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Research Article

A New Singular Impulsive Delay Differential Inequality and Its Application

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A new singular impulsive delay differential inequality is established. Using this inequality, the invariant and attracting sets for impulsive neutral neural networks with delays are obtained. Our results can extend and improve earlier publications.

1. Introduction

It is well known that inequality technique is an important tool for investigating dynamical behavior of differential equation. The significance of differential and integral inequalities in the qualitative investigation of various classes of functional equations has been fully illustrated during the last 40 years [1–3]. Various inequalities have been established such as the delay integral inequality in [4], the differential inequalities in [5, 6], the impulsive differential inequalities in [7–10], Halanay inequalities in [11–13], and generalized Halanay inequalities in [14–17]. By using the technique of inequality, the invariant and attracting sets for differential systems have been studied by many authors [9, 18–21].

However, the inequalities mentioned above are ineffective for studying the invariant and attracting sets of impulsive nonautonomous neutral neural networks with time-varying delays. On the basis of this, this article is devoted to the discussion of this problem.

Motivated by the above discussions, in this paper, a new singular impulsive delay differential inequality is established. Applying this equality and using the methods in [10, 22], some sufficient conditions ensuring the invariant set and the global attracting set for a class of neutral neural networks system with impulsive effects are obtained.

2. Preliminaries

Throughout the paper, E_n means n-dimensional unit matrix, $\mathbb{R}$the set of real numbers, $\mathbb{N}$ the set of positive integers, and $\mathcal{N} \triangleq \{1,2,\ldots,n\}$. $A \geq B$ $(A > B)$ means that each pair of corresponding elements of A and B satisfies the inequality "$\geq$ $(>)$". Especially, A is called a nonnegative matrix if $A \geq 0$.

$C(X,Y)$ denotes the space of continuous mappings from the topological space X to the topological space Y. In particular, let $C \triangleq C([-\tau,0],\mathbb{R}^n)$, where $\tau > 0$ is a constant.

$PC([a,b],\mathbb{R}^n)$ denotes the space of piecewise continuous functions $\psi(s) : [a,b] \to \mathbb{R}^n$ with at most countable discontinuous points and at this points $\psi(s)$ are right continuous. Especially, let $PC \triangleq PC([-\tau,0],\mathbb{R}^n)$. Furthermore, put $PC([a,b),\mathbb{R}^n) = \bigcup_{c\in[a,b)} PC([a,c],\mathbb{R}^n)$.

$PC^1([a,b],\mathbb{R}^n) = \{\psi(s) : [a,b] \to \mathbb{R}^n \mid \psi(s), \dot{\psi}(s) \in PC([a,b],\mathbb{R}^n)\}$, where $\dot{\psi}(s)$ denotes the derivative of $\psi(s)$. In particular, let $PC^1 \triangleq PC^1([a,b],\mathbb{R}^n)$.

$\mathscr{H} = \{h(t) : \mathbb{R} \to \mathbb{R} \mid h(t)$ is a positive integrable function and satisfies $\sup_{a\leq t<b}\int_{t-\tau}^{t} h(s)ds = \sigma < \infty$ and $\lim_{t\to\infty}\int_{a}^{t} h(s)ds = \infty\}$.

For $x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$, $\varphi \in C$ or $\varphi \in PC$, we define $[x]^+ = (|x_1|,\ldots,|x_n|)^T$, $[A]^+ = (|a_{ij}|)_{n\times n}$, $[\varphi(t)]_\tau = ([\varphi_1(t)]_\tau,\ldots,[\varphi_n(t)]_\tau)^T$, $[\varphi(t)]_\tau^+ = [[\varphi(t)]^+]_\tau$, $[\varphi_i(t)]_\tau = \sup_{-\tau\leq\theta<0}\{\varphi_i(t+\theta)\}$. And we introduce the following norm, respectively,

$$\|x\| = \max_{1\leq i\leq n}|x_i|, \qquad \|A\| = \max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}|, \qquad \|\varphi\|_\tau = \sup_{-\tau\leq s\leq 0}\|\varphi(s)\|. \tag{2.1}$$

For any $\varphi \in PC^1$, we define the following norm:

$$\|\varphi\|_{1\tau} = \max\left\{\|\varphi\|_\tau, \|\varphi'\|_\tau\right\}. \tag{2.2}$$

For an M-matrix D defined in [23], we denote

$$\Omega_M(D) \triangleq \left\{z \in \mathbb{R}^n \mid Dz > 0,\ z > 0\right\}. \tag{2.3}$$

It is a cone without conical surface in $\mathbb{R}^n$. We call it an "M-cone".

3. Singular Impulsive Delay Differential Inequality

For convenience, we introduce the following conditions.

