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Research Article

Weak Contractions, Common Fixed Points, and Invariant Approximations

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The existence of common fixed points is established for the mappings, where T is (f, θ, L)-weak contraction on a nonempty subset of a Banach space. As application, some results on the invariant best approximation are proved. Our results unify and substantially improve several recent results given by some authors.

1. Introduction and Preliminaries

Let M be a subset of a normed space $(X, \|\cdot\|)$. The set

$$P_M(u) = \{x \in M : \|x - u\| = \operatorname{dist}(u, M)\} \tag{1.1}$$

is called *the set of best approximants* to $u \in X$ out of M, where

$$\operatorname{dist}(u, M) = \inf\{\|y - u\| : y \in M\}. \tag{1.2}$$

We denote $\mathbb{N}$ and $\operatorname{cl}(M)$ (resp., $\operatorname{wcl}(M)$) by the set of positive integers and the closure (resp., weak closure) of a set M in X, respectively. Let $f,\ T : M \to M$ be mappings. The set of fixed points of T is denoted by $F(T)$. A point $x \in M$ is a coincidence point (resp., common fixed point) of f and T if $fx = Tx$ (resp., $x = fx = Tx$). The set of coincidence points of f and T is denoted by $C(f, T)$.

The pair $\{f, T\}$ is said to be

(1) *commuting* [1] if $Tfx = fTx$ for all $x \in M$,

(2) *compatible* [2, 3] if $\lim_{n\to\infty}\|Tfx_n - fTx_n\| = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty}Tx_n = \lim_{n\to\infty}fx_n = t$ for some t in M,

(3) *weakly compatible* if they commute at their coincidence points, that is, if $fTx = Tfx$ whenever $fx = Tx$,

(4) a *Banach operator pair* if the set $F(f)$ is T-invariant, namely, $T(F(f)) \subseteq F(f)$.

Obviously, the commuting pair (T, f) is a Banach operator pair, but converse is not true in general (see [4, 5].) If (T, f) is a Banach operator pair, then (f, T) needs not be a Banach operator pair (see [4, Example 1]).

The set M is said to be *q-starshaped* with $q \in M$ if the segment $[q, x] = \{(1-k)q + kx : 0 \le k \le 1\}$ joining q to x is contained in M for all $x \in M$. The mapping f defined on a q-starshaped set M is said to be *affine* if

$$f\big((1-k)q + kx\big) = (1-k)fq + kfx, \quad \forall x \in M. \tag{1.3}$$

Suppose that the set M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then T and f are said to be

(5) *C_q-commuting* [3, 6] if $fTx = Tfx$ for all $x \in C_q(f, T)$, where $C_q(f, T) = \cup\{C(f, T_k) : 0 \le k \le 1\}$ where $T_kx = (1-k)q + kTx$,

(6) *pointwise R-subweakly commuting* [7] if, for given $x \in M$, there exists a real number $R > 0$ such that $\|fTx - Tfx\| \le R\,\mathrm{dist}(fx, [q, Tx])$,

(7) *R-subweakly commuting* on M [8] if, for all $x \in M$, there exists a real number $R > 0$ such that $\|fTx - Tfx\| \le R\,\mathrm{dist}(fx, [q, Tx])$.

In 1963, Meinardus [9] employed Schauder's fixed point theorem to prove a result regarding invariant approximation. Further, some generalizations of the result of Meinardus were obtained by Habiniak [10], Jungck and Sessa [11], and Singh [12].

Since then, Al-Thagafi [13] extended these works and proved some results on invariant approximations for commuting mappings. Hussain and Jungck [8], Hussain [5], Jungck and Hussain [3], O'Regan and Hussain [7], Pathak and Hussain [14], and Pathak et al. [15] extended the work of Al-Thagafi [13] for more general noncommuting mappings.

Recently, Chen and Li [4] introduced the class of Banach operator pairs as a new class of noncommuting mappings and it has been further studied by Hussain [5], Khan and Akbar [16], and Pathak and Hussain [14].

