Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 419845, 24 pages
doi:10.1155/2009/419845

Research Article

Sharp Hardy-Sobolev Inequalities with General Weights and Remainder Terms

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Received 18 December 2008; Accepted 10 August 2009

Recommended by Panayiotis Siafarikas

We consider a class of sharp Hardy-Sobolev inequality, where the weights are functions of the distance from a surface. It is proved that the Hardy-Sobolev inequality can be successively improved by adding to the right-hand side a lower-order term with optimal weight and constant.

1. Introduction

The classical Hardy inequality says

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}x \leq \left| \frac{p}{N-p} \right|^p \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x, \quad u \in C_0^{\infty}\left(\mathbb{R}^N \setminus \{0\}\right), \tag{1.1}$$

where the constant $|p/(N-p)|^p$ is optimal but never attained; see, for example, [1–4]. This suggests that one might look for an error term. Brezis and Vázques [5] showed that if Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with $0 \in \Omega$, then there exists a positive constant λ_Ω such that

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \geq \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} \mathrm{d}x + \lambda_\Omega \int_\Omega u^2 \mathrm{d}x, \quad u \in H_0^1(\Omega). \tag{1.2}$$

This result was extended to the L^p setting by Gazzola et al. [6]. Adimurthi et al. proved that Hardy's inequality can be successively improved by adding lower order terms; see [7, 8] for details. Abdellaoui et al. [9] obtained Hardy's inequality with the type of weight $|x|^{-p\gamma}$. See [10, 11] for the case of general weight $\phi(|x|)$.

Another type of Hardy's inequality contains weight involving the distant to the boundary of the domain. For a convex domain $\Omega \subset \mathbb{R}^N$ with smooth boundary the Hardy inequality

$$\int_\Omega \frac{|u|^p}{d^p}\mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla u|^p \mathrm{d}x, \quad u \in C_0^\infty(\Omega) \tag{1.3}$$

is valid with $(p/(p-1))^p$ being the best constant, where d is the distance to the boundary $\partial\Omega$, that is, $d = d(x) = \operatorname{dist}(x, \partial\Omega)$, cf [12, 13]. Brezis and Marcus [14] proved that for bounded and convex domain Ω there holds

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \ge \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\mathrm{d}x + \frac{1}{4L^2}\int_\Omega u^2 \mathrm{d}x, \quad u \in C_0^\infty(\Omega), \tag{1.4}$$

where $L = \operatorname{diam}\Omega$.

Throughout this paper, $p > 1$, Ω is a domain in $\mathbb{R}^N$, $N \ge 2$, and $K \subset \overline{\Omega}$ is a piecewise smooth closed and connected surface of codimension $k = 1, \dots, N$. The distance from K is denoted by d, that is $d = d(x) = \operatorname{dist}(x, K)$. Then d is a Lipschitz continuous function with $|\nabla d| = 1$ a.e.

Suppose that for $p \neq k$, the following inequality holds in the weak sense:

$$\Delta_p d^{(p-k)/(p-1)} \le 0, \quad \text{in } \Omega \setminus \mathrm{K}. \tag{C}$$

Define $X_1(t) = (1 - \log t)^{-1}$ for $t \in (0,1)$, and recursively $X_k(t) = X_1(X_{k-1}(t))$ for $k = 2, 3, \dots$. Barbatis et al. [15] proved that if $\sup_{x\in\Omega} d(x) < \infty$, then there exists a positive constant $D_0 = D_0(k,p) \ge \sup_{x\in\Omega} d(x)$ such that for any $D \ge D_0$, $m \in \mathbb{N}$ and all $u \in W_0^{1,p}(\Omega \setminus K)$ there holds

$$\begin{aligned}\int_\Omega |\nabla u|^p \mathrm{d}x - \left|\frac{k-p}{p}\right|^p \int_\Omega \frac{|u|^p}{d^p}\mathrm{d}x \\ \ge \frac{p-1}{2p}\left|\frac{k-p}{p}\right|^{p-2}\left(\sum_{i=1}^m \int_\Omega \frac{|u|^p}{d^p} X_1^2\left(\frac{d}{D}\right)\cdots X_i^2\left(\frac{d}{D}\right)\mathrm{d}x\right),\end{aligned} \tag{1.5}$$

and the constants in front of integrals are optimal. The authors also obtained the result for the degenerate case of $p = k$.

Let ϕ be positive and continuous in $(0, \infty)$. In this paper, we are concerned with a general class of sharp Hardy inequality with general weight $\phi(d)$. Define

$$\overline{h}(r_1, r_2) = c_0 \int_{r_1}^{r_2} \left(\phi r^{k-1}\right)^{-1/(p-1)} \mathrm{d}r \tag{1.6}$$

for $0 \le r_1 \le r_2 \le \infty$, where c_0 is a positive constant. Let us consider three cases:

(A_1) $\overline{h}(r,\infty) < \infty$ and $\overline{h}(0,r) = \infty$ for $r > 0$;

(A_2) $\overline{h}(r,\infty) = \infty$ and $\overline{h}(0,r) = \infty$ for $r > 0$;

(A_3) $\overline{h}(r,\infty) = \infty$ and $\overline{h}(0,r) < \infty$ for $r > 0$.

Definition 1.1. Let $p > 1$. If (A_1) or (A_2) occurs, we denote by $W_0^{1,p}(\Omega,\phi)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p,\phi} = \left(\int_\Omega \phi(d)|\nabla u|^p \mathrm{d}x\right)^{1/p}. \tag{1.7}$$

If (A_3) occurs, we denote by $W_0^{1,p}(\Omega \setminus K,\phi)$ the completion of $C_0^\infty(\Omega \setminus K)$ with respect to the above norm. For simplicity, we use W to denote $W_0^{1,p}(\Omega,\phi)$ or $W_0^{1,p}(\Omega \setminus K,\phi)$.

Let $r > 0$, define

$$\overline{h}(r) = \begin{cases} \overline{h}(r,\infty), & \text{if } (A_1) \text{ occurs}, \\ \overline{h}(r,D), & \text{if } (A_2) \text{ occurs}, \\ \overline{h}(0,r), & \text{if } (A_3) \text{ occurs}, \end{cases} \tag{1.8}$$

$$h(r) = \overline{h}^{(p-1)/p}(r) = \begin{cases} \left(c_0\displaystyle\int_r^\infty \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)/p}, & \text{if } (A_1) \text{ occurs}, \\ \left(c_0\displaystyle\int_r^D \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)/p}, & \text{if } (A_2) \text{ occurs}, \\ \left(c_0\displaystyle\int_0^r \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)/p}, & \text{if } (A_3) \text{ occurs}, \end{cases} \tag{1.9}$$

where D is a positive constant such that $\Omega \subset B_D(0)$.

Theorem 1.2. *Let $p > 1$, Ω be a bounded domain in $\mathbb{R}^N$ and K a piecewise smooth surface of codimension k, $k = 1,\ldots,N$. Assume*

$$\operatorname{div}\left(\phi(d)\left|\nabla\overline{h}(d)\right|^{p-2}\nabla\overline{h}(d)\right) \le 0 \quad \text{in } \Omega \setminus K. \tag{C*}$$

Then for all $u \in W$,

$$\int_\Omega \psi(d)|u|^p \mathrm{d}x \le \int_\Omega \phi(d)|\nabla u|^p \mathrm{d}x, \tag{1.10}$$

where $\psi = \phi|h'/h|^p$. Moreover, the constant 1 is optimal, that is,

$$1 = \inf_{u \in W\setminus\{0\}} \frac{\int_\Omega \psi(d)|u|^p}{\int_\Omega \phi(d)|\nabla u|^p \mathrm{d}x}. \tag{1.11}$$

Example 1.3. Let $K = \mathbb{R}^{N-k}$. Then $d = |x'| = (x_1^2 + \cdots + x_k^2)^{1/2}$. If $\phi(r) = r^\alpha$ with $\alpha > p - k$, then

$$h(r) = r^{(p-k-\alpha)/p}, \qquad \varphi(r) = \left(\frac{k+\alpha-p}{p}\right)^p r^{\alpha-p}, \tag{1.12}$$

and we have by Theorem 1.2

$$\int_{\mathbb{R}^N} |u|^p \left|x'\right|^\alpha \mathrm{d}x \le \left(\frac{p}{\alpha+k}\right)^p \int_{\mathbb{R}^N} |\nabla u|^p \left|x'\right|^{\alpha+p} \mathrm{d}x, \quad u \in C_0^\infty\left(\mathbb{R}^N\right), \tag{1.13}$$

see also Secchi et al. [16]. If $\phi(r) = r^\alpha$ with $\alpha = p - k$, then

$$h(r) = \left(\ln\frac{D}{r}\right)^{(p-1)/p}, \qquad \varphi(r) = \left(\frac{p}{p-1}\right)^p r^{\alpha-p}\left(\ln\frac{D}{r}\right)^{-p}. \tag{1.14}$$

If $\phi(r) = r^\alpha$ with $\alpha = p - k = 0$, Theorem 1.2 turns to be Theorem 4.2 in [17].