(C_1) Let the r-dimensional diagonal matrix $K = \mathrm{diag}\{k_1,\ldots,k_r\}$ satisfy

$$k_i > 0, \quad i \in S \subset \mathcal{N}^* \triangleq \{1,\ldots,r\}, \qquad k_i = 0, \quad i \in S^* \triangleq \mathcal{N}^* - S. \tag{3.1}$$

(C_2) Let $U = -(P+Q)$ be an M-matrix, where $Q = (q_{ij})_{r\times r} \geq 0$ and $P = (p_{ij})_{r\times r}$ satisfies

$$p_{ij} \geq 0, \quad i \neq j, \qquad p_{ij} = 0, \quad i \neq j,\ i \in \mathcal{N}^*,\ j \in S^*. \tag{3.2}$$

Theorem 3.1. *Assume the conditions* (C_1) *and* (C_2) *hold. Let* $L = (L_1, \dots, L_r)$ *and* $u(t) = (u_1(t), \dots, u_r(t))^T$ *be a solution of the following singular delay differential inequality with the initial conditions* $u(t) \in PC([a-\tau, a], \mathbb{R}^r)$*:*

$$KD^{+}u(t) \leq h(t)\left[Pu(t) + Q\left[u(t)\right]_{\tau} + L\right], \quad t \in [a,b), \tag{3.3}$$

where $\tau > 0$, $a < b \leq +\infty$, *and* $u_i(t) \in C([a,b),\mathbb{R})$, $i \in S$, $u_i(t) \in PC([a,b),\mathbb{R})$, $i \in S^*$, $h(t) \in \mathscr{H}$. *Then*

$$u(t) \leq dze^{-\lambda\int_a^t h(s)\mathrm{d}s} - (P+Q)^{-1}L, \quad t \in [a,\, b), \tag{3.4}$$

provided that the initial conditions satisfy

$$u(t) \leq dze^{-\lambda\int_a^t h(s)\mathrm{d}s} - (P+Q)^{-1}L, \quad t \in [a-\tau, a], \tag{3.5}$$

where $d \geq 0$, $z = (z_1, \dots, z_r)^T \in \Omega_M(U)$ *and the positive number* λ *satisfies the following inequality:*

$$\left[\lambda K + P + Qe^{\lambda\sigma}\right]z < 0, \quad t \in [a,b). \tag{3.6}$$

Proof. By the conditions (C_2) and the definition of M-matrix, there is a constant vector $z = (z_1, \dots, z_r)^T$ such that $(P+Q)z < 0$, $-(P+Q)^{-1}$ exists and $-(P+Q)^{-1} \geq 0$.

By using continuity, we obtain that there must exist a positive constant λ satisfying the inequality (3.6), that is,

$$\sum_{j=1}^{r}\left[p_{ij} + q_{ij}e^{\lambda\sigma}\right]z_j < -\lambda k_i z_i, \quad i \in \mathcal{N}^*. \tag{3.7}$$

Denote by

$$v(t) = \left(v_1(t), \dots, v_r(t)\right)^T = u(t) + (P+Q)^{-1}L, \quad t \in [a-\tau,\, b). \tag{3.8}$$

It follows from (3.3) and (3.5) that

$$\begin{aligned} KD^{+}v(t) &\leq h(t)\left[Pu(t) + Q\left[u(t)\right]_{\tau} + L\right] \\ &\leq h(t)\left[Pv(t) + Q\left[v(t)\right]_{\tau}\right], \quad t \in [a,b), \\ v(t) &\leq dze^{-\lambda\int_a^t h(s)\mathrm{d}s}, \quad t \in [a-\tau,\, a]. \end{aligned} \tag{3.9}$$

In the following, we will prove that for any positive constant ε,

$$v_i(t) \leq (d+\varepsilon)z_i e^{-\lambda\int_a^t h(s)\mathrm{d}s} \triangleq \omega_i(t), \quad t \in [a,\, b),\ i \in \mathcal{N}^*. \tag{3.10}$$

Let

$$\wp = \{i \in \mathcal{N}^* \mid v_i(t) > w_i(t) \text{ for some } t \in [a,b)\},$$
$$\theta_i = \inf\{t \in [a,b) \mid v_i(t) > w_i(t),\ i \in \wp\}. \tag{3.11}$$

If inequality (3.10) is not true, then $\wp$ is a nonempty set and there must exist some integer $m \in \wp$ such that $\theta_m = \min_{i\in\wp}\{\theta_i\} \in [a,b)$.

If $m \in S$, by $v_m(t) \in C([a,b),\mathbb{R})$ and the inequality (3.5), we can get

$$\theta_m > a, \qquad v_m(\theta_m) = w_m(\theta_m), \qquad D^+ v_m(\theta_m) \geq \dot{w}_m(\theta_m), \tag{3.12}$$

$$v_i(t) \leq w_i(t), \quad t \in [a-\tau,\ \theta_m),\ i \in \mathcal{N}^*, \qquad v_i(\theta_m) \leq w_i(\theta_m), \quad i \in S. \tag{3.13}$$

