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Research Article

On Solvability of a Generalized Nonlinear Variational-Like Inequality

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A new generalized nonlinear variational-like inequality is introduced and studied. By applying the auxiliary principle technique and KKM theory, we construct a new iterative algorithm for solving the generalized nonlinear variational-like inequality. By means of the Banach fixed-point theorem, we establish the existence and uniqueness of solution for the generalized nonlinear variational-like inequality. The convergence of the sequence generated by the iterative algorithm is also discussed.

1. Introduction

It is well known that variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in many diverse fields. One of the most interesting and important problems in the variational inequality theory is the development of an efficient iterative algorithm to compute approximate solutions. The researchers in [1–10] suggested a lot of iterative algorithms for solving various variational inequalities and variational-like inequalities. By using the auxiliary principle technique, Ding and Yao [3], Ding et al. [4], Huang and Deng [5], Liu et al. [8, 9], and others studied several classes of nonlinear variational inequalities and variational-like inequalities in reflexive Banach and Hilbert spaces, respectively, suggested some iterative algorithms to compute approximate solutions for these nonlinear variational inequalities and variational-like inequalities, and proved the existence of solutions for the nonlinear variational inequalities and variational-like inequalities involving different monotone mappings.

Motivated and inspired by the research work in [1–10], we introduce and study a generalized nonlinear variational-like inequality. By using the auxiliary principle technique and KKM theorem due to Zhang and Xiang [2], we suggest a new iterative scheme for solving the generalized nonlinear variational-like inequality. Utilizing the Banach fixed-point theorem, we prove the existence and uniqueness of solution for the generalized nonlinear variational-like inequality. Under certain conditions, we discuss the convergence of the iterative sequence generalized by the iterative algorithm.

2. Preliminaries

Throughout this paper, let H be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$, respectively, and $\mathbb{R} = (-\infty, +\infty)$. Let K be a nonempty closed convex subset of H. Assume that $a : K \times K \to \mathbb{R}$ is a coercive continuous bilinear form, that is, there exist positive constants $c, d > 0$ such that

(a1) $a(v, v) \geq c\|v\|^2$, for all $v \in K$;

(a2) $|a(u, v)| \leq d\|u\|\|v\|$, for all $u, v \in K$.

Remark 2.1. It follows from (a1) and (a2) that $c \leq d$.

Let $b : K \times K \to \mathbb{R}$ be nondifferentiable and satisfy the following conditions:

(b1) b is linear in the first argument,

(b2) b is convex in the second argument,

(b3) b is bounded, that is, there exists a constant $l > 0$ satisfying

$$|b(u, v)| \leq l\|u\|\|v\|, \quad \forall\, u, v \in K, \tag{2.1}$$

(b4) $b(u, v) - b(u, w) \leq b(u, v - w)$, $\forall u, v, w \in K$.

Remark 2.2. It follows that

$$|b(u, v) - b(u, w)| \leq l\|u\|\|v - w\|, \quad \forall u, v, w \in K, \tag{2.2}$$

which implies that b is continuous in the second argument.

Let $A, B, C, D, S : K \to H, N : H \times H \to H, \eta : K \times K \to H$ be mappings and $f \in H$. Now we consider the following generalized nonlinear variational-like inequality.

Find $u \in K$ such that

$$\langle N(Au, Bu) - Su - f, \eta(v, u)\rangle + a(u, v - u) \geq b(u, u) - b(u, v), \quad \forall v \in K. \tag{2.3}$$

It is clear that for appropriate and suitable choices of the mappings $N, M, A, B, C, D, \eta, a, b$ and $f \in H$, the generalized nonlinear variational-like inequality (2.3) includes some variational inequalities and variational-like inequalities in [1–10] as special cases.

Recall the following concepts and results.

Definition 2.3. Let $A : K \to H, N : H \times H \to H$, and $\eta : K \times K \to H$ be mappings.