Let $r > 0$, define

$$h_1(r) = \begin{cases} \dfrac{p}{(p-1)c_0} \ln \dfrac{h(r)}{h(D)}, & \text{if } (\mathrm{A}_1) \text{ occurs}, \\ \dfrac{p}{(p-1)c_0} \ln h(r), & \text{if } (\mathrm{A}_2) \text{ occurs}, \\ \dfrac{p}{(p-1)c_0} \ln \dfrac{h(D)}{h(r)}, & \text{if } (\mathrm{A}_3) \text{ occurs}, \end{cases} \tag{1.15}$$

and $h_{i+1}(r) = \ln \mathrm{e}h_i(r)$ for $i = 1, 2, \ldots$.

For convenience, we write

$$I_{m,\phi}[u] = \int_\Omega \phi(d)|\nabla u|^p \mathrm{d}x - \int_\Omega \varphi(d)|u|^p \mathrm{d}x - \frac{p}{2(p-1)c_0^2}\sum_{i=1}^m \int_\Omega \varphi h_1^{-2}(d)\cdots h_i^{-2}(d)|u|^p \mathrm{d}x. \tag{1.16}$$

Theorem 1.4. *Let $p > 1$, Ω be a bounded domain in $\mathbb{R}^N$ and K a piecewise smooth surface of codimension k, $k = 1, \ldots, N$. Assume that* (C^*) *holds, then*

(1) *there exists a positive constant $D_0 = D_0(k,p) > \sup_{x\in\Omega} d(x)$ such that for all $D \ge D_0$ and $u \in W$, there holds*

$$\int_\Omega \phi(d)|\nabla u|^p \mathrm{d}x - \int_\Omega \varphi(d)|u|^p \mathrm{d}x \ge \frac{p}{2(p-1)c_0^2}\sum_{i=1}^m \int_\Omega \varphi(d) h_1^{-2}(d)\cdots h_i^{-2}(d)|u|^p \mathrm{d}x \tag{1.17}$$

where $\varphi = \phi|h'/h|^p$, if in addition $p \ge 2$ and (A_1) *occurs, then one can take $D_0 = \sup_{x\in\Omega} d(x)$.*

(2) *the constants in* (1.17) *are optimal, that is,*

$$\frac{p}{2(p-1)c_0^2} = \inf_{u\in W\setminus\{0\}} \frac{I_{m-1,\phi}[u]}{\int_\Omega \varphi(d)h_1^{-2}(d)\cdots h_m^{-2}(d)|u|^p \mathrm{d}x}. \tag{1.18}$$

Remark 1.5. Let $\phi(r) = r^\alpha$. Then (A_1) occurs if $k > p-\alpha$, (A_2) occurs if $k = p-\alpha$ and (A_3) occurs if $k < p-\alpha$. There are three cases for k and K: (1) $k = 1$ and $K = \partial\Omega$; (2) $2 \le k \le N-1$ and $\Omega \cap K \neq \emptyset$; (3) $k = N$ and $K = \{0\} \subset \Omega$. If $\alpha = 0$ and $k = 1$, then neither (A_1) nor (A_2) occurs because of $p > 1$.

Remark 1.6. Theorem 1.4 extends the inequality (1.5) to Sobolev space with general weight $\phi(d)$. Moreover, it also includes the results of [18, 19].

Example 1.7. Let $\phi(r) = r^\alpha$. If $\alpha > p-k$, we have

$$\begin{gathered} c_0 = \frac{k+\alpha-p}{p-1}, \qquad h(r) = r^{(p-k-\alpha)/p}, \qquad \varphi(r) = \left(\frac{k+\alpha-p}{p}\right)^p r^{\alpha-p}, \\ h_1(r) = \ln\frac{D}{r}, \qquad h_{i+1}(r) = \ln \mathrm{e}h_i(r), \quad i = 1,2,\ldots. \end{gathered} \tag{1.19}$$

Then it follows from Theorem 1.4 that for all $u \in W_0^{1,p}(\Omega, d^\alpha)$,

$$\begin{aligned} &\int_\Omega d^\alpha |\nabla u|^p \mathrm{d}x - \left|\frac{k+\alpha-p}{p}\right|^p \int_\Omega d^{\alpha-p}|u|^p \mathrm{d}x \\ &\quad \ge \frac{p-1}{2p}\left|\frac{k+\alpha-p}{p}\right|^{p-2} \sum_{i=1}^m \int_\Omega d^{\alpha-p} h_1^{-2}(d)\cdots h_i^{-2}(d)|u|^p \mathrm{d}x, \end{aligned} \tag{1.20}$$

which is (1.5) [15, Theorem A], if $\alpha = 0$ (i.e., $p < k$). If $\alpha < p-k$, the above inequality holds for all $u \in W_0^{1,p}(\Omega \setminus K, d^\alpha)$. If $\alpha = p-k$, we have

$$\begin{aligned} &\int_\Omega d^\alpha |\nabla u|^p \mathrm{d}x - \left(\frac{p-1}{p}\right)^p \int_\Omega d^{\alpha-p}\left(\ln\frac{D}{d}\right)^{-p} |u|^p \mathrm{d}x \\ &\quad \ge \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \sum_{i=2}^m \int_\Omega d^{\alpha-p}\left(\ln\frac{D}{d}\right)^{-p} h_1^{-2}(d)\cdots h_i^{-2}(d)|u|^p \mathrm{d}x. \end{aligned} \tag{1.21}$$

This is the result of Theorem B in [15] if $\alpha = 0$.

2. Preliminary Lemmas

Lemma 2.1. *If* (A_1) *or* (A_2) *occurs, then*

$$\operatorname{div}\left(\phi h^{\alpha}(-h')^{p-1}\nabla d\right) = (1-\alpha)\phi h^{\alpha-1}(-h')^{p} + \phi h^{\alpha}(-h')^{p-1}\left(\Delta d - \frac{k-1}{d}\right). \tag{2.1}$$

If (A_3) *occurs, then*

$$\operatorname{div}\left(\phi h^{\alpha}(-h')^{p-1}\nabla d\right) = (\alpha-1)\phi h^{\alpha-1}(-h')^{p} + \phi h^{\alpha}(-h')^{p-1}\left(\Delta d - \frac{k-1}{d}\right). \tag{2.2}$$

Proof. Note that

$$h = \left(c_0\int_d^b (\phi r^{k-1})^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)/p}. \tag{2.3}$$

where $b=\infty$ for the case (A_1) and $b=D$ for the case (A_2), then

$$\phi h^{\alpha}(-h')^{p-1} = \left(\frac{p-1}{p}\right)^{p-1} c_0^{p-1}\left(c_0\int_d^b (\phi r^{k-1})^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)(\alpha-1)/p} d^{1-k}. \tag{2.4}$$

Hence

$$\begin{aligned}
&\left(\phi h^{\alpha}(-h')^{p-1}\right)' \\
&\quad= \left(\frac{p-1}{p}\right)^{p}(1-\alpha)c_0^{p}\left(c_0\int_d^b \left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)(\alpha-1)/p-1} \\
&\qquad\times\left(\phi d^{k-1}\right)^{-1/(p-1)} d^{1-k} + \phi h^{\alpha}(-h')^{p-1}(1-k)d^{-1}.
\end{aligned} \tag{2.5}$$