In this paper, we extend and improve the recent common fixed point and invariant approximation results of Al-Thagafi [13], Al-Thagafi and Shahzad [17], Berinde [18], Chen and Li [4], Habiniak [10], Jungck and Sessa [11], Pathak and Hussain [14], and Singh [12] to the class of (f, θ, L)-weak contractions. The applications of the fixed point theorems are remarkable in diverse disciplines of mathematics, statistics, engineering, and economics in dealing with the problems arising in approximation theory, potential theory, game theory, theory of differential equations, theory of integral equations, and others (see [14, 15, 19, 20]).

2. Main Results

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *weak contraction* if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx), \quad \forall x, y \in X. \tag{2.1}$$

Remark 2.1. Due to the symmetry of the distance, the weak contraction condition (2.1) includes the following:

$$d(Tx, Ty) \leq \theta d(x, y) + L(x, Ty), \quad \forall x, y \in X, \tag{2.2}$$

which is obtained from (2.1) by formally replacing $d(Tx, Ty)$, $d(x, y)$ by $d(Ty, Tx)$, $d(y, x)$, respectively, and then interchanging x and y.

Consequently, in order to check the weak contraction of T, it is necessary to check both (2.1) and (2.2). Obviously, a Banach contraction satisfies (2.1) and hence is a weak contraction. Some examples of weak contractions are given in [18, 21, 22]. The next example shows that a weak contraction needs not to be continuous.

Example 2.2 (see [18, 22]). Let $[0, 1]$ be the unit interval with the usual norm and let $T : [0, 1] \to [0, 1]$ be given by $Tx = 2/3$ for all $x \in [0, 1)$ and $T1 = 0$. Then T satisfies the inequality (2.1) with $1 > \theta \geq 2/3$ and $L \geq \theta$ and T has a unique fixed point $x = 2/3$, but T is not continuous.

Let f be a self-mapping on X. A mapping $T : X \to X$ is said to be *f-weak contraction* or *(f, θ, L)-weak contraction* if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \theta d(fx, fy) + Ld(fy, Tx), \quad \forall x, y \in X. \tag{2.3}$$

Berinde [18] introduced the notion of a (θ, L)-weak contraction and proved that a lot of the well-known contractive conditions do imply the (θ, L)-weak contraction. The concept of (θ, L)-weak contraction does not ask $\theta + L$ to be less than 1 as happens in many kinds of fixed point theorems for the contractive conditions that involve one or more of the displacements $d(x, y)$, $d(x, Tx)$, $d(y, Ty)$, $d(x, Ty)$, $d(y, Tx)$. For more details, we refer to [18, 21] and references cited in these papers.

The following result is a consequence of the main theorem of Berinde [18].

Lemma 2.3. *Let M be a nonempty subset of a metric space (X, d) and let T be a self-mapping of M. Assume that $\operatorname{cl} T(M) \subset M$, $\operatorname{cl} T(M)$ is complete, and T is a (θ, L)-weak contraction. Then $M \cap F(T)$ is nonempty.*

Theorem 2.4. *Let M be a nonempty subset of a metric space (X, d) and let T, f be self-mappings of M. Assume that $F(f)$ is nonempty, $\operatorname{cl} T(F(f)) \subseteq F(f)$, $\operatorname{cl}(T(M))$ is complete, and T is an (f, θ, L)-weak contraction. Then $M \cap F(T) \cap F(f) \neq \emptyset$.*

Proof. Since $\mathrm{cl}(T(F(f)))$ is a closed subset of $\mathrm{cl}(T(M))$, $\mathrm{cl}(T(F(f)))$ is complete. Further, by the (f,θ,L)-weak contraction of T, for all $x,y \in F(f)$, we have

$$d(Tx,Ty) \le \theta d(fx,fy) + L \cdot d(fy,Tx) = d(x,y) + L \cdot d(y,Tx). \tag{2.4}$$

Hence T is a (θ,L)-weak contraction on $F(f)$ and $\mathrm{cl}\,T(F(f)) \subseteq F(f)$. Therefore, by Lemma 2.3, T has a fixed point z in $F(f)$ and so $M \cap F(T) \cap F(f) \neq \varnothing$. □

Corollary 2.5. *Let M be a nonempty subset of a metric space (X,d) and let (T,f) be a Banach operator pair on M. Assume that $\mathrm{cl}(T(M))$ is complete, T is (f,θ,L)-weak contraction, and $F(f)$ is nonempty and closed. Then $M \cap F(T) \cap F(f) \neq \varnothing$.*

In Theorem 2.4 and Corollary 2.5, if $L = 0$, then we easily obtain the following result, which improves Lemma 3.1 of Chen and Li [4].