(1) A is said to be *Lipschitz continuous* if there exists a constant $t > 0$ such that

$$\|Ax - Ay\| \leq t\|x - y\|, \quad \forall x, y \in K. \tag{2.4}$$

(2) A is said to be *η-relaxed Lipschitz* if there exists a constant $\lambda > 0$ such that

$$\langle Su - Sv, \eta(u, v)\rangle \leq -\lambda\|u - v\|^2, \quad \forall u, v \in K. \tag{2.5}$$

(3) N is said to be *η-strongly monotone* with respect to A in the first argument if there exists a constant $\beta > 0$ such that

$$\langle N(Au, y) - N(Av, y), \eta(u, v)\rangle \geq \beta\|u - v\|^2, \quad \forall u, v \in K, \ \forall y \in H. \tag{2.6}$$

(4) N is said to be *η-monotone* with respect to A in the second argument if

$$\langle N(y, Au) - N(y, Av), \eta(u, v)\rangle \geq 0, \quad \forall u, v \in K, \ \forall y \in H. \tag{2.7}$$

(5) N is said to be *η-relaxed cocoercive* with respect to A in the second argument if there exists a constant $s > 0$ such that

$$\langle N(y, Au) - N(y, Av), \eta(u, v)\rangle \geq -s\|N(y, Au) - N(y, Av)\|^2, \quad \forall u, v \in K, \ \forall y \in H. \tag{2.8}$$

(6) N is said to be *Lipschitz continuous* in the second argument if there exists a constant $\gamma > 0$ such that

$$\|N(y, u) - N(y, v)\| \leq \gamma\|u - v\|, \quad \forall u, v, y \in H. \tag{2.9}$$

(7) η is said to be *Lipschitz continuous* if there exists a constant $\delta > 0$ such that

$$\|\eta(u, v)\| \leq \delta\|u - v\|, \quad \forall u, v \in K. \tag{2.10}$$

(8) N and S are said to be *η-hemicontinuous* with respect to A and B in K if for any $x, y, u, v \in K$, the mapping $t \to \langle N(A(tx + (1 - t)y), B(tx + (1 - t)y)) - S(tx + (1 - t)y) - f, \eta(u, v)\rangle$ is continuous on $[0, 1]$.

Lemma 2.4 (see [2]). *Let X be a nonempty closed convex subset of a Hausdorff linear topological space E, and let $\phi, \psi : X \times X \to \mathbb{R}$ be mappings satisfying the following conditions:*

(a) $\psi(x, y) \leq \phi(x, y)$, *for all* $x, y \in X$, *and* $\psi(x, x) \geq 0$, *for all* $x \in X$;

(b) *for each* $x \in X, \phi(x, \cdot)$ *is upper semicontinuous on* X;

(c) *for each* $y \in X$, *the set* $\{x \in X : \psi(x, y) < 0\}$ *is a convex set;*

(d) *there exists a nonempty compact set* $Y \subset X$ *and* $x_0 \in Y$ *such that* $\psi(x_0, y) < 0$, *for all* $y \in X \setminus Y$.

Then there exists $\hat{y} \in Y$ *such that* $\phi(x, \hat{y}) \geq 0$, *for all* $x \in X$.

Lemma 2.5 (see [11]). *Let* $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, *and* $\{c_n\}_{n\geq 0}$ *be nonnegative sequences satisfying*

$$a_{n+1} \leq (1 - \lambda_n) a_n + \lambda_n b_n + c_n, \quad \forall n \geq 0, \tag{2.11}$$

where

$$\{\lambda_n\}_{n=0}^{\infty} \subset [0, 1], \qquad \sum_{n=0}^{\infty} \lambda_n = +\infty, \qquad \sum_{n=0}^{\infty} c_n < +\infty, \qquad \lim_{n\to\infty} b_n = 0. \tag{2.12}$$

Then $\lim_{n\to\infty} a_n = 0$.

Assumption 2.6. Let $\eta : K \times K \to H$ satisfy that

(1) $\eta(x, y) = -\eta(y, x)$, for all $x, y \in K$;

(2) for any $x, y \in K$, the mapping $v \mapsto \langle N(Ax, Bx) - Sx - f, \eta(v, y)\rangle$ is convex and lower semicontinuous in K.