Then

$$\begin{aligned}
\operatorname{div}\left(\phi h^{\alpha}(-h')^{p-1}\nabla d\right) &= \left(\phi h^{\alpha}(-h')^{p-1}\right)'|\nabla d|^2 + \phi h^{\alpha}(-h')^{p-1}\Delta d \\
&= (1-\alpha)\phi h^{\alpha-1}(-h')^{p} + \phi h^{\alpha}(-h')^{p-1}\left(\Delta d - \frac{k-1}{d}\right).
\end{aligned} \tag{2.6}$$

The same argument can give the corresponding result if (A_3) occurs. □

Lemma 2.2. *Let $1 < p < N$ and $K = \{0\} \subset \Omega$. If (A_1) is satisfies, then $\overline{h}$ is the fundamental solution for the p-Laplace operator with weight ϕ, that is,*

$$-\operatorname{div}\left(\phi\left|\nabla\overline{h}\right|^{p-2}\nabla\overline{h}\right) = c_0^{p-1}\omega_N\delta(x), \tag{2.7}$$

where $\delta(x)$ is the Dirac measure, and ω_N denotes the volume of the unit sphere in $\mathbb{R}^N$.

Proof. Since $h = \left(c_0\int_r^\infty(\phi r^{N-1})^{-1/(p-1)}\mathrm{d}r\right)^{(p-1)/p}$, we have

$$\begin{aligned}-\operatorname{div}\left(\phi\left|\nabla\overline{h}\right|^{p-2}\nabla\overline{h}\right) &= -\operatorname{div}\left(\phi\left|\overline{h'}\right|^{p-1}\frac{x}{|x|}\right)\\ &= -c_0^{p-1}\operatorname{div}\left(\phi\left(\phi r^{N-1}\right)^{-1}\frac{x}{|x|}\right)\\ &= -c_0^{p-1}\operatorname{div}\left(\frac{x}{|x|^N}\right) = -c_0^{p-1}\omega_N\delta(x),\end{aligned} \tag{2.8}$$

where the last equality sign is because of $-\operatorname{div}(x/|x|^N) = \omega_N\delta(x)$. □

Proposition 2.3. *Let $1 < p < N$ and $K = \{0\} \subset \Omega$. If (A_1) is satisfies, then h_i is the fundamental solution for the Laplace operator with weight ϕ_i, that is, for $i = 1, 2, \ldots$,*

$$-\operatorname{div}(\phi_i\nabla h_i) = -c_i\omega_N\delta(x). \tag{2.9}$$

Proof. The result follows by the following equalities:

$$-\operatorname{div}(\phi_i\nabla h_i) = -\operatorname{div}\left(\phi_i h_i'\frac{x}{|x|}\right) = -c_i\operatorname{div}\left(\frac{x}{|x|^N}\right) = -c_i\omega_N\delta(x). \tag{2.10}$$

□

Set $Y_1^{-1}(r) = h_1(r)$, it follows from (1.15) that, if (A_1) or (A_2) occurs,

$$-\frac{Y_1'}{Y_1^2} = (h_1)' = \frac{p}{(p-1)c_0}\frac{h'}{h}. \tag{2.11}$$

Define $Y_i^{-1}(r) = h_i(r)$, then

$$-Y_i^{-2}Y_i' = (h_i)' = (\ln h_{i-1})' = \frac{1}{h_{i-1}}(h_{i-1})' = \frac{p}{(p-1)c_0}\frac{h'}{h}Y_{i-1}\cdots Y_1, \tag{2.12}$$

that is,

$$Y_i' = -\frac{p}{(p-1)c_0}\frac{h'}{h}Y_1\cdots Y_{i-1}Y_i^2, \tag{2.13}$$

and so for any $\beta \neq -1$, $i \in \mathbb{N}$, we have

$$\left(Y_i^{\beta}\right)' = -\beta \frac{p}{(p-1)c_0} \frac{h'}{h} Y_1 \cdots Y_{i-1} Y_i^{\beta+1}. \tag{2.14}$$

Let $m \in \mathbb{N}$ and write

$$\eta(r) = \sum_{i=1}^{m} Y_1 \cdots Y_i, \qquad B(r) = \sum_{i=1}^{m} Y_1^2 \cdots Y_m^2, \tag{2.15}$$

then a simple calculation gives

$$\eta' = -\frac{p}{(p-1)c_0} \frac{h'}{h} \left(\frac{1}{2} \left(B(d) + \eta^2(d) \right) \right). \tag{2.16}$$

Similarly, if (A_3) occurs, we have

$$\begin{aligned} -\frac{Y_1'}{Y_1^2} &= -\frac{p}{(p-1)c_0} \frac{h'}{h}, \\ Y_i' &= \frac{p}{(p-1)c_0} \frac{h'}{h} Y_1 \cdots Y_{i-1} Y_i^2. \end{aligned} \tag{2.17}$$

Then

$$\eta' = \frac{p}{(p-1)c_0} \frac{h'}{h} \left(\frac{1}{2} \left(B(d) + \eta^2(d) \right) \right). \tag{2.18}$$

3. Proof of Theorems

Proof of Theorem 1.2. Define a C^1 vector field as

$$T = \begin{cases} \phi\left(-\dfrac{h'}{h}\right)^{p-1} \nabla d & \text{if } (A_1) \text{ or } (A_2) \text{ occurs}, \\ -\phi\left(\dfrac{h'}{h}\right)^{p-1} \nabla d & \text{if } (A_3) \text{ occurs}. \end{cases} \tag{3.1}$$

Then we can prove (1.10) analogous to the following proof of Theorem 1.4 (1). As to the best constant, we fix small positive parameter α and define the functions

$$w(x) = h^{1-(\alpha/(p-1)c_0)}. \tag{3.2}$$

The rest is similar to the following proof of Theorem 1.4(2). □

Proof of Theorem 1.4(1) . We will make use of a suitable vector field as in [15]. To proceed we now make a specific choice of T. Firstly, we consider the cases (A_1) and (A_2). We take

$$\begin{aligned} T &= \phi\left(-\frac{h'}{h}\right)^{p-1}\left(1+c_0^{-1}\eta+a\eta^2\right)\nabla d \\ &= \phi\left(-\frac{h'}{h}\right)^{p-1}\nabla d + c_0^{-1}\phi\left(-\frac{h'}{h}\right)^{p-1}\eta\nabla d + a\phi\left(-\frac{h'}{h}\right)^{p-1}\eta^2\nabla d \\ &=: T_1+T_2+T_3, \end{aligned} \tag{3.3}$$

where a is a free parameter to be chosen later. In any cases a will be such that the quantity $1+c_0^{-1}\eta+a\eta^2$ is positive on Ω. Note that T is singular at $x\in K$, but $\operatorname{div} T$ and T are integrable if (A_1) or (A_2) occurs.

Let $u\in C_0^\infty(\Omega)$ if (A_1) or (A_2) occurs. We integrate by parts to obtain, for any positive ϵ,

$$\int_{\Omega\setminus\Omega_\epsilon}\operatorname{div} T|u|^p\mathrm{d}x = -p\int_{\Omega\setminus\Omega_\epsilon}(T\cdot\nabla u)|u|^{p-2}u\,\mathrm{d}x + p\int_{\Omega\cap\{d=\epsilon\}} T\cdot|u|^p\nabla d\cdot\vec{n}\mathrm{d}S, \tag{3.4}$$

where $\Omega_\epsilon=\{x\in\Omega\mid d(x)<\epsilon\}$, and $\vec{n}$ denotes the unit outer normal to $\partial\Omega_\epsilon$. Note that

$$I\equiv\left|\int_{\Omega\cap\{d=\epsilon\}} T\cdot|u|^p\nabla d\cdot\vec{n}\mathrm{d}S\right| \le \int_{\Omega\cap\{d=\epsilon\}}|T||u|^p\mathrm{d}S = \int_{\Omega\cap\{d=\epsilon\}}\phi\left|\frac{h'}{h}\right|^{p-1}|u|^p\mathrm{d}S. \tag{3.5}$$