Corollary 2.6 (see [17, Theorem 2.2]). *Let M be a nonempty subset of a metric space (X,d) and let T,f be self-mappings of M. Assume that $F(f)$ is nonempty, $\mathrm{cl}(T(F(f))) \subseteq F(f)$, $\mathrm{cl}(T(M))$ is complete, and T is an f-contraction. Then $M \cap F(T) \cap F(f)$ is a singleton.*

The following result properly contains [4, Theorems 3.2-3.3] and improves [13, Theorem 2.2], [10, Theorem 4], and [11, Theorem 6].

Theorem 2.7. *Let M be a nonempty subset of a normed (resp., Banach) space X and let T, f be self-mappings of M. Suppose that $F(f)$ is q-starshaped, $\mathrm{cl}\,T(F(f)) \subseteq F(f)$ (resp., $\mathrm{wcl}\,T(F(f)) \subseteq F(f)$), $\mathrm{cl}(T(M))$ is compact (resp., $\mathrm{wcl}(T(M))$ is weakly compact, and either $I - T$ is demiclosed at 0 or X satisfies Opial's condition, where I stands for the identity mapping), and there exists a constant $L \ge 0$ such that*

$$\|Tx - Ty\| \le \|fx - fy\| + L \cdot \mathrm{dist}\big(fy, [q,Tx]\big), \quad \forall x,y \in M. \tag{2.5}$$

Then $M \cap F(T) \cap F(f) \neq \varnothing$.

Proof. For each $n \in \mathbb{N}$, define $T_n : F(f) \to F(f)$ by $T_n x = (1-k_n)q + k_n Tx$ for all $x \in F(f)$ and a fixed sequence $\{k_n\}$ of real numbers $(0 < k_n < 1)$ converging to 1. Since $F(f)$ is q-starshaped and $\mathrm{cl}\,T(F(f)) \subseteq F(f)$ (resp., $\mathrm{wcl}\,T(F(f)) \subseteq F(f)$), we have $\mathrm{cl}\,T_n(F(f)) \subseteq F(f)$ (resp., $\mathrm{wcl}\,T_n(F(f)) \subseteq F(f)$) for each $n \in \mathbb{N}$. Also, by the inequality (2.5),

$$\begin{aligned}\big\|T_n x - T_n y\big\| &= k_n\|Tx - Ty\| \\ &\le k_n\|fx - fy\| + k_n L \cdot \mathrm{dist}\big(fy, [q,Tx]\big) \\ &\le k_n\|fx - fy\| + L_n \cdot \big\|fy - T_n x\big\|\end{aligned} \tag{2.6}$$

for all $x,y \in F(f)$, $L_n := k_n L$, and $0 < k_n < 1$. Thus, for $n \in \mathbb{N}$, T_n is a (f,k_n,L_n)-weak contraction, where $L_n \ge 0$.

If $\mathrm{cl}(T(M))$ is compact, then, for each $n \in \mathbb{N}$, $\mathrm{cl}(T_n(M))$ is compact and hence complete. By Theorem 2.4, for each $n \in \mathbb{N}$, there exists $x_n \in F(f)$ such that $x_n = fx_n = T_n x_n$. The compactness of $\mathrm{cl}(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that

$Tx_m \to z \in \mathrm{cl}(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f))$ and $\mathrm{cl}\,T(F(f)) \subseteq F(f)$, we have $z \in F(f)$. Further, it follows that

$$x_m = T_m x_m = (1-k_m)q + k_m T x_m \longrightarrow z, \quad \|x_m - Tx_m\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.7}$$

Moreover, we have

$$\begin{aligned}\|Tx_m - Tz\| &\le \|fx_m - fz\| + L \cdot \operatorname{dist}(fz, [q, Tx_m]) \\ &= \|x_m - z\| + L \cdot \operatorname{dist}(z, [q, Tx_m]) \\ &\le \|x_m - Tx_m\| + \|Tx_m - z\| + L \cdot \|z - Tx_m\|.\end{aligned} \tag{2.8}$$

Taking the limit as $m \to \infty$, we get $z = Tz$ and so $M \cap F(T) \cap F(f) \neq \emptyset$.