3. Auxiliary Problem and Algorithm

Now we consider the following auxiliary problem with respect to the generalized nonlinear variational-like inequality (2.3). For each $u \in K$, find $\hat{w} \in K$ such that

$$\begin{aligned} \langle \hat{w}, v - \hat{w}\rangle \geq{} & \langle u, v - \hat{w}\rangle - \rho\langle N(A\hat{w}, B\hat{w}) - S\hat{w} - f, \eta(v, \hat{w})\rangle \\ & - \rho a(\hat{w}, v - \hat{w}) - \rho b(u, v) + \rho b(u, \hat{w}), \quad \forall v \in K, \end{aligned} \tag{3.1}$$

where $\rho > 0$ is a constant.

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space $H, f \in H$, and let $\eta : K \times K \to H$ be Lipschitz continuous with constant δ. Assume that $a : K \times K \to \mathbb{R}$ is a coercive continuous bilinear form satisfying (a1) and (a2), $b : K \times K \to \mathbb{R}$ satisfies (b1)–(b4). Let $A, B, C, D, S : K \to H$ and $N : H \times H \to H$ be mappings such that B is Lipschitz continuous with constant s, N is η-strongly monotone with respect to A in the first argument with constant α, η-relaxed cocoercive with respect to B and Lipschitz continuous in the second argument with constants β and σ, respectively, S is η-relaxed Lipschitz with constant λ, and N and S are η-hemicontinuous* *with respect to A and B in K. Assume that Assumption 2.6 holds and there exists a positive constant ρ satisfying*

$$\rho\left(\beta\sigma^2 s^2 - \alpha - \lambda - c\right) < 1. \tag{3.2}$$

Then for each $u \in K$, the auxiliary problem (3.1) *has a unique solution in K.*

Proof. Let u be in K. Define two functionals ϕ and $\psi : K \times K \to \mathbb{R}$ by

$$\begin{aligned}
\phi(v,w) &= \langle v, v-w\rangle - \langle u, v-w\rangle + \rho\langle N(Av,Bv) - Sv - f, \eta(v,w)\rangle \\
&\quad + \rho a(v, v-w) - \rho b(u,w) + \rho b(u,v), \\
\psi(v,w) &= \langle w, v-w\rangle - \langle u, v-w\rangle + \rho\langle N(Aw,Bw) - Sw - f, \eta(v,w)\rangle \\
&\quad + \rho a(w, v-w) - \rho b(u,w) + \rho b(u,v)
\end{aligned} \tag{3.3}$$

for all $v, w \in K$.

Now we prove that the functionals ϕ and ψ satisfy all the conditions of Lemma 2.4 in the weak topology. It is easy to see for all $v, w \in K$,

$$\begin{aligned}
\phi(v,w) - \psi(v,w) &= \|v-w\|^2 + \rho\langle N(Av,Bv) - N(Aw,Bv), \eta(v,w)\rangle \\
&\quad + \rho\langle N(Aw,Bv) - N(Aw,Bw), \eta(v,w)\rangle - \rho\langle Sv - Sw, \eta(v,w)\rangle\rangle \\
&\quad + \rho a(v-w, v-w) \\
&\ge \left[1 - \rho\left(\beta\sigma^2 s^2 - \alpha - \lambda - c\right)\right]\|v-w\|^2 \\
&\ge 0, \\
\psi(v,v) &= 0,
\end{aligned} \tag{3.4}$$

which imply that ϕ and ψ satisfy condition (a) of Lemma 2.4. Since a is a coercive continuous bilinear form, b is convex and continuous in the second argument, and for given $w \in K$, the mapping $v \mapsto \langle N(Aw,Bw) - Sw - f, \eta(v,w)\rangle$ is convex and lower semicontinuous in K, it follows that for each $v \in K, \phi(v,\cdot)$ is weakly upper semicontinuous in the second argument and the set $\{v \in K : \psi(v,w) < 0\}$ is convex for each $w \in K$. That is, the conditions (b) and (c) of Lemma 2.4 hold. Let $\overline{v} \in K$,

$$\begin{aligned}
E &= \left[1 - \rho\left(\beta\sigma^2 s^2 - \alpha - \lambda - c\right)\right]^{-1} \\
&\quad \times \left[\|\overline{v} - u\| + \rho d\|\overline{v}\| + \rho\delta\left\|N(A\overline{v}, B\overline{v}) - M(C\overline{v}, D\overline{v}) - f\right\| + \rho l\|u\| + 1\right], \\
Y &= \{w \in K : \|w - \overline{v}\| \le E\}.
\end{aligned} \tag{3.5}$$