By using (C_2), (3.3), (3.7), (3.12), (3.13), and $[v_i(t)]_\tau = \sup_{-\tau\leq\theta<0}\{u_i(t+\theta)\}$, $i \in \mathcal{N}^*$, we obtain that

$$\begin{aligned}
k_m D^+ v_m(\theta_m) &\leq h(\theta_m)\sum_{j=1}^{r}\left[p_{mj}v_j(\theta_m) + q_{mj}\left[v_j(\theta_m)\right]_\tau\right]\\
&= h(\theta_m)\Bigg[p_{mm}v_m(\theta_m) + \sum_{j\neq m,\, j\in S} p_{mj}v_j(\theta_m)\\
&\qquad + \sum_{j\in S^*} p_{mj}v_j(\theta_m) + \sum_{j=1} q_{mj}\left[v_j(\theta_m)\right]_\tau\Bigg]\\
&\leq h(\theta_m)\left[\sum_{j\in S} p_{mj}(d+\varepsilon)z_j e^{-\lambda\int_a^{\theta_m} h(s)\mathrm{d}s} + \sum_{j=1}^{r} q_{mj}(d+\varepsilon)z_j e^{-\lambda\int_a^{\theta_m-\tau} h(s)\mathrm{d}s}\right]\\
&\leq (d+\varepsilon)h(\theta_m)\sum_{j=1}^{r}\left[p_{mj} + q_{mj}e^{\lambda\sigma}\right]z_j e^{-\lambda\int_a^{\theta_m} h(s)\mathrm{d}s}\\
&< -(d+\varepsilon)\lambda k_m z_m h(\theta_m) e^{-\lambda\int_a^{\theta_m} h(s)\mathrm{d}s}.
\end{aligned} \tag{3.14}$$

Since $m \in S$, we have $k_m > 0$ by (H_1). Then (3.14) becomes

$$D^+ u_m(\theta_m) < -(d+\varepsilon)\lambda z_m h(\theta_m) e^{-\lambda\int_a^{\theta_m} r(s)\mathrm{d}s} = \dot{w}_m(\theta_m), \tag{3.15}$$

which contradicts the second inequality in (3.12).

If $m \in S^*$, then $k_m = 0$ by (C_1) and $v_m(t) \in PC([a,b),\mathbb{R})$. From the inequality (3.5), we can get

$$\theta_m > a, \qquad v_m(\theta_m) \geq w_m(\theta_m), \qquad v_i(\theta_m) \leq w_i(\theta_m), \quad i \in S,$$
$$v_i(t) \leq w_i(t), \quad t \in [a-\tau,\ \theta_m),\ i \in \mathcal{N}^*. \tag{3.16}$$

By using (C_2), (3.3), (3.7), (3.16), and $[v_i(t)]_\tau = \sup_{-\tau\le\theta<0}\{v_i(t+\theta)\}$, $i \in \mathcal{N}^*$, we obtain that

$$\begin{aligned}
0 &\le \sum_{j=1}^{r} p_{mj} v_j(\theta_m) + \sum_{j=1} q_{mj}\left[v_j(\theta_m)\right]_\tau \\
&= \sum_{j\in S} p_{mj} v_j(\theta_m) + p_{mm} v_m(\theta_m) + \sum_{j\ne m,\, j\in S^*} p_{mj} v_j(\theta_m) + \sum_{j=1}^{r} q_{mj}\left[v_j(\theta_m)\right]_\tau \\
&\le (d+\varepsilon)\left[\sum_{j\in S} p_{mj} z_j + p_{mm} z_m + \sum_{j=1} q_{mj} z_j e^{\lambda\int_{\theta_m-\tau}^{\theta_m} h(s)\mathrm{d}s}\right] e^{-\lambda\int_a^{\theta_m} h(s)\mathrm{d}s} \\
&\le (d+\varepsilon)\sum_{j=1}^{r}\left[p_{mj} z_j + q_{mj} z_j e^{\lambda\sigma}\right] e^{-\lambda\int_a^{\theta_m} h(s)\mathrm{d}s} \\
&< -(d+\varepsilon) k_m z_m h(\theta_m) e^{-\lambda\int_a^{\theta_m} h(s)\mathrm{d}s} \\
&= 0.
\end{aligned} \tag{3.17}$$

This is a contradiction. Thus the inequality (3.10) holds. Therefore, letting $\varepsilon \to 0$ in (3.10), we have

$$v(t) = u(t) + (P+Q)^{-1} L \le dz e^{-\lambda\int_a^t h(s)\mathrm{d}s}, \quad t \in [a,b). \tag{3.18}$$

The proof is complete. □

Remark 3.2. In order to overcome the difficulty that $u(t)$ in (3.3) may be discontinuous, we introduce the notation $[u_i(t)]_\tau = \sup_{-\tau\le s<0}\{u_i(t+s)\}$ which is different from the notation $[u_i(t)]_\tau = \sup_{-\tau\le s\le 0}\{u_i(t+s)\}$ in [7]. However, when $u_i(t)$ is continuous in t, we have

$$\left[u_i(t)\right]_\tau = \sup_{-\tau\le s<0}\left\{u_i(t+s)\right\} = \sup_{-\tau\le s\le 0}\left\{u_i(t+s)\right\}, \quad i \in \mathcal{N}^*. \tag{3.19}$$

So we can get [7, Lemma 1] when we choose $K = E_r$, $S = \mathcal{N}^*$, $h(t) \equiv 1$ in Theorem 3.1.