Next, the weak compactness of $\mathrm{wcl}(T(M))$ implies that $\mathrm{wcl}(T_n(M))$ is weakly compact and hence complete due to completeness of X (see [3]). From Theorem 2.4, for each $n \in \mathbb{N}$, there exists $x_n \in F(f)$ such that $x_n = fx_n = T_n x_n$. The weak compactness of $\mathrm{wcl}(T(M))$ implies that there is a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ converging weakly to $y \in \mathrm{wcl}(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f))$, we have $y \in \mathrm{wcl}(T(F(f))) \subseteq F(f)$. Also, we have $x_m - Tx_m \to 0$ as $m \to \infty$. If $I - T$ is demiclosed at 0, then $y = Ty$ and so $M \cap F(T) \cap F(f) \neq \emptyset$.

If $fy \neq Ty$, then we have

$$\begin{aligned}&\liminf_{m\to\infty}\|fx_m - fy\| \\ &\quad< \liminf_{m\to\infty}\|fx_m - Ty\| \\ &\quad\le \liminf_{m\to\infty}\|fx_m - Tx_m\| + \liminf_{m\to\infty}\|Tx_m - Ty\| \\ &\quad\le \liminf_{m\to\infty}\|fx_m - Tx_m\| + \liminf_{m\to\infty}\|fx_m - fy\| + \liminf_{m\to\infty} L \cdot \operatorname{dist}(fy, [q, Tx_m]) \\ &\quad\le \liminf_{m\to\infty}\|fx_m - fy\| + \liminf_{m\to\infty}\|fx_m - Tx_m\| + L \cdot \liminf_{m\to\infty}\|y - Tx_m\| \\ &\quad= \liminf_{m\to\infty}\|fx_m - fy\|,\end{aligned} \tag{2.9}$$

which is a contradiction. Thus $Ty = fy = y$ and hence $M \cap F(T) \cap F(f) \neq \emptyset$. This completes the proof. □

Obviously, f-nonexpansive mappings satisfy the inequality (2.5) and so we obtain the following.

Corollary 2.8 (see [17, Theorem 2.4]). *Let M be a nonempty subset of a normed (resp., Banach) space X and let T, f be self-mappings of M. Suppose that $F(f)$ is q-starshaped,* $\mathrm{cl}\,T(F(f)) \subseteq F(f)$

(resp., $\operatorname{wcl} T(F(f)) \subseteq F(f)$*),* $\operatorname{cl}(T(M))$ *is compact, (resp.,* $\operatorname{wcl}(T(M))$ *is weakly compact, and either* $I - T$ *is demiclosed at* 0 *or* X *satisfies Opial's condition), and* T *is* f*-nonexpansive on* M*. Then* $M \cap F(T) \cap F(f) \neq \varnothing$*.*

Corollary 2.9 (see [4, Theorems 3.2-3.3]). *Let* M *be a nonempty subset of a normed (resp., Banach) space* X *and let* T*,* f *be self-mappings of* M*. Suppose that* $F(f)$ *is* q*-starshaped and closed (resp., weakly closed),* $\operatorname{cl}(T(M))$ *is compact (resp.,* $\operatorname{wcl}(T(M))$ *is weakly compact, and either* $I - T$ *is demiclosed at* 0 *or* X *satisfies Opial's condition),* (T, f) *is a Banach operator pair, and* T *is* f*-nonexpansive on* M*. Then* $M \cap F(T) \cap F(f) \neq \varnothing$*.*

Corollary 2.10 (see [13, Theorem 2.1]). *Let* M *be a nonempty closed and* q*-starshaped subset of a normed space* X *and let* T*,* f *be self-mappings of* M *such that* $T(M) \subseteq f(M)$*. Suppose that* T *commutes with* f *and* $q \in F(f)$*. If* $\operatorname{cl}(T(M))$ *is compact,* f *is continuous, linear, and* T *is* f*-nonexpansive on* M*, then* $M \cap F(T) \cap F(f) \neq \varnothing$*.*

Let $C = P_M(u) \cap C_M^f(u)$, where $C_M^f(u) = \{x \in M : fx \in P_M(u)\}$.