Clearly, Y is a weakly compact subset of K. For each $w \in K \setminus Y$, we infer that

$$\begin{aligned}
\varphi(\overline{v}, w) &= \langle w, \overline{v} - w\rangle - \langle u, \overline{v} - w\rangle + \rho\langle N(Aw, Bw) - Sw - f, \eta(\overline{v}, w)\rangle \\
&\quad + \rho a(w, \overline{v} - w) - \rho b(u, w) + \rho b(u, \overline{v}) \\
&= -\langle \overline{v} - w, \overline{v} - w\rangle + \langle \overline{v} - u, \overline{v} - w\rangle - \rho\langle N(A\overline{v}, Bw) - N(Aw, Bw), \eta(\overline{v}, w)\rangle \\
&\quad - \rho\langle N(A\overline{v}, B\overline{v}) - N(A\overline{v}, Bw), \eta(\overline{v}, w)\rangle + \rho\langle S\overline{v} - Sw, \eta(\overline{v}, w)\rangle \\
&\quad + \rho\langle N(A\overline{v}, B\overline{v}) - S\overline{v} - f, \eta(\overline{v}, w)\rangle - \rho a(\overline{v} - w, \overline{v} - w) \\
&\quad + \rho a(\overline{v}, \overline{v} - w) - \rho b(u, w) + \rho b(u, \overline{v}) \\
&\leq -\|\overline{v} - w\|\Big\{\Big[1 - \rho\Big(\beta\sigma^2 s^2 - \alpha - \lambda - c\Big)\Big]\|\overline{v} - w\| - \|\overline{v} - u\| \\
&\quad -\rho\delta\big\|N(A\overline{v}, B\overline{v}) - M(C\overline{v}, D\overline{v}) - f\big\| - \rho d\|\overline{v}\| - \rho l\|u\|\Big\} \\
&< 0,
\end{aligned} \tag{3.6}$$

which means that the condition (d) of Lemma 2.4 holds. Thus Lemma 2.4 ensures that there exists $\widehat{w} \in Y \subseteq K$ such that $\phi(v, \widehat{w}) \geq 0$ for all $v \in K$, that is,

$$\begin{aligned}
\langle v, v - \widehat{w}\rangle &\geq \langle u, v - \widehat{w}\rangle - \rho\langle N(Av, Bv) - Sv - f, \eta(v, \widehat{w})\rangle \\
&\quad - \rho a(v, v - \widehat{w}) - \rho b(u, v) + \rho b(u, \widehat{w}), \quad \forall v \in K.
\end{aligned} \tag{3.7}$$

Put $v_t = tv + (1-t)\widehat{w}$ for $t \in (0,1]$ and $v \in K$. Replacing v by v_t in (3.7), we obtain that

$$\begin{aligned}
\langle v_t, v_t - \widehat{w}\rangle &\geq \langle u, v_t - \widehat{w}\rangle - \rho\langle N(Av_t, Bv_t) - Sv_t - f, \eta(v_t, \widehat{w})\rangle \\
&\quad - \rho a(v_t, v_t - \widehat{w}) - \rho b(u, v_t) + \rho b(u, \widehat{w}), \quad \forall v \in K.
\end{aligned} \tag{3.8}$$

Notice that b is convex in the second argument. It follows from Assumption 2.6 and (3.8) that

$$\begin{aligned}
t\langle v_t, v - \widehat{w}\rangle &\geq t\big[\langle u, v - \widehat{w}\rangle - \rho\langle N(Av_t, Bv_t) - Sv_t - f, \eta(v, \widehat{w})\rangle \\
&\quad -\rho a(v_t, v - \widehat{w}) - \rho b(u, v) + \rho b(u, \widehat{w})\big], \quad \forall v \in K,
\end{aligned} \tag{3.9}$$

which implies that

$$\begin{aligned}
\langle v_t, v - \widehat{w}\rangle &\geq \langle u, v - \widehat{w}\rangle - \rho\langle N(Av_t, Bv_t) - Sv_t - f, \eta(v, \widehat{w})\rangle \\
&\quad - \rho a(v_t, v - \widehat{w}) - \rho b(u, v) + \rho b(u, \widehat{w}), \quad \forall v \in K.
\end{aligned} \tag{3.10}$$