Remark 3.3. Suppose that $L = (L_1, \ldots, L_r)^T = 0$ and $h(t) \equiv 1$ in Theorem 3.1, then we can get [10, Theorem 3.1].

4. Applications

The singular impulsive delay differential inequality obtained in Section 3 can be widely applied to study the dynamics of impulsive neutral differential equations. To illustrate the theory, we consider the following nonautonomous impulsive neutral neural networks with delays

$$\begin{aligned}
\dot{x}(t) &= -D(t)x(t) + A(t)F(x(t)) + B(t)G(x(t-\tau(t))) \\
&\qquad + C(t)H(\dot{x}(t-r(t))) + J(t), \quad t \ne t_k, \\
x(t) &= I_k(t, x(t^-)), \quad t = t_k,
\end{aligned} \tag{4.1}$$

where $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ is the neural state vector; $D(t) = \operatorname{diag}\{d_1(t), \dots, d_n(t)\} > 0$, $A(t) = (a_{ij}(t))_{n\times n}$, $B(t) = (b_{ij}(t))_{n\times n}$, $C(t) = (c_{ij}(t))_{n\times n}$ are the interconnection matrices representing the weighting coefficients of the neurons; $F(x) = (f_1(x_1), \dots, f_n(x_n))^T$, $G(x) = (g_1(x_1), \dots, g_n(x_n))^T$, $H(x) = (h_1(x_1), \dots, h_n(x_n))^T$ are activation functions; $\tau(t) = (\tau_{ij}(t))_{n\times n}$, $r(t) = (r_{ij}(t))_{n\times n}$ are transmission delays; $J(t) = (J_1(t), \dots, J_n(t))^T$ denotes the external inputs at time t. $I_k(t, y) = (I_{1k}(t, y), \dots, I_{nk}(t, y))^T$ represents impulsive perturbations; the fixed moments of time t_k satisfy $t_k < t_{k+1}$, $\lim_{k\to+\infty} t_k = +\infty$, $k \in \mathbb{N}$.

The initial condition for (4.1) is given by

$$x(t_0 + s) = \varphi(s) \in PC^1, \quad t_0 \in \mathbb{R}, \ -\tau \le s \le 0. \tag{4.2}$$

We always assume that for any $\varphi \in PC^1$, (4.1) has at least one solution through (t_0, φ), denoted by $x(t, t_0, \varphi)$ or $x_t(t_0, \varphi)$ (simply $x(t)$ or x_t if no confusion should occur).

Definition 4.1. The set $S \subset PC^1$ is called a positive invariant set of (4.1), if for any initial value $\varphi \in S$, we have the solution $x_t(t_0, \varphi) \in S$ for $t \ge t_0$.

Definition 4.2. The set $S \subset PC^1$ is called a global attracting set of (4.1), if for any initial value $\varphi \in PC^1$, the solution $x_t(t_0, \varphi)$ converges to S as $t \to +\infty$. That is,

$$\operatorname{dist}(x_t, S) \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty, \tag{4.3}$$

where $\operatorname{dist}(\phi, S) = \inf_{\psi\in S} \operatorname{dist}(\phi, \psi)$, $\operatorname{dist}(\phi, \psi) = \sup_{s\in[-\tau,0]} |\phi(s) - \psi(s)|$, for $\phi \in PC^1$.

Throughout this section, we suppose the following.

(H_1) $D(t) \in PC(\mathbb{R}, \mathbb{R}^n)$, $A(t), B(t), C(t), \tau(t), r(t)$ are continuous. Moreover, $0 \le \tau_{ij}(t) \le \tau$ and $0 < r_{ij}(t) \le \tau$ $(i, j \in \mathcal{N})$.

(H_2) There exist nonnegative matrices $\widetilde{D}_1 = \operatorname{diag}\{\widehat{d}_{11}, \dots, \widehat{d}_{1n}\}$, $\widetilde{D}_2 = \operatorname{diag}\{\widehat{d}_{21}, \dots, \widehat{d}_{2n}\}$, $\widehat{J} = (\widehat{J}_1, \dots, \widehat{J}_n)^T$, $h(s) \in \mathscr{H}$ and a constant $\delta > 0$ such that

$$\widetilde{D}_1 h(t) \le D(t) \le \widetilde{D}_2 h(t), \qquad 0 < h(t) \le \frac{1}{\delta}, \qquad [J(t)]^+ \le \widehat{J} h(t). \tag{4.4}$$