It follows from (1.9) that

$$\phi\left|\frac{h'}{h}\right|^{p-1} \le \left(\int_d^b\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-(p-1)} d^{k-1}, \tag{3.6}$$

where $b=\infty$ if (A_1) occurs, or $b=D$ if (A_2) occurs. Since

$$c_1 r^{k-1} \le \int_{\Omega\cap\{d=r\}}\mathrm{d}S \le c r^{k-1} \tag{3.7}$$

for some positive constants c and c_1, (A_1) or (A_2) implies that

$$\left(\int_\epsilon^b\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-(p-1)} \longrightarrow 0 \tag{3.8}$$

as ϵ tends to 0. Since η is bounded, we know $I\to 0$ as $\epsilon\to 0$. Hence,

$$\int_\Omega\operatorname{div} T|u|^p\mathrm{d}x = -p\int_\Omega(T\cdot\nabla u)|u|^{p-2}u\,\mathrm{d}x. \tag{3.9}$$

By Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}\int_\Omega \operatorname{div} T|u|^p \mathrm{d}x &= -p\int_\Omega (T\cdot\nabla u)|u|^{p-2}u\,\mathrm{d}x\\ &\le p\left(\int_\Omega \phi|\nabla u|^p\mathrm{d}x\right)^{1/p}\left(\int_\Omega \left|T\phi^{-1/p}\right|^{p/(p-1)}|u|^p\mathrm{d}x\right)^{(p-1)/p}\\ &\le \int_\Omega \phi|\nabla u|^p\mathrm{d}x + (p-1)\int_\Omega \left|T\phi^{-1/p}\right|^{p/p-1}|u|^p\mathrm{d}x.\end{aligned}\tag{3.10}$$

We therefore arrive at

$$\int_\Omega \phi|\nabla u|^p\mathrm{d}x \ge \int_\Omega\left(\operatorname{div} T - (p-1)\left|T\phi^{-1/p}\right|^{p/(p-1)}\right)|u|^p\mathrm{d}x.\tag{3.11}$$

If (A_3) occurs, the above inequality is obvious for $u\in C_0^\infty(\Omega\setminus K)$.

By Lemma 2.1 and condition (C^*), we have

$$\operatorname{div} T_1 = p\phi\left(-\frac{h'}{h}\right)^p + \phi h^{1-p}(-h')^{p-1}\left(\Delta d - \frac{k-1}{d}\right) \ge p\phi\left(-\frac{h'}{h}\right)^p.\tag{3.12}$$

Similarly, it follows from Lemma 2.1, condition (C^*) and (2.16)

$$\begin{aligned}\operatorname{div} T_2 &= c_0^{-1}\eta\operatorname{div}\phi\left(-\frac{h'}{h}\right)^{p-1}\nabla d + c_0^{-1}\phi\left(-\frac{h'}{h}\right)^{p-1}|\nabla d|^2\eta'\\ &\ge pc_0^{-1}\eta\phi\left(-\frac{h'}{h}\right)^p + \frac{p}{(p-1)c_0^2}\phi\left(-\frac{h'}{h}\right)^p\left(\frac{B}{2}+\frac{\eta^2}{2}\right),\end{aligned}\tag{3.13}$$

$$\operatorname{div} T_3 \ge ap\eta^2\phi\left(-\frac{h'}{h}\right)^p + \frac{p}{(p-1)c_0^2}\phi\left(-\frac{h'}{h}\right)^p\eta\left(B+\eta^2\right).\tag{3.14}$$

Combining (3.12)–(3.14), we obtain

$$\operatorname{div} T \ge \phi\left(-\frac{h'}{h}\right)^p\left[\left(p + pc_0^{-1}\eta + ap\eta^2\right) + \frac{p}{2(p-1)c_0^2}\left(B+\eta^2\right) + \frac{ap}{(p-1)c_0}\left(B\eta+\eta^3\right)\right].\tag{3.15}$$

Next we compute $(p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)}$. We set for convenience

$$g(\eta) = \left(1 + c_0^{-1}\eta + a\eta^2\right)^{p/(p-1)}. \tag{3.16}$$

When $\eta > 0$ is small, the Taylor expansion of $g(\eta)$ about $\eta = 0$ gives

$$\begin{aligned} g(\eta) &= 1 + \frac{p}{(p-1)c_0}\eta + \frac{1}{2}\left(\frac{p}{(p-1)^2c_0^2} + \frac{2pa}{p-1}\right)\eta^2 \\ &\quad + \frac{1}{6}\left(\frac{p(2-p)}{(p-1)^3c_0^3} + \frac{6pa}{(p-1)^2c_0}\right)\eta^3 + O\left(\eta^4\right), \end{aligned} \tag{3.17}$$

and so

$$\begin{aligned} (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} = -\phi\left(-\frac{h'}{h}\right)^p\Bigg[&(p-1) + \frac{p}{c_0}\eta + \left(\frac{p}{2(p-1)c_0^2} + pa\right)\eta^2 \\ &+ \left(\frac{p(2-p)}{(p-1)^2c_0^3} + \frac{pa}{(p-1)c_0}\right)\eta^3 + O\left(\eta^4\right)\Bigg]. \end{aligned} \tag{3.18}$$

Hence

$$\begin{aligned} &\operatorname{div} T - (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} \\ &\quad\ge \phi\left(-\frac{h'}{h}\right)^p\left[1 + \frac{pB}{2(p-1)c_0^2} + \frac{pa}{(p-1)c_0}B\eta - \frac{p(2-p)}{(p-1)^2c_0^3}\eta^3 + O\left(\eta^4\right)\right]. \end{aligned} \tag{3.19}$$

If we show

$$\frac{ap}{(p-1)c_0} \ge \left(\frac{p(2-p)}{(p-1)^2c_0^3} + O(\eta)\right)\frac{\eta^2}{B} \tag{3.20}$$

then we obtain

$$\operatorname{div} T - (p-1)\phi^{-1/p-1}|T|^{p/p-1} \ge \phi\left(-\frac{h'}{h}\right)^p\left[1 + \frac{p}{2(p-1)c_0^2}B\right]. \tag{3.21}$$

From the definition of η and B it follows easily that

$$m \ge \frac{\eta^2}{B} \ge 1. \tag{3.22}$$

(a) If $1 < p < 2$, we assume that η is small for the case (A_1). Since

$$h_1 = \frac{p}{(p-1)c_0} \ln \frac{h(r)}{h(D)} \tag{3.23}$$

and $\Omega \subset B_{r_0}(0)$ is bounded, we can choose D large enough such that $h_1^{-1}(r)$ is small enough if $r < r_0$, and then η is small. It is enough to show that we can choose a such that (3.20) holds. In view of (3.22), we see that for (3.20) to be valid, it is enough to take a to be big and positive. It is similar for the case (A_2).

(b) If $p \geq 2$, we choose $a = 0$, then

$$\left(1 + c_0^{-1}\eta\right)^{p/(p-1)} = 1 + \frac{p}{(p-1)c_0}\eta + \frac{p}{2(p-1)^2 c_0^2}\eta^2 + \frac{p(2-p)}{6(p-1)^3 c_0^3}\left(1 + c_0^{-1}\xi\right)^{(3-2p)/(p-1)}\eta^3 \tag{3.24}$$

for some $\xi \in (0, \eta)$, without any smallness assumption. Since $2 - p \leq 0$, we have

$$\left(1 + c_0^{-1}\eta\right)^{p/(p-1)} \leq 1 + \frac{p}{(p-1)c_0}\eta + \frac{p}{2(p-1)^2 c_0^2}\eta^2. \tag{3.25}$$

It follows from (3.15) that

$$\operatorname{div} T \geq \phi\left(-\frac{h'}{h}\right)^p \left[p\left(1 + c_0^{-1}\eta\right) + \frac{p(B+\eta^2)}{2(p-1)c_0^2}\right]. \tag{3.26}$$

Hence

$$\operatorname{div} T - (p-1)\phi^{1/(p-1)}|T|^{p/(p-1)} \geq \phi\left(-\frac{h'}{h}\right)^p \left(1 + \frac{Bp}{2(p-1)c_0^2}\right). \tag{3.27}$$

Then (1.17) follows by inserting the above inequality into (3.11).