Letting $t \to 0^+$ in (3.10), we conclude that

$$\begin{aligned}\langle \widehat{w}, v-\widehat{w}\rangle \geq{}& \langle u, v-\widehat{w}\rangle - \rho\langle N(A\widehat{w}, B\widehat{w}) - S\widehat{w} - f, \eta(v,\widehat{w})\rangle \\ &- \rho a(\widehat{w}, v-\widehat{w}) - \rho b(u,v) + \rho b(u,\widehat{w}), \quad \forall v \in K.\end{aligned} \tag{3.11}$$

That is, $\widehat{w} \in K$ is a solution of the auxiliary problem (3.1).

Now we prove the uniqueness of solution for the auxiliary problem (3.1). Suppose that $w_1, w_2 \in K$ are two solutions of the auxiliary problem (3.1) with respect to u. It follows that

$$\begin{aligned}\langle w_1, v-w_1\rangle \geq{}& \langle u, v-w_1\rangle - \rho\langle N(Aw_1, Bw_1) - Sw_1 - f, \eta(v,w_1)\rangle \\ &- \rho a(w_1, v-w_1) - \rho b(u,v) + \rho b(u,w_1), \quad \forall v \in K,\end{aligned} \tag{3.12}$$

$$\begin{aligned}\langle w_2, v-w_2\rangle \geq{}& \langle u, v-w_2\rangle - \rho\langle N(Aw_2, Bw_2) - Sw_2 - f, \eta(v,w_2)\rangle \\ &- \rho a(w_2, v-w_2) - \rho b(u,v) + \rho b(u,w_2), \quad \forall v \in K.\end{aligned} \tag{3.13}$$

Taking $v = w_2$ in (3.12) and $v = w_1$ in (3.13), we get that

$$\begin{aligned}\langle w_1, w_2-w_1\rangle \geq{}& \langle u, w_2-w_1\rangle - \rho\langle N(Aw_1, Bw_1) - Sw_1 - f, \eta(w_2,w_1)\rangle \\ &- \rho a(w_1, w_2-w_1) - \rho b(u,w_2) + \rho b(u,w_1),\end{aligned} \tag{3.14}$$

$$\begin{aligned}\langle w_2, w_1-w_2\rangle \geq{}& \langle u, w_1-w_2\rangle - \rho\langle N(Aw_2, Bw_2) - Sw_2 - f, \eta(w_1,w_2)\rangle \\ &- \rho a(w_2, w_1-w_2) - \rho b(u,w_1) + \rho b(u,w_2).\end{aligned} \tag{3.15}$$

Adding (3.14) and (3.15), we deduce that

$$\begin{aligned}\|w_1-w_2\|^2 \leq{}& -\rho\langle N(Aw_1, Bw_1) - N(Aw_2, Bw_1), \eta(w_1,w_2)\rangle \\ &- \rho\langle N(Aw_2, Bw_1) - N(Aw_2, Bw_2), \eta(w_1,w_2)\rangle \\ &+ \rho\langle Sw_1 - Sw_2, \eta(w_1,w_2)\rangle - \rho a(w_1-w_2, w_1-w_2) \\ \leq{}& \rho\Big(\beta\sigma^2 s^2 - \alpha - \lambda - c\Big)\|w_1-w_2\|^2,\end{aligned} \tag{3.16}$$

which yields that $w_1 = w_2$ by (3.2). That is, $\widehat{w}$ is the unique solution of the auxiliary problem (3.1). This completes the proof. □

The proof of the below result is similar to that of Theorem 3.1 and is omitted.

