} \tag{2.54}$$

where $\beta(\lambda, \alpha)$ is given by (2.50). Also, for the case

$$\left\{p(z_0)\right\}^{1/\alpha} = -ix \quad (x > 0), \tag{2.55}$$

we obtain

$$\begin{aligned} \arg\left((1-\lambda)p(z_0) + \lambda z_0 p'(z_0)\right) &\le -\frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\frac{\lambda\alpha}{1-\lambda}\right) \\ &= -\frac{\pi}{2}\beta(\lambda, \alpha), \end{aligned} \tag{2.56}$$

where $\beta(\lambda, \alpha)$ is given by (2.50). These contradict the assumption of Theorem 2.14 and so we complete the proof of Theorem 2.14. □

Corollary 2.15. *Let* $f \in \mathcal{A}$ *with* $f(z)f'(z)/z \neq 0$ *in* $\mathbb{U}$ *and* $0 < \alpha < 1$*. If*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right\}\right| < \frac{\pi}{2}(\alpha+1) \quad (z \in \mathbb{U}), \tag{2.57}$$

then $f \in \mathcal{SCS}(\alpha)$*.*

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