(H_3) There exist nonnegative matrices $\widetilde{A} = (\widehat{a}_{ij})_{n\times n}$, $\widetilde{B} = (\widehat{b}_{ij})_{n\times n}$, $\widetilde{C} = (\widehat{c}_{ij})_{n\times n}$ such that

$$[A(t)]^+ \le \widetilde{A} h(t), \qquad [B(t)]^+ \le \widetilde{B} h(t), \qquad [C(t)]^+ \le \widetilde{C} h(t). \tag{4.5}$$

(H_4) There exist nonnegative matrices $\widetilde{F} = \operatorname{diag}\{\alpha_1, \dots, \alpha_n\}$, $\widetilde{G} = \operatorname{diag}\{\beta_1, \dots, \beta_n\}$, $\widetilde{H} = \operatorname{diag}\{\gamma_1, \dots, \gamma_n\}$ such that for all $u \in \mathbb{R}^n$ the activating functions $F(\cdot), G(\cdot)$ and $H(\cdot)$ satisfy

$$[F(u)]^+ \le \widetilde{F}[u]^+, \qquad [G(u)]^+ \le \widetilde{G}[u]^+, \qquad [H(u)]^+ \le \widetilde{H}[u]^+. \tag{4.6}$$

(H_5) There exists nonnegative matrix $\widehat{I}_k = (\widehat{I}_{ij}^k)_{n\times n}$, such that for all $u \in \mathbb{R}^n$, $i \in \mathcal{N}$ and $k \in \mathbb{N}$

$$[I_k(t, u)]^+ \le \widehat{I}_k [u]^+. \tag{4.7}$$

(H_6) Denote by

$$\begin{gathered}
\widetilde{U} = \widetilde{A}\widetilde{F}, \qquad \widetilde{V} = \widetilde{B}\widetilde{G}, \qquad \widetilde{W} = \widetilde{C}\widetilde{H}, \\
\overline{K} = \begin{pmatrix} E_n & 0 \\ 0 & 0 \end{pmatrix} \triangleq \operatorname{diag}\{\widehat{k}_1, \dots, \widehat{k}_{2n}\}, \qquad \overline{L} = (\widehat{J}, \widehat{J})^T, \\
\overline{P} = \begin{pmatrix} -\widetilde{D}_1 + \widetilde{U} & 0 \\ \widetilde{D}_2 + \widetilde{U} & -\delta E_n \end{pmatrix} \triangleq (\widehat{p}_{ij}(t))_{2n\times 2n}, \qquad \overline{Q} = \begin{pmatrix} \widetilde{V} & \widetilde{W} \\ \widetilde{V} & \widetilde{W} \end{pmatrix} \triangleq (\widehat{q}_{ij}(t))_{2n\times 2n},
\end{gathered} \tag{4.8}$$

and let $\overline{D} = -(\overline{P} + \overline{Q})$ be an M-matrix, and $\varrho = -(\overline{P} + \overline{Q})^{-1}\overline{L} = (\varrho_1, \varrho_2)^T \ge 0$, $\varrho_1, \varrho_2 \in \mathbb{R}^n$.

(H_7) There exists a constant ν such that

$$\ln \eta_k \le \nu \int_{t_{k-1}}^{t_k} h(s)\mathrm{d}s, \quad k \in \mathbb{N}, \qquad \mu = \sum_{k=1}^{\infty} \ln \mu_k < \infty, \tag{4.9}$$

where $\nu < \lambda$, and the scalar $\lambda > 0$ is determined by the inequality

$$[\lambda\overline{K} + \overline{P} + \overline{Q}e^{\lambda\sigma}]z^* < 0, \tag{4.10}$$

where $z^* = (z_1, \dots, z_{2n})^T \in \Omega_M(\overline{D})$, and

$$\eta_k, \mu_k \ge 1, \qquad \eta_k z_x^* \ge \widehat{I}_k z_x^*, \qquad \mu_k \varrho_1 \ge \widehat{I}_k \varrho_1, \quad k \in \mathbb{N},\ z_x^* = (z_1, \dots, z_n)^T. \tag{4.11}$$

Theorem 4.3. *Assume that* (H_1)–(H_7) *hold. Then* $S = \{\phi \in PC^1 \mid [\phi]_\tau^+ \le e^{\mu}\varrho_1\}$ *is a global attracting set of* (4.1).

Proof. Denote $\dot{x}(t) = y(t)$. Let $\operatorname{sgn}(\cdot)$ be the sign function. For $x = (x_1, \dots, x_n)^T$, define $\operatorname{Sgn}(x) = \operatorname{diag}\{\operatorname{sgn}(x_1), \dots, \operatorname{sgn}(x_n)\}$.