Corollary 2.11. *Let* X *be a normed (resp., Banach) space* X *and let* T*,* f *be self-mappings of* X*. If* $u \in X$*,* $D \subseteq C$*,* $D_0 := D \cap F(f)$ *is* q*-starshaped,* $\operatorname{cl}(T(D_0)) \subseteq D_0$ *(resp.,* $\operatorname{wcl}(T(D_0)) \subseteq D_0$*),* $\operatorname{cl}(T(D))$ *is compact, (resp.,* $\operatorname{wcl}(T(D))$ *is weakly compact, and* $I - T$ *is demiclosed at* 0*). If the inequality* (2.5) *holds for all* $x, y \in D$*, then* $P_M(u) \cap F(T) \cap F(f) \neq \varnothing$*.*

Corollary 2.12. *Let* X *be a normed (resp., Banach) space* X *and let* T*,* f *be self-mappings of* X*. If* $u \in X$*,* $D \subseteq P_M(u)$*,* $D_0 := D \cap F(f)$ *is* q*-starshaped,* $\operatorname{cl}(T(D_0)) \subseteq D_0$ *(resp.,* $\operatorname{wcl}(T(D_0)) \subseteq D_0$*),* $\operatorname{cl}(T(D))$ *is compact, (resp.,* $\operatorname{wcl}(T(D))$ *is weakly compact, and* $I - T$ *is demiclosed at* 0*). If the inequality* (2.5) *holds for all* $x, y \in D$*, then* $P_M(u) \cap F(T) \cap F(f) \neq \varnothing$*.*

Corollary 2.13 (see [11, Theorem 7]). *Let* f*,* T *be self-mappings of a Banach space* X *with* $u \in F(f) \cap F(T)$ *and* $M \subset X$ *with* $T(\partial M) \subset M$*. Suppose that* $D = P_M(u)$ *is* q*-starshaped with* $q \in F(f)$*,* $f(D) = D$*, and* f *is affine, continuous in the weak and strong topology on* D*. If* f *and* T *are commuting on* D *and* T *is* f*-nonexpansive on* $D \cup \{u\}$*, then* $P_M(u) \cap F(T) \cap F(f) \neq \varnothing$ *provided either* (i) D *is weakly compact and* $(f - T)$ *is demiclosed or* (ii) D *is weakly compact and* X *satisfies Opial's condition.*

Remark 2.14. Corollary 2.5 in [17] and Theorems 4.1-4.2 of Chen and Li [4] are special cases of Corollaries 2.11-2.12

We denote $\Im_0$ by the class of closed convex subsets of X containing 0. For any $M \in \Im_0$, we define $M_u = \{x \in M : \|x\| \leq 2\|u\|\}$. It is clear that $P_M(u) \subset M_u \in \Im_0$ (see [8, 13]).

Theorem 2.15. *Let* f*,* T *be self-mappings of a normed (resp., Banach) space* X*. If* $u \in X$ *and* $M \in \Im_0$ *such that* $T(M_u) \subseteq M$*,* $\operatorname{cl}(T(M_u))$ *is compact (resp.,* $\operatorname{wcl}(T(M_u))$ *is weakly compact), and* $\|Tx - u\| \leq \|x - u\|$ *for all* $x \in M_u$*, then* $P_M(u)$ *is nonempty closed and convex with* $T(P_M(u)) \subseteq P_M(u)$*. If, in addition,* $D \subseteq P_M(u)$*,* $D_0 := D \cap F(f)$ *is* q*-starshaped,* $\operatorname{cl}(T(D_0)) \subseteq D_0$ *(resp.,* $\operatorname{wcl}(T(D_0)) \subseteq D_0$*, and* $I - T$ *is demiclosed at* 0*), and the inequality* (2.5) *holds for all* $x, y \in D$*, then* $P_M(u) \cap F(T) \cap F(f) \neq \varnothing$*.*