Theorem 3.2. *Let K be a nonempty closed convex subset of a real Hilbert space H, $f \in H$, and $\eta : K \times K \to H$ Lipschitz continuous with constant δ. Assume that $a : K \times K \to \mathbb{R}$ is a coercive continuous bilinear form satisfying (a1) and (a2), $b : K \times K \to \mathbb{R}$ satisfies (b1)–(b4). Let $A, B, C, D, S : K \to H$ and $N : H \times H \to H$ be mappings such that N is η-strongly monotone with respect to A in the first argument with constant α, η-monotone with respect to B in the second argument, S is Lipschitz* *continuous with constant ξ, and N and S are η-hemicontinuous with respect to A and B in K. Assume that Assumption 2.6 holds and there exists a positive constant ρ satisfying*

$$\rho(\xi\delta - \alpha - c) < 1. \tag{3.17}$$

Then for each $u \in K$, the auxiliary problem (3.1) *has a unique solution in K.*

Based on Theorems 3.1 and 3.2, we suggest the following iterative algorithm with errors for solving the generalized nonlinear variational-like inequality (2.3).

Algorithm 3.3. For given $u_0 \in K$, compute sequence $\{u_n\}_{n\geq 0} \subset K$ by the following iterative scheme:

$$\begin{aligned}
&\langle u_{n+1}, v - u_{n+1}\rangle \\
&\quad \geq \langle u_n, v - u_{n+1}\rangle - \rho\langle N(Au_{n+1}, Bu_{n+1}) - Su_{n+1} - f, \eta(v, u_{n+1})\rangle \\
&\qquad - \rho a(u_{n+1}, v - u_{n+1}) - \rho b(u_n, v) + \rho b(u_n, u_{n+1}) + \langle e_n, \eta(v, u_{n+1})\rangle, \quad \forall v \in K,\ n \geq 0,
\end{aligned} \tag{3.18}$$

where $\rho > 0$ is a constant and $\{e_n\}_{n\geq 0}$ is a sequence in K introduced to take into account possible inexact computation and satisfies that

$$\lim_{n\to\infty} \|e_n\| = 0. \tag{3.19}$$

4. Existence and Convergence

In this section, we prove the existence of solution for the generalized nonlinear variational-like inequality (2.3) and discuss the convergence of the sequence generated by Algorithm 3.3.

Theorem 4.1. *Let K be a nonempty closed convex subset of a real Hilbert space $H, f \in H$, and $\eta : K \times K \to H$ Lipschitz continuous with constant δ. Assume that $a : K \times K \to \mathbb{R}$ is a coercive continuous bilinear form satisfying (a1) and (a2), $b : K \times K \to \mathbb{R}$ satisfies (b1)–(b4). Let $A, B, C, D, S : K \to H$ and $N : H \times H \to H$ be mappings such that B is Lipschitz continuous with constant s, N is η-strongly monotone with respect to A in the first argument with constant α, η-relaxed cocoercive with respect to B and Lipschitz continuous in the second argument with constants β and σ, respectively, S is η-relaxed Lipschitz with constant λ, and N and S are η-hemicontinuous with respect to A and B in K. Assume that Assumption 2.6 holds and*

$$\beta\sigma^2 s^2 + l < \alpha + \lambda + c. \tag{4.1}$$

Then the generalized nonlinear variational-like inequality (2.3) *possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.3 converges strongly to u.*

Proof. Note that (4.1) implies that (3.2) holds. It follows from Theorem 3.1 that there exists a mapping $F : K \to K$ such that for each $u \in K, F(u) = \widehat{w}$ is the unique solution of the auxiliary problem (3.1). Next we show that F is a contraction mapping in K. Let u_1 and u_2 be arbitrary elements in K and $\rho > 0$ a constant. Using (3.1), we deduce that

$$\begin{aligned}\langle Fu_1, v - Fu_1\rangle \geq \langle u_1, v - Fu_1\rangle - \rho\langle N(AFu_1, BFu_1) - SFu_1 - f, \eta(v, Fu_1)\rangle \\ - \rho a(Fu_1, v - Fu_1) - \rho b(u_1, v) + \rho b(u_1, Fu_1), \quad \forall v \in K,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\langle Fu_2, v - Fu_2\rangle \geq \langle u_2, v - Fu_2\rangle - \rho\langle N(AFu_2, BFu_2) - SFu_2 - f, \eta(v, Fu_2)\rangle \\ - \rho a(Fu_2, v - Fu_2) - \rho b(u_2, v) + \rho b(u_2, Fu_2), \quad \forall v \in K.\end{aligned} \tag{4.3}$$