Calculating the upper right derivative $D^+[x(t)]^+$ along system (4.1). From (4.1), (H_2) and (H_3) we have

$$\begin{aligned}
D^+[x(t)]^+ \le h(t)\{&(-D_1 + \widetilde{U})[x(t)]^+ + \widetilde{V}G[x(t)]_\tau^+ \\
&+ \widetilde{W}[y(t)]_\tau^+ + \widehat{J}\}, \quad t \in [t_{k-1}, t_k),\ t \ge \sigma,\ k \in \mathbb{N}.
\end{aligned} \tag{4.12}$$

On the other hand, from (4.1) and (H_2)–(H_4), we have

$$\begin{aligned}0 &\le h(t)\left\{-\frac{1}{h(t)}[y(t)]^+ + (\widetilde{D}_2+\widetilde{U})[x(t)]^+ + \widetilde{V}[x(t)]_\tau^+ + \widetilde{W}[y(t)]_\tau^+ + \widehat{J}\right\} \\ &\le h(t)\{-\delta[y(t)]^+ + (\widetilde{D}_2+\widetilde{U})[x(t)]^+ + \widetilde{V}[x(t)]_\tau^+ + \widetilde{W}[y(t)]_\tau^+ + \widehat{J}\}, \quad t\in[t_{k-1},t_k),\ k\in\mathbb{N}.\end{aligned} \tag{4.13}$$

Let

$$u(t) = (x(t),y(t))^T \in \mathbb{R}^{2n}, \tag{4.14}$$

then from (4.12)–(4.14) and (H_6), we have

$$\overline{K}D^+[u(t)]^+ \le h(t)[\overline{P}[u(t)]^+ + \overline{Q}[u(t)]_\tau^+ + \overline{L}], \quad t\in[t_{k-1},t_k),\ k\in\mathbb{N}. \tag{4.15}$$

By the conditions (H_6) and the definition of M-matrix, we may choose a vector $z^* = (z_1,\dots,z_{2n})^T \in \Omega_M(\overline{D})$ such that

$$\overline{D}z^* = -(\overline{P}+\overline{Q})z^* > 0. \tag{4.16}$$

By using continuity, we obtain that there must be a positive constant λ satisfying the inequality (4.10). Let $z_x^* = (z_1,\dots,z_n)^T$ and $z_y^* = (z_{n+1},\dots,z_{2n})^T$, then $z^* = (z_x^*,z_y^*)^T$. Since $z^*>0$, denote

$$d = \frac{1}{\min_{1\le i\le 2n}\{z_i\}}, \tag{4.17}$$

then $dz^* \ge e_{2n} = (1,\dots,1)^T \in \mathbb{R}^{2n}$. From the property of M-cone, we have, $dz^* \in \Omega_M(\overline{D})$.

For the initial conditions $x(t_0+s)=\varphi(s)$, $s\in[-\tau,0]$, where $\varphi\in PC^1$ and $t_0\in\mathbb{R}$ (no loss of generality, we assume $t_0\le t_1$), and $t\in[t_0-\tau,t_0]$, we can get

$$\begin{aligned}[x(t)]^+ &\le \|\varphi(t)\|_\tau e^{-\lambda\int_{t_0}^t h(s)\mathrm{d}s} dz_x^*, \\ [y(t)]^+ &\le \|\varphi'(t)\|_\tau e^{-\lambda\int_{t_0}^t h(s)\mathrm{d}s} dz_y^*,\end{aligned} \tag{4.18}$$

Then (4.18) yield

$$[u(t)]^+ \le dz^*\|\varphi\|_{1\tau} e^{-\lambda\int_{t_0}^t h(s)\mathrm{d}s} + \varrho, \quad t_0-\tau\le t\le t_0. \tag{4.19}$$

Let $\mathcal{N}^* = \{1,\dots,2n\}$, $S=\{1,\dots,n\}=\mathcal{N}$ and $S^*=\{n+1,\dots,2n\}=\mathcal{N}^*-S$. Thus, all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, we have

$$[u(t)]^+ \le dz^*\|\varphi\|_{1\tau} e^{-\lambda\int_{t_0}^t h(s)\mathrm{d}s} + \varrho, \quad t_0\le t< t_1. \tag{4.20}$$

Suppose that for all $m = 1, \dots, k$, the inequalities

$$[u(t)]^+ \leq \prod_{j=0}^{m-1} \eta_j d z^* \|\varphi\|_{1\tau} e^{-\lambda \int_{t_0}^{t} h(s)\mathrm{d}s} + \prod_{j=0}^{m-1} \mu_j \varrho, \quad t_{m-1} \leq t < t_m,\ t \geq t_0, \tag{4.21}$$

hold, where $\eta_0 = \mu_0 = 1$.