Now we consider the case (A_3). In this case, $h' > 0$, that is,

$$h = \left(c_0 \int_0^d \left(\phi r^{k-1}\right)^{-1/(p-1)} \mathrm{d}r\right)^{(p-1)/p}. \tag{3.28}$$

We take

$$T = -\phi\left(\frac{h'}{h}\right)^{p} \nabla d\left(1 - c_0^{-1}\eta + a\eta^2\right), \tag{3.29}$$

where a is a free parameter to be chosen later. In any case a will be such that the quantity $1 - c_0^{-1}\eta + a\eta^2$ is positive on Ω. Note that T is singular at $x \in K$, but since $u \in C_0^{\infty}(\Omega \setminus K)$ all previous calculations are legitimate. Analogues to the calculations before, by Lemma 2.1 and (2.18), we have

$$\begin{aligned}
\operatorname{div} T &= p\phi\left(\frac{h'}{h}\right)^{p}\left(1-c_0^{-1}\eta + a\eta^2\right)-\left(\Delta d-\frac{k-1}{d}\right)\phi\left(\frac{h'}{h}\right)^{p}\left(1-c_0^{-1}\eta+a\eta^2\right) \\
&\quad + \frac{p}{2(p-1)c_0^2}\phi\left(\frac{h'}{h}\right)^{p}\left(\eta^2 + B\right) - \frac{ap}{(p-1)c_0}\phi\left(\frac{h'}{h}\right)^{p}\left(B\eta + \eta^3\right) \\
&\geq \phi\left(\frac{h'}{h}\right)^{p}\left[p\left(1-c_0^{-1}\eta+a\eta^2\right)+\frac{p}{2(p-2)c_0^2}\left(\eta^2+B-2ac_0\left(B\eta+\eta^3\right)\right)\right], \\
(p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} &\geq \phi\left(\frac{h'}{h}\right)^{p}\left[(p-1) - \frac{p}{c_0}\eta + \left(\frac{p}{2(p-1)c_0^2} + pa\right)\eta^2\right. \\
&\quad \left. + \frac{1}{6}\left(-\frac{p(2-p)}{(p-1)^2c_0^3} - \frac{6pa}{(p-1)c_0}\right)\eta^3\right] + O\left(\eta^4\right).
\end{aligned} \tag{3.30}$$

Hence

$$\begin{aligned}
&\operatorname{div} T - (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} \\
&\quad \geq \phi\left(\frac{h'}{h}\right)^{p}\left(1 + \frac{pB}{2(p-1)c_0^2} - \frac{apB}{(p-1)c_0}\eta + \frac{p(2-p)}{6(p-1)^2c_0^3}\eta^3\right) + O\left(\eta^4\right).
\end{aligned} \tag{3.31}$$

If $1 < p < 2$, since

$$h_1 = \frac{p}{(p-1)c_0}\ln\frac{h(D)}{h(d)} \tag{3.32}$$

when (A_1) occurs, we can choose D large enough such that $h(D)$ is so large and $h_1^{-1}(r)$ is small, then we know that η is small. By taking $a = 0$, we obtain

$$\operatorname{div} T - (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} \geq \phi\left(\frac{h'}{h}\right)^{p}\left(1 + \frac{pB}{2(p-1)c_0}\right). \tag{3.33}$$

If $p \geq 2$, taking a be negative with $|a|$ large enough, we also arrive at (3.33) by using (3.21). The result (1.17) then follows from (3.11) and (3.33). □

Proof of Theorem 1.4(2). All our analysis will be local, say, in a fixed ball of radius δ (denoted by B_δ) centered at the origin, for some fixed small δ. The proof we present works for any $k = 1, 2, \dots, N$. We note however that for $k = N$ (distant from a point) the subsequent calculations are substantially simplified, whereas for $k = 1$ (distant from the boundary) one should replace B_δ by $B_\delta \cap \Omega$. This last change entails some minor modifications, the arguments otherwise being the same. Without any loss of generality we may assume that $0 \in K \cap \Omega$ ($k \neq 1$), or $0 \in \partial\Omega$ if $k = 1$. We divide the proof into several steps.

Step 1. Let $\theta \in C_0^\infty(B_\delta)$ be such that $0 \le \theta \le 1$ in B_δ and $\theta = 1$ in $B_{\delta/2}$. We fix small positive parameters $\alpha_0, \alpha_1, \dots, \alpha_m$ and define the functions

$$\begin{gathered} w(x) = h^{1-\alpha_0/(p-1)c_0} h_1^{(1-\alpha_1)/p} \cdots h_m^{(1-\alpha_m)/p}(d), \\ u(x) = \theta(x) w(x). \end{gathered} \tag{3.34}$$

Let (A_1) or (A_2) happen. Hence $u \in W_0^{1,p}(\Omega, \phi)$. To prove the proposition we will estimate the corresponding Rayleigh quotient of u in the limit $\alpha_0 \to 0$, $\alpha_1 \to 0, \dots, \alpha_m \to 0$ in this order.

It is easily seen that

$$\nabla w = \frac{p}{(p-1)c_0} h^{-\alpha_0/(p-1)c_0} h' \nabla d Y_1^{(-1+\alpha_1)/p} \cdots Y_m^{(-1+\alpha_m)/p} \left(\frac{(p-1)c_0}{p} + \frac{\overline{\eta}}{p} \right), \tag{3.35}$$

where $Y_i = h_i^{-1}$ and $\overline{\eta} = -\alpha_0 + (1-\alpha_1) Y_1 + \cdots + (1-\alpha_m) Y_1 \cdots Y_m$.

Now $\nabla u = \theta \nabla w + w \nabla \theta$ and hence, using the elementary inequality

$$|a+b|^p \le |a|^p + c_p \left(|a|^{p-1}|b| + |b|^p \right), \quad a, b \in \mathbb{R}^N \tag{3.36}$$

for $p > 1$, we obtain

$$\begin{aligned} \int_\Omega \phi |\nabla u|^p \mathrm{d}x &\le \int_\Omega \phi \theta^p |\nabla w|^p \mathrm{d}x + c_p \int_\Omega \phi \theta^{p-1} |\nabla \theta| |w| |\nabla w|^{p-1} \mathrm{d}x + c_p \int_\Omega \phi |\nabla \theta|^p |w|^p \mathrm{d}x \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.37}$$

We claim that

$$I_2, I_3 = O(1) \quad \text{uniformly as } \alpha_0, \alpha_1, \dots, \alpha_{\mathrm{m}} \text{ tend to zero.} \tag{3.38}$$

Let us give the proof for I_2:

$$
\begin{aligned}
I_2 \le C\int_{B_\delta} & \phi h^{-\alpha_0/c_0}|\nabla h|^{p-1}\Upsilon_1^{(-1+\alpha_1)(p-1)/p}\cdots\Upsilon_m^{(-1+\alpha_m)(p-1)/p} \\
&\times\left[(p-1)c_0+\alpha_0+(1-\alpha_1)\Upsilon_1+\cdots+(1-\alpha_m)\Upsilon_1\cdots\Upsilon_m\right]^{p-1} \\
&\times h^{1-\alpha_0/(p-1)c_0}\Upsilon_1^{(-1+\alpha_1)/p}\cdots\Upsilon_m^{(-1+\alpha_m)/p}\mathrm{d}x \\
\le C\int_{B_\delta} & \phi h^{1-\alpha_0 p/(p-1)c_0}|\nabla h|^{p-1}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m} \\
&\times\left[(p-1)c_0+\alpha_0+(1-\alpha_1)\Upsilon_1+\cdots+(1-\alpha_m)\Upsilon_1\cdots\Upsilon_m\right]^{p-1}\mathrm{d}x.
\end{aligned}
\tag{3.39}
$$

It follows from the definition of h that

$$
h' = \frac{p-1}{p}\left(c_0\int_d^b\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-1/p}\left(-c_0\left(\phi d^{k-1}\right)^{-1/(p-1)}\right). \tag{3.40}
$$

Then

$$
\phi(-h')^{p-1}h = \left(\frac{c_0(p-1)}{p}\right)^{p-1}d^{1-k}. \tag{3.41}
$$

Hence, by the coarea formula and the fact that

$$
c_1 r^{k-1} \le \int_{\{d=r\}\cap B_\delta}\mathrm{d}S \le c_2 r^{k-1} \tag{3.42}
$$

we have

$$
\begin{aligned}
I_2 &\le C\int_{B_\delta} d^{1-k}h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\mathrm{d}x \\
&= C\int_0^\delta \mathrm{d}r\int_{\{d=r\}}\frac{d^{1-k}h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}}{|\nabla d|}\mathrm{d}S \\
&= C\int_0^\delta \mathrm{d}r\int_{\{d=r\}} d^{1-k}h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\mathrm{d}S \\
&\le C\int_0^\delta h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}(r)\mathrm{d}r.
\end{aligned}
\tag{3.43}
$$

The boundedness of $h^{-1}(r)$ together with the fact $h^{-1}(0) = 0$ implies that I_2 is uniformly bounded. The integral I_3 is treated similarly.