Letting $v = Fu_2$ in (4.2) and $v = Fu_1$ in (4.3), and adding these inequalities, we arrive at

$$\begin{aligned}\|Fu_1 - Fu_2\|^2 &= \langle Fu_1 - Fu_2, Fu_1 - Fu_2\rangle \\ &\leq \langle u_1 - u_2, Fu_1 - Fu_2\rangle \\ &\quad - \rho\langle N(AFu_1, BFu_1) - SFu_1 - N(AFu_2, BFu_2) + SFu_2, \eta(Fu_1, Fu_2)\rangle \\ &\quad - \rho a(Fu_1 - Fu_2, Fu_1 - Fu_2) + \rho b(u_1 - u_2, Fu_2 - Fu_1) \\ &\leq \|u_1 - u_2\|\|Fu_1 - Fu_2\| \\ &\quad - \rho\langle N(AFu_1, BFu_1) - N(AFu_2, BFu_1), \eta(Fu_1, Fu_2)\rangle \\ &\quad - \rho\langle N(AFu_2, BFu_1) - N(AFu_2, BFu_2), \eta(Fu_1, Fu_2)\rangle \\ &\quad + \rho\langle SFu_1 - SFu_2, \eta(Fu_1, Fu_2)\rangle \\ &\quad - \rho a(Fu_1 - Fu_2, Fu_1 - Fu_2) + \rho b(u_1 - u_2, Fu_2 - Fu_1) \\ &\leq (1 + \rho l)\|u_1 - u_2\|\|Fu_1 - Fu_2\| + \rho\left(\beta\sigma^2 s^2 - \alpha - \lambda - c\right)\|Fu_1 - Fu_2\|^2,\end{aligned} \tag{4.4}$$

that is,

$$\|Fu_1 - Fu_2\| \leq \theta\|u_1 - u_2\|, \tag{4.5}$$

where

$$\theta = \frac{1 + \rho l}{1 - \rho(\beta\sigma^2 s^2 - \alpha - \lambda - c)} < 1 \tag{4.6}$$

by (4.1). Therefore, $F : K \to K$ is a contraction mapping. It follows from the Banach fixed-point theorem that F has a unique fixed point $u \in K$. In light of (3.1), we get that

$$\begin{aligned}\langle u, v - u\rangle \geq \langle u, v - u\rangle - \rho\langle N(Au, Bu) - Su - f, \eta(v, u)\rangle \\ - \rho a(u, v - u) - \rho b(u, v) + \rho b(u, u), \quad \forall v \in K,\end{aligned} \tag{4.7}$$

which implies that

$$\langle N(Au,Bu) - Su - f, \eta(v,u)\rangle + a(u,v-u) \geq b(u,u) - b(u,v), \quad \forall v \in K, \tag{4.8}$$

that is, $u \in K$ is a solution of the generalized nonlinear variational-like inequality (2.3).

Now we prove the uniqueness. Suppose that the generalized nonlinear variational-like inequality (2.3) has two solutions $\widehat{u}, u_0 \in K$. It follows that

$$\left\langle N(A\widehat{u},B\widehat{u}) - S\widehat{u} - f, \eta(v,\widehat{u})\right\rangle + a(\widehat{u},v-\widehat{u}) \geq b(\widehat{u},\widehat{u}) - b(\widehat{u},v), \tag{4.9}$$

$$\langle N(Au_0,Bu_0) - Su_0 - f, \eta(v,u_0)\rangle + a(u_0,v-u_0) \geq b(u_0,u_0) - b(u_0,v) \tag{4.10}$$

for all $v \in K$. Taking $v = u_0$ in (4.9) and $v = \widehat{u}$ in (4.10), we obtain that

$$\begin{aligned} &\langle N(A\widehat{u},B\widehat{u}) - S\widehat{u} - f, \eta(u_0,\widehat{u})\rangle + a(\widehat{u},u_0-\widehat{u}) \geq b(\widehat{u},\widehat{u}) - b(\widehat{u},u_0), \\ &\langle N(Au_0,Bu_0) - Su_0 - f, \eta(\widehat{u},u_0)\rangle + a(u_0,\widehat{u}-u_0) \geq b(u_0,u_0) - b(u_0,\widehat{u}). \end{aligned} \tag{4.11}$$