From (4.21), (H_5), and (H_7), we can get

$$\begin{aligned}[x(t_k)]^+ &\leq \widehat{I}_k [x(t_k^-)]^+ \\ &\leq \prod_{j=0}^{k} \eta_j\, d z_x^* \|\varphi(t)\|_{1\tau} e^{-\int_{t_0}^{t_k} h(s)\mathrm{d}s} + \prod_{j=0}^{k} \mu_j \varrho_1.\end{aligned} \tag{4.22}$$

Since $\varrho = -(\overline{P} + \overline{Q})^{-1} \overline{L}$, we have

$$(\widetilde{D}_2 + \widetilde{U} + \widetilde{V}) \varrho_1 + \widetilde{W} \varrho_2 + \widehat{J} = \delta \varrho_2. \tag{4.23}$$

On the other hand, it follows from (H_7) that

$$(\widetilde{D}_2 + \widetilde{U}) z_x^* + (\widetilde{V} z_x^* + \widetilde{W} z_y^*) e^{\lambda \sigma} < \delta z_y^*. \tag{4.24}$$

Then from (4.21)–(4.24), we have

$$[y(t_k)]^+ \leq \prod_{j=0}^{k} \eta_j d z_y^* \|\varphi\|_{1\tau} e^{-\lambda \int_{t_0}^{t_k} h(s)\mathrm{d}s} + \prod_{j=0}^{k} \mu_j \varrho_2, \tag{4.25}$$

which together with (4.22) yields that

$$[u(t_k)]^+ \leq \prod_{j=0}^{k} \eta_j d z^* \|\varphi\|_{1\tau} e^{-\lambda \int_{t_0}^{t_k} h(s)\mathrm{d}s} + \prod_{j=0}^{k} \mu_j \varrho. \tag{4.26}$$

Then, it follows from (4.21) and (4.26) that

$$\begin{aligned}[u(t)]^+ &\leq \prod_{j=0}^{k} \eta_j d z^* \|\varphi\|_{1\tau} e^{-\lambda \int_{t_0}^{t} h(s)\mathrm{d}s} + \prod_{j=0}^{k} \mu_j \varrho \\ &= \prod_{j=0}^{k} \eta_j d z^* \|\varphi\|_{1\tau} e^{-\lambda \int_{t_0}^{t_k} h(s)\mathrm{d}s} e^{-\lambda \int_{t_k}^{t} h(s)\mathrm{d}s} + \prod_{j=0}^{k} \mu_j \varrho, \quad \forall\, t \in [t_k - \tau, t_k].\end{aligned} \tag{4.27}$$

Using Theorem 3.1 again, we have

$$\begin{aligned}[u(t)]^{+} &\le \prod_{j=0}^{k}\eta_j dz^{*}\|\varphi\|_{1\tau}e^{-\lambda\int_{t_0}^{t_k}h(s)\mathrm{d}s}e^{-\lambda\int_{t_k}^{t}h(s)\mathrm{d}s}+\prod_{j=0}^{k}\mu_j\varrho \\ &= \prod_{j=0}^{k}\eta_j dz^{*}\|\varphi\|_{1\tau}e^{-\lambda\int_{t_0}^{t}h(s)\mathrm{d}s}+\prod_{j=0}^{k}\mu_j\varrho, \quad t_k\le t<t_{k+1}.\end{aligned}\tag{4.28}$$

By mathematical induction, we can conclude that

$$[u(t)]^{+} \le \prod_{j=0}^{k}\eta_j dz^{*}\|\varphi\|_{1\tau}e^{-\lambda\int_{t_0}^{t}h(s)\mathrm{d}s}+\prod_{j=0}^{k}\mu_j\varrho, \quad t_k\le t<t_{k+1},\ k\in\mathbb{N}.\tag{4.29}$$

Noticing that $\eta_k \le e^{\nu\int_{t_{k-1}}^{t_k}h(s)\mathrm{d}s}$, by (H_7), we can use (4.29) to conclude that

$$\begin{aligned}[u(t)]^{+} &\le dz^{*}\|\varphi\|_{1\tau}e^{\nu\int_{t_0}^{t}h(s)\mathrm{d}s}e^{-\lambda\int_{t_0}^{t}h(s)\mathrm{d}s}+e^{\mu}\varrho \\ &= dz^{*}\|\varphi\|_{1\tau}e^{-(\lambda-\nu)\int_{t_0}^{t}h(s)\mathrm{d}s}+e^{\mu}\varrho, \quad t_{k-1}\le t<t_k,\ k\in\mathbb{N}.\end{aligned}\tag{4.30}$$

This implies that the conclusion of the theorem holds. □

By using Theorem 4.3 with $d=0$, we can obtain a positive invariant set of (4.1), and the proof is similar to that of Theorem 4.3.

Theorem 4.4. *Assume that* (H_1)–(H_7) *with* $\widehat{I}_k = E_n$ *hold. Then* $S=\{\phi\in PC^1 \mid [\phi]_\tau^{+}\le \varrho_1\}$ *is a positive invariant set and also a global attracting set of* (4.1).

Remark 4.5. Suppose that $\widehat{c}_{ij}\equiv 0$, $i,j\in\mathcal{N}$ in (H_5), and $h(t)\equiv 1$, then we can get Theorems 1 and 2 in [9].

Remark 4.6. If $I_k(t,x(t^-)) = x\in\mathbb{R}^n$ then (4.1) becomes the nonautonomous neutral neural networks without impulses, we can get Theorem 4.1 in [22].

5. Illustrative Example

The following illustrative example will demonstrate the effectiveness of our results.