Step 2. We will repeatedly deal with integrals of the form

$$Q = \int_{\Omega} \theta^p \phi h^{-\beta_0 p/(p-1)c_0} (-h')^p Y_1^{1+\beta_1} \cdots Y_m^{1+\beta_m} \mathrm{d}x. \tag{3.44}$$

By (1.9), we have

$$\begin{gathered} \phi h^{-\beta_0 p/(p-1)c_0} (-h)'^p = \phi \left(c_0 \int_d^b \left(\phi r^{k-1} \right)^{-1/(p-1)} \mathrm{d}r \right)^{-\beta_0/c_0 - 1} \left(c_0 \phi d^{k-1} \right)^{-p/(p-1)}, \\ d^{k-1} \phi h^{-\beta_0 p/(p-1)c_0} (-h)'^p = c \left(\int_d^b \left(\phi r^{k-1} \right)^{-1/(p-1)} \mathrm{d}r \right)^{-\beta_0/c_0 - 1} \left(\phi d^{k-1} \right)^{-1/(p-1)}. \end{gathered} \tag{3.45}$$

Using the coarea formula and (3.41), if $\beta_0 = \cdots = \beta_{m-1} = 0$ and $\beta_m > 0$, by (1.15), (2.14) and $h'/h = ((p-1)/p)(\overline{h}'/\overline{h})$, we have

$$C \int_0^{\delta/2} \left| \frac{h'}{h} \right| Y_1 \cdots Y_{m-1} Y_m^{1+\beta_m} \mathrm{d}r \le Q \le C \int_0^{\delta} \left| \frac{h'}{h} \right| Y_1 \cdots Y_{m-1} Y_m^{1+\beta_m} \mathrm{d}r = \frac{C}{\beta_m} Y_m^{\beta_m} \Big|_0^{\delta} < +\infty. \tag{3.46}$$

Analogue arguments arrive at

$$Q < \infty \Longleftrightarrow \begin{cases} \beta_0 > 0, & \text{or} \\ \beta_0 = 0, \ \beta_1 > 0, & \text{or} \\ \vdots & \\ \beta_0 = \beta_1 = \cdots = \beta_{m-1}, \ \beta_m > 0. & \end{cases} \tag{3.47}$$

Moreover, if $\beta_m \to 0$, we have

$$Q \longrightarrow \infty. \tag{3.48}$$

Step 3. We introduce some auxiliary quantities and prove some simple relations about them. For $0 \le i \le j \le m$ we define

$$\begin{gathered} A_0 = \int_{\Omega} \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x, \\ A_i = \int_{\Omega} \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{1+\alpha_1} \cdots Y_i^{1+\alpha_i} Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x, \\ \Gamma_{0i} = \int_{\Omega} \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{\alpha_1} \cdots Y_i^{\alpha_i} Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x, \\ \Gamma_{ij} = \int_{\Omega} \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{1+\alpha_1} \cdots Y_i^{1+\alpha_i} Y_{i+1}^{\alpha_{i+1}} \cdots Y_j^{\alpha_j} Y_{j+1}^{-1+\alpha_{j+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x. \end{gathered} \tag{3.49}$$

with $\Gamma_{ii} = A_i$. We have the following two identities. Let $0 \le i \le m-1$ be given and assume that $\alpha_0 = \alpha_1 = \cdots = \alpha_{i-1} = 0$, then

$$\alpha_i A_i = \sum_{j=i+1}^{m} (1-\alpha_j)\Gamma_{ij} + O(1), \tag{3.50}$$

$$\alpha_i \Gamma_{ij} = -\sum_{k=i+1}^{j} \alpha_k \Gamma_{kj} + \sum_{k=j+1}^{m} (1-\alpha_k)\Gamma_{jk} + O(1), \tag{3.51}$$

where the $O(1)$ is uniform as the α_i's tend to zero. Let us give the proof for (3.50). Firstly, we discuss the case of $i = 0$. By Lemma 2.1, we have

$$\begin{aligned}
&\phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p \\
&\quad = \frac{(p-1)c_0}{p\alpha_0} \left[\operatorname{div}\left(\phi h^{1-\alpha_0\, p/(p-1)c_0} (-h')^{p-1} \nabla d \right) - \phi h^{1-\alpha_0 p/(p-1)c_0} (-h')^{p-1} \left(\Delta d - \frac{k-1}{d} \right) \right].
\end{aligned} \tag{3.52}$$

Multiplying the above equality by $\theta^p \Upsilon_1^{-1+\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m}$ and integrating over Ω, we obtain

$$\begin{aligned}
A_0 &= \frac{(p-1)c_0}{p\alpha_0} \int_\Omega \operatorname{div}\left(\phi h^{1-\alpha_0 p/(p-1)c_0} (-h')^{p-1} \nabla d \right) \theta^p \Upsilon_1^{-1+\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m} \mathrm{d}x \\
&\quad - \frac{(p-1)c_0}{p\alpha_0} \int_\Omega \phi h^{1-\alpha_0 p/(p-1)c_0} (-h')^{p-1} \left(\Delta d - \frac{k-1}{d} \right) \theta^p \Upsilon_1^{-1+\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m} \mathrm{d}x \\
&=: A_{01} + A_{02}.
\end{aligned} \tag{3.53}$$

Let us estimate A_{01}. Using integration by parts, we obtain

$$\begin{aligned}
A_{01} &= -\frac{(p-1)c_0}{p\alpha_0} \int_\Omega \phi h^{1-\alpha_0 p/(p-1)c_0} (-h')^{p-1} \nabla d \nabla \left(\theta^p \Upsilon_1^{-1+\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m} \right) \mathrm{d}x \\
&= -\frac{(p-1)c_0}{p\alpha_0} \int_\Omega \phi h^{1-\alpha_0 p/(p-1)c_0} (-h')^{p-1} \\
&\quad \times \left\{ \theta^p \left[\left(\Upsilon_1^{-1+\alpha_1} \right)' \Upsilon_2^{-1+\alpha_2} \cdots \Upsilon_m^{-1+\alpha_m} + \cdots + \Upsilon_1^{-1+\alpha_1} \cdots \Upsilon_{m-1}^{-1+\alpha_{m-1}} \left(\Upsilon_m^{-1+\alpha_m} \right)' \right] \right. \\
&\qquad \left. + \Upsilon_1^{-1+\alpha_1} \cdots \Upsilon_m^{-1+\alpha_m} \cdot p \nabla \theta^p \nabla d \right\} \mathrm{d}x.
\end{aligned} \tag{3.54}$$

It follows from (2.14) that

$$\left(Y_i^{-1+\alpha_i}\right)' = (1-\alpha_i)\frac{p}{(p-1)c_0}\frac{h'}{h}Y_1\cdots Y_{i-1}Y_i^{\alpha_i}. \tag{3.55}$$

Then we have

$$\begin{aligned} A_{01} &= (1-\alpha_0)\Gamma_{01}+\cdots+(1-\alpha_m)\Gamma_{0m} \\ &\quad -\frac{p(p-1)c_0}{p\alpha_0}\int_\Omega \phi h^{1-\alpha_0 p/(p-1)c_0}(-h')^{p-1}\nabla\theta^p\nabla d Y_1^{-1+\alpha_1}\cdots Y_m^{-1+\alpha_m}\mathrm{d}x. \end{aligned} \tag{3.56}$$

Hence, by (3.41), (3.42) and condition (A_1) (or (A_2)), we obtain

$$A_{01} = (1-\alpha_0)\Gamma_{01}+\cdots+(1-\alpha_m)\Gamma_{0m}+O(1). \tag{3.57}$$

For A_{02}, note that it is a direct consequence of [20, Theorem 3.2], that

$$d\Delta d + 1 - k = O(d) \tag{3.58}$$

as d tends to zero, a similar argument as before, we can obtain $A_{02} = O(1)$.