Adding (4.11), we deduce that

$$\begin{aligned} &\left(\alpha + \lambda + c - \beta\sigma^2 s^2 - l\right)\|\widehat{u} - u_0\|^2 \\ &\quad \leq \langle N(A\widehat{u},B\widehat{u}) - N(Au_0,Bu_0), \eta(\widehat{u},u_0)\rangle - \langle S\widehat{u} - Su_0, \eta(\widehat{u},u_0)\rangle \\ &\qquad + a(\widehat{u}-u_0,\widehat{u}-u_0) - b(\widehat{u}-u_0,\widehat{u}-u_0) \\ &\quad \leq 0, \end{aligned} \tag{4.12}$$

which together with (4.1) implies that $\widehat{u} = u_0$. That is, the generalized nonlinear variational-like inequality (2.3) has a unique solution in K.

Next we discuss the convergence of the iterative sequence generated by Algorithm 3.3. Taking $v = u_{n+1}$ in (4.7) and $v = u$ in (3.18), and adding these inequalities, we infer that

$$\begin{aligned} \|u_{n+1} - u\|^2 &\leq \langle u_n - u, u_{n+1} - u\rangle - \rho\langle N(Au_{n+1},Bu_{n+1}) - N(Au,Bu_{n+1}), \eta(u_{n+1},u)\rangle \\ &\quad - \rho\langle N(Au,Bu_{n+1}) - N(Au,Bu), \eta(u_{n+1},u)\rangle \\ &\quad + \rho\langle Su_{n+1} - Su, \eta(u_{n+1},u)\rangle - \rho a(u_{n+1}-u, u_{n+1}-u) \\ &\quad + \rho b(u_n - u, u - u_{n+1}) + \langle e_n, \eta(u_{n+1},u)\rangle \\ &\leq (1+\rho l)\|u_n - u\|\|u_{n+1} - u\| - \rho\left(\beta\sigma^2 s^2 - \alpha - \lambda - c\right)\|u_{n+1} - u\|^2 \\ &\quad + \delta\|e_n\|\|u_{n+1} - u\|, \quad \forall n \geq 0. \end{aligned} \tag{4.13}$$

That is,

$$\|u_{n+1} - u\| \leq \theta \|u_n - u\| + \frac{\delta}{1 - \rho(\beta\sigma^2 s^2 - \alpha - \lambda - c)} \|e_n\|, \quad \forall n \geq 0, \tag{4.14}$$

where θ is defined by (4.6). It follows from (3.19), (4.1), (4.14), and Lemma 2.5 that the iterative sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.3 converges strongly to u. This completes the proof. □

As in the proof of Theorem 4.1, we have the following theorem.

Theorem 4.2. *Let K be a nonempty closed convex subset of a real Hilbert space H, $f \in H$, and $\eta : K \times K \to H$ Lipschitz continuous with constant δ. Assume that $a : K \times K \to \mathbb{R}$ is a coercive continuous bilinear form satisfying (a1) and (a2), $b : K \times K \to \mathbb{R}$ satisfies (b1)–(b4). Let $A, B, C, D, S : K \to H$ and $N : H \times H \to H$ be mappings such that N is η-strongly monotone with respect to A in the first argument with constant α, η-monotone with respect to B in the second argument, S is Lipschitz continuous with constant ξ, and N and S are η-hemicontinuous with respect to A and B in K. Assume that Assumption 2.6 holds and*

$$\xi\delta + l < \alpha + c. \tag{4.15}$$

Then the generalized nonlinear variational-like inequality (2.3) *possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.3 converges strongly to u.*

Remark 4.3. The conditions of Theorems 3.1, 3.2, 4.1, and 4.2 are different from the conditions of the results in [1–10]. In particular, the mappings B and N with respect to the second argument in Theorems 3.2 and 4.2 are Lipschitz continuous, but other mappings in Theorems 3.2 and 4.2 are not Lipschitz continuous.

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