Example 5.1. Consider nonlinear impulsive neutral neural networks:

$$\begin{aligned}\dot{x}_1(t) &= -(7+\cos^2 t)x_1(t)+\sin t\tan(x_1(t-\tau_{11}(t)))-\frac{1}{4}\cos t\left|\dot{x}_2(t-r_{12}(t))\right|-1.5\cos t, \\ \dot{x}_2(t) &= -(6+\sin^2 t)x_2(t)-2\cos t\left|x_2(t-\tau_{22}(t))\right|+\frac{1}{4}\sin t\tan(\dot{x}_1(t-r_{21}(t)))+2.5\sin t,\end{aligned}\quad t\ne k,\tag{5.1}$$

with

$$x_1(t) = I_1(t, x(t^-)), \quad x_2(t) = I_2(t, x(t^-)), \qquad t = k, \; k \in \mathbb{N}, \tag{5.2}$$

where $\tau_{ij}(t) = (1/4)|\cos((i+j)t)| \le 1/4 \stackrel{\Delta}{=} \tau$, $r_{ij}(t) = 1/4 - (1/8)|\sin((i+j)t)|$, $i, j = 1, 2$, $I_k(t,x) = (a_{1k}(t)x_1 + b_{1k}(t)x_2), a_{2k}(t)x_1 + b_{2k}(t)x_2)^T$, $k \in \mathbb{N}$.

The parameters of conditions (H_3)–(H_9) are as follows:

$$h(t) = \delta = 1, \qquad \widetilde{D}_1 = \mathrm{diag}\{7,6\}, \qquad \widetilde{D}_2 = \mathrm{diag}\{8,7\}, \qquad \sigma = \frac{1}{4}, \qquad \widehat{J} = (1.5, 2.5)^T,$$

$$\widetilde{U} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \widetilde{V} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \widetilde{W} = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\overline{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \overline{P} = \begin{pmatrix} -\widetilde{D}_1 + \widetilde{U} & 0 \\ \widetilde{D}_2 + \widetilde{U} & -\delta E_2 \end{pmatrix} = \begin{pmatrix} -7 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 8 & 0 & -1 & 0 \\ 0 & 7 & 0 & -1 \end{pmatrix},$$

$$\overline{Q} = \begin{pmatrix} \widetilde{V} & \widetilde{W} \\ \widetilde{V} & \widetilde{W} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 2 & \frac{1}{4} & 0 \\ 1 & 0 & 0 & \frac{1}{4} \\ 0 & 2 & \frac{1}{4} & 0 \end{pmatrix}, \tag{5.3}$$

$$\overline{D} = -(\overline{P} + \overline{Q}) = \begin{pmatrix} 6 & 0 & 0 & -\frac{1}{4} \\ 0 & 4 & -\frac{1}{4} & 0 \\ -9 & 0 & 1 & -\frac{1}{4} \\ 0 & -9 & -\frac{1}{4} & 1 \end{pmatrix}.$$

It is easy to prove that $\overline{D}$ is an M-matrix and

$$\Omega_M(\overline{D}) = \left\{ (z_1, z_2, z_3, z_4)^T > 0 \mid 9z_2 + \frac{1}{4}z_3 < z_4 < 24z_1,\; 9z_1 + \frac{1}{4}z_4 < z_3 < 16z_2 \right\}. \tag{5.4}$$

Let $z^* = (1, 1, 15, 20)^T$, then $z^* \in \Omega_M(\overline{D})$ and $z_x^* = (1,1)^T$. Let $\lambda = 0.1$ which satisfies the inequality

$$\left[\lambda \overline{K} + \overline{P} + \overline{Q} e^{\lambda\sigma}\right] z^* < 0. \tag{5.5}$$

Now, we discuss the asymptotical behavior of the system (5.1) as follows.

(i) If $a_{1k}(t) = b_{2k}(t) = 0$, $b_{1k}(t) = (1/2)e^{1/5^{2k}}(1+\sin t)$, $a_{2k}(t) = (1/2)e^{1/5^{2k}}(1-\cos t)$, then

$$\widehat{I}_k = e^{1/5^{2k.}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.6}$$

Thus $\eta_k = \mu_k = e^{1/5^{2k}} \geq 1$, $\ln \eta_k = e^{1/5^{2k}} \leq 0.04$, $\nu = 0.04 < \lambda$, and $\mu = 1/24$. Clearly, all conditions of Theorem 4.3 are satisfied, by Theorem 4.3, $S = \{\phi \in PC^1 \mid [\phi]_\tau^+ \leq e^{1/24}\varrho_1 \approx e^{1/24}(1.196, 1.746)^T\}$ is a global attracting set of (5.1).

(ii) If $a_{1k}(t) = \cos t$, $b_{2k}(t) = \sin t$, $b_{1k}(t) = a_{2k}(t) = 0$, then $\widehat{I}_k = E_2$. By Theorem 4.4, $S = \{\phi \in PC^1 \mid [\phi]_\tau^+ \leq \varrho_1 \approx (1.196, 1.746)^T\}$ is a positive invariant set of (5.1).

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