Now we assume that $\alpha_0 = \alpha_1 = \cdots = \alpha_{i-1} = 0$. By Lemma 2.1 and (2.14), we have

$$\begin{aligned} &\operatorname{div}\left(\phi h(-h')^{p-1}Y_i^{\alpha_i}\nabla d\right) \\ &\quad = Y_i^{\alpha_i}\operatorname{div}\left(\phi h(-h')^{p-1}\nabla d\right)+\phi h(-h')^{p-1}(Y_i^{\alpha_i})'|\nabla d|^2 \\ &\quad = Y_i^{\alpha_i}\phi h(-h')^{p-1}\left(\Delta d-\frac{k-1}{d}\right)-\alpha_i\phi h(-h')^{p-1}Y_1\cdots Y_{i-1}Y_i^{1+\alpha_i}\cdot\frac{p}{(p-1)c_0}\frac{h'}{h}, \end{aligned} \tag{3.59}$$

that is,

$$\begin{aligned} &\alpha_i\phi(-h')^pY_1\cdots Y_{i-1}Y_i^{1+\alpha_i} \\ &\quad = \frac{(p-1)c_0}{p}\operatorname{div}\left(\phi h(-h')^{p-1}Y_i^{\alpha_i}\nabla d\right)-\frac{(p-1)c_0}{p}\phi h(-h')^{p-1}Y_i^{\alpha_i}\left(\Delta d-\frac{k-1}{d}\right). \end{aligned} \tag{3.60}$$

Hence, we have

$$\begin{aligned}
\alpha_i A_i &= \alpha_i \int_\Omega \theta^p \phi(-h')^p Y_1 \cdots Y_{i-1} Y_i^{1+\alpha_i} Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\
&= \int_\Omega \theta^p \left[\frac{(p-1)c_0}{p} \operatorname{div}\left(\phi h (-h')^{p-1} Y_i^{\alpha_i} \nabla d \right) - \frac{(p-1)c_0}{p} \phi h (-h')^{p-1} Y_i^{\alpha_i} \left(\Delta d - \frac{k-1}{d} \right) \right] \\
&\quad \times Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\
&=: E_1 + E_2.
\end{aligned} \tag{3.61}$$

Integration by parts gives

$$\begin{aligned}
E_1 &= -\frac{(p-1)c_0}{p} \int_\Omega \theta^p \phi h (-h')^{p-1} Y_i^{\alpha_i} \nabla d \cdot \nabla \left(Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \right) \mathrm{d}x \\
&\quad + \frac{(p-1)c_0}{p} \int_\Omega \phi h (-h')^{p-1} Y_i^{\alpha_i} \nabla d \nabla \theta^p Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\
&=: E_{11} + E_{22}.
\end{aligned} \tag{3.62}$$

Since

$$\left(Y_{i+1}^{-1+\alpha_{i+1}} \cdots Y_m^{-1+\alpha_m} \right)' = -\sum_{j=i+1}^{m} (-1+\alpha_j) \frac{ph'}{(p-1)c_0 h} \cdot Y_1 \cdots Y_i Y_{i+1}^{\alpha_{i+1}} \cdots Y_j^{\alpha_j} Y_{j+1}^{-1+\alpha_{j+1}} \tag{3.63}$$

we have

$$\begin{aligned}
E_{11} &= \sum_{j=i+1}^{m} (1-\alpha_j) \int_\Omega \theta^p \phi (-h)^p Y_1 \cdots Y_i Y_{i+1}^{\alpha_{i+1}} \cdots Y_j^{\alpha_j} Y_{j+1}^{-1+\alpha_{j+1}} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\
&= \sum_{j=i+1}^{m} (1-\alpha_j) \Gamma_{ij}.
\end{aligned} \tag{3.64}$$

A similar argument as the estimation of I_2 in Step 1 shows that

$$E_{12} = O(1). \tag{3.65}$$

For E_2, since

$$d\Delta d + (1-k) = O(d) \tag{3.66}$$

as $d \to 0$, by (3.41) and (3.42), we can know that E_2 is bounded uniformly in $\alpha_0, \ldots, \alpha_m$. Hence (3.50) has been proved. To prove (3.51), we use (3.59) once more and proceed similarly, we omit the details.

Step 4. We proceed to estimate I_1:

$$
\begin{aligned}
I_1 &= \int_\Omega \phi\theta^p |\nabla w|^p \mathrm{d}x \\
&\leq \left(\frac{p}{(p-1)c_0}\right)^p \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \left(\frac{(p-1)c_0}{p} + \frac{\overline{\eta}}{p}\right)^p \mathrm{d}x,
\end{aligned}
\tag{3.67}
$$

where $\overline{\eta} = -\alpha_0 + (1-\alpha_1)Y_1 + \cdots + (1-\alpha_m)Y_1 \cdots Y_m$. Since $\overline{\eta}$ is small compared to $(p-1)c_0/p$ we may use Taylor's expansion to obtain

$$
\left(\frac{(p-1)c_0}{p} + \frac{\overline{\eta}}{p}\right)^p \leq \left(\frac{(p-1)c_0}{p}\right)^p + \left(\frac{(p-1)c_0}{p}\right)^{p-1} \overline{\eta} + \frac{p-1}{2p}\left(\frac{(p-1)c_0}{p}\right)^{p-2} \overline{\eta}^2 + C\overline{\eta}^3,
\tag{3.68}
$$

Using this inequality we can bound I_1 by

$$
I_1 \leq I_{10} + I_{11} + I_{12} + I_{13},
\tag{3.69}
$$

where

$$
\begin{aligned}
I_{10} &= \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\
&= \int_\Omega \theta^p \psi h^{p-\alpha_0 p/(p-1)c_0} Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \mathrm{d}x \\
&= \int_\Omega \theta^p \psi |w|^p \mathrm{d}x = \int_\Omega \psi |u|^p \mathrm{d}x, \\
I_{12} &= \left(\frac{p}{(p-1)c_0}\right)^p \frac{p-1}{2p} \left(\frac{(p-1)c_0}{p}\right)^{p-2} \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \overline{\eta}^2 \mathrm{d}x \\
&= \frac{p-1}{2p} \left(\frac{p}{(p-1)c_0}\right)^2 \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \overline{\eta}^2 \mathrm{d}x \\
&= \frac{p}{2(p-1)c_0^2} \int_\Omega \theta^p \phi h^{-\alpha_0 p/(p-1)c_0} (-h')^p Y_1^{-1+\alpha_1} \cdots Y_m^{-1+\alpha_m} \overline{\eta}^2 \mathrm{d}x.
\end{aligned}
\tag{3.70}
$$

We will prove that

$$
I_{11}, I_{13} = O(1) \quad \text{uniformly in } \alpha_0, \alpha_1, \ldots, \alpha_{\mathrm{m}}.
\tag{3.71}
$$

Firstly,

$$\begin{aligned} I_{11} &= \left(\frac{p}{(p-1)c_0}\right)^p\left(\frac{(p-1)c_0}{p}\right)^{p-1}\int_\Omega \theta^p\phi(-h')^p h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\overline{\eta}\,\mathrm{d}x \\ &= \frac{p}{(p-1)c_0}\left[-\alpha_0\int_\Omega \theta^p\phi(-h')^p h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\mathrm{d}x\right. \\ &\qquad + (1-\alpha_1)\int_\Omega \theta^p\phi(-h')^p h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\mathrm{d}x+\cdots \\ &\qquad \left. + (1-\alpha_m)\int_\Omega \theta^p\phi(-h')^p h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{\alpha_1}\cdots\Upsilon_m^{\alpha_m}\mathrm{d}x\right]+O(1) \\ &= \frac{p}{(p-1)c_0}(-\alpha_0 A_0+(1-\alpha_1)\Gamma_{01}+\cdots+(1-\alpha_m)\Gamma_{0m})+O(1). \end{aligned} \tag{3.72}$$

To estimate I_{13}, by (1.15) and $\Upsilon_i = h_i^{-1}$, we have

$$\Upsilon_1\cdots\Upsilon_i \le C\Upsilon_1 \quad \text{for some C} > 0 \tag{3.73}$$

and thus obtain

$$\begin{aligned} I_{13} &\le C\alpha_0^3\int_\Omega \theta^p\phi(-h')^p h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{-1+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\mathrm{d}x \\ &\quad + C\int_\Omega \theta^p\phi(-h')^p h^{-\alpha_0 p/(p-1)c_0}\Upsilon_1^{2+\alpha_1}\cdots\Upsilon_m^{-1+\alpha_m}\mathrm{d}x \\ &=: I'_{13}+I''_{13}. \end{aligned} \tag{3.74}$$

Note that

$$\Upsilon_1^{-2} = \begin{cases} \dfrac{p^2}{(p-1)^2c_0^2}\left[\ln\dfrac{h(r)}{h(D)}\right]^2 & \text{if } (\mathrm{A}_1) \text{ occurs,} \\ \dfrac{p^2}{(p-1)^2c_0^2}\ln h(d) & \text{if } (\mathrm{A}_2) \text{ occurs.} \end{cases} \tag{3.75}$$

hence, if (A_1) occurs, by the coarea formula and (3.42) we have

$$\begin{aligned} I'_{13} &\le C\alpha_0^3\int_\Omega \theta^p\phi h^{-\alpha_0 p/(p-1)c_0}(-h')^p\Upsilon_1^{-2}\mathrm{d}x \\ &\le C\alpha_0^3\int_\Omega \theta^p\phi\left(\int_d^\infty\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-\alpha_0/c_0} \\ &\quad \times\left(\int_d^\infty(\phi r^{k-1})^{-1/(p-1)}\mathrm{d}r\right)^{-1}\left(\phi d^{k-1}\right)^{-p/(p-1)}\left[\ln\frac{h(d)}{h(D)}\right]^2 d^{k-1}\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\leq C\alpha_0^3\int_0^{\delta}\left(\phi r^{k-1}\right)^{1-p/(p-1)}\left(\int_r^{\infty}\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-1-\alpha_0/c_0}\left[\ln\frac{h(r)}{h(D)}\right]^2\mathrm{d}r\\
&= C\alpha_0^3\int_0^{\delta}\left(\int_r^{\infty}\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-1-\alpha_0/c_0}\left[\ln\frac{h(r)}{h(D)}\right]^2\mathrm{d}\left(\int_r^{\infty}\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)\\
&\leq C\alpha_0^2 c_0\int_0^{\delta}\left[\ln\frac{h(r)}{h(D)}\right]^2\mathrm{d}\left(\int_r^{\infty}\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-\alpha_0/c_0}.
\end{aligned}
\tag{3.76}
$$

Denote

$$
s=\left(\int_d^{\infty}\left(\phi r^{k-1}\right)^{-1/(p-1)}\mathrm{d}r\right)^{-\alpha_0/c_0}
\tag{3.77}
$$

and then we have

$$
I_{13}'\leq C\alpha_0^2\int_0^{\delta}\left[C-\frac{(p-1)c_0}{p\alpha_0}\ln s\right]^2\mathrm{d}s\leq O(1).
\tag{3.78}
$$

The boundedness of $Y_1^{2+\alpha_1}Y_2^{-1+\alpha_2}\cdots Y_m^{-1+\alpha_m}$ implies that I_{13}'' is bounded uniformly in the α_i's. Hence we conclude that

$$
\int_{\Omega}\phi|\nabla u|^p\mathrm{d}x-\int_{\Omega}\psi|u|^p\mathrm{d}x\leq I_{12}+O(1)
\tag{3.79}
$$

uniformly in the α_i's. If (A_2) occurs, we can also obtain the above estimate by similar arguments with ∞ being replaced by D.

Step 5. Recalling the definition of $I_{m-1,\phi}[\cdot]$ we obtain from (3.79)

$$
\begin{aligned}
I_{m-1,\phi}[u]&\leq\frac{p}{2(p-1)c_0^2}\int_{\Omega}\theta^p\phi(-h')^p h^{-\alpha_0 p/(p-1)c_0}Y_1^{-1+\alpha_1}\cdots Y_m^{-1+\alpha_m}\\
&\quad\times\left(\overline{\eta}^2-\sum_{i=1}^{m-1}Y_1^2\cdots Y_i^2\right)\mathrm{d}x+O(1)\\
&=\frac{p}{2(p-1)c_0^2}J+O(1),
\end{aligned}
\tag{3.80}
$$

where

$$
\begin{aligned}
J &= \int_\Omega \theta^p \phi(-h')^p h^{-\alpha_0 p/(p-1)c_0} Y_1^{-1+\alpha_1}\cdots Y_m^{-1+\alpha_m} \\
&\quad\times \left(\alpha_0^2 + \sum_{i=1}^{m}(1-\alpha_i)^2 Y_1^2\cdots Y_i^2 - \sum_{i=1}^{m-1} Y_1^2\cdots Y_i^2 - 2\alpha_0 \sum_{j=1}^{m}(1-\alpha_j) Y_1\cdots Y_j \right. \\
&\qquad\quad \left. + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(1-\alpha_i)(1-\alpha_j) Y_1^2\cdots Y_i^2 Y_{i+1}\cdots Y_j \right) \mathrm{d}x \\
&= \alpha_0^2 A_0 + A_m + \sum_{i=1}^{m}\left(\alpha_i^2 - 2\alpha_i\right) A_i - 2\alpha_0 \sum_{j=1}^{m}(1-\alpha_j)\Gamma_{0j} \\
&\quad + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} 2(1-\alpha_i)(1-\alpha_j)\Gamma_{ij}.
\end{aligned} \tag{3.81}
$$

Step 6. We intend to take the limit $\alpha_0 \to 0$ in (3.81). By (3.50) and (3.51), analogues to [15, Step 7], we have

$$
\alpha_0^2 A_0 - 2\alpha_0 \sum_{j=1}^{m}(1-\alpha_j)\Gamma_{0j} = \sum_{i=1}^{m}\left(\alpha_i - \alpha_i^2\right) A_i + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(2\alpha_i - 1)(1-\alpha_j)\Gamma_{ij} + O(1). \tag{3.82}
$$

All the terms in the last expression remain bounded as $\alpha_0 \to 0$, taking the limit in (3.81) we obtain

$$
J = A_m - \sum_{i=1}^{m}\alpha_i A_i + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(1-\alpha_j)\Gamma_{ij} + O(1) \quad (\alpha_0 = 0), \tag{3.83}
$$

where the $O(1)$ is uniform with respect to $\alpha_1, \ldots, \alpha_m$. Next taking $\alpha_i \to 0, \ldots, \alpha_{m-1} \to 0$ in order, the same argument as before gives

$$
J = (1-\alpha_m) A_m + O(1) \quad (\alpha_0 = \alpha_1 = \cdots = \alpha_{m-1} = 0) \tag{3.84}
$$

uniformly in α_m. Combing (3.80) and (3.84), we conclude that

$$
\frac{I_{m-1,\phi}[u]}{\int_\Omega \psi h_1^{-2}\cdots h_m^{-2}|u|^p \mathrm{d}x} \le \frac{p}{2(p-1)c_0^2}\,\frac{(1-\alpha_m)A_m + O(1)}{A_m} \longrightarrow \frac{p}{2(p-1)c_0^2} \tag{3.85}
$$

as $\alpha_m \to 0$, since $A_m \to \infty$ as $\alpha_m \to 0$ by (3.48). This completes the proof. □

Acknowledgment

This project was supported by the NSFC (no. 10771074, 10726060) and the NSF (no. 04020077).

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