Proof. We may assume that $u \notin M$. If $x \in M \setminus M_u$, then $\|x\| > 2\|u\|$. Note that

$$\|x - u\| \geq \|x\| - \|u\| > \|u\| \geq \operatorname{dist}(u, M). \tag{2.10}$$

Thus $\mathrm{dist}(u, M_u) = \mathrm{dist}(u, M) \leq \|u\|$. If $\mathrm{cl}(T(M_u))$ is compact, then, by the continuity of the norm, we get $\|z - u\| = \mathrm{dist}(u, \mathrm{cl}(T(M_u)))$ for some $z \in \mathrm{cl}(T(M_u))$. If we assume that $\mathrm{wcl}(T(M_u))$ is weakly compact, then, using [23, Lemma 5.5, page 192], we can show the existence of a $z \in \mathrm{wcl}(T(M_u))$ such that $\mathrm{dist}(u, \mathrm{wcl}(T(M_u))) = \|z - u\|$. Thus, in both cases, we have

$$\mathrm{dist}(u, M_u) \leq \mathrm{dist}(u, \mathrm{cl}\, T(M_u)) \leq \mathrm{dist}(u, T(M_u)) \leq \|Tx - u\| \leq \|x - u\| \tag{2.11}$$

for all $x \in M_u$. Hence $\|z - u\| = \mathrm{dist}(u, M)$ and so $P_M(u)$ is nonempty closed and convex with $T(P_M(u)) \subseteq P_M(u)$. The compactness of $\mathrm{cl}(T(M_u))$ (resp., the weak compactness of $\mathrm{wcl}(T(M_u))$) implies that $\mathrm{cl}(T(D))$ is compact (resp., $\mathrm{wcl}(T(D))$ is weakly compact). Therefore, the result now follows from Corollary 2.12. This completes the proof. □

Corollary 2.16. *Let f, T be self-mappings of a normed (resp., Banach) space X. If $u \in X$ and $M \in \Im_0$ such that $T(M_u) \subseteq M$, $\mathrm{cl}(T(M_u))$ is compact (resp., $\mathrm{wcl}(T(M_u))$ is weakly compact), and $\|Tx - u\| \leq \|x - u\|$ for all $x \in M_u$, then $P_M(u)$ is nonempty closed and convex with $T(P_M(u)) \subseteq P_M(u)$. If, in addition, $D \subseteq P_M(u)$, $D_0 := D \cap F(f)$ is q-starshaped and closed (resp., weakly closed and $I - T$ is demiclosed at 0), (T, f) is a Banach operator pair on D, and the inequality* (2.5) *holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \neq \varnothing$.*

Remark 2.17. Theorem 2.15 and Corollary 2.16 extend [13, Theorems 4.1 and 4.2], [17, Theorem 2.6], and [10, Theorem 8].

Banach's Fixed Point Theorem states that if (X, d) is a complete metric space, K is a nonempty closed subset of X, and $T : K \to K$ is a self-mapping satisfying the following condition: there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y), \quad \forall x, y \in K, \tag{2.12}$$

then T has a unique fixed point, say z in K, and the Picard iterative sequence $\{T^n x\}$ converges to the point z for all $x \in K$. Since then, Ćirić [24] introduced and studied self-mappings on K satisfying the following condition: there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda m(x, y), \quad \forall x, y \in K, \tag{2.13}$$

where

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{2.14}$$

Further, many investigations were developed by Berinde [19], Jungck [1, 2], Hussain and Jungck [8], Hussain and Rhoades [6], O'Regan and Hussain [7], and many other mathematicians (see [14, 25] and references therein). Recently, Jungck and Hussain [3] proved the following extension of the result of Ćirić [24].

Theorem 2.18 (see [3, Theorem 2.1]). *Let M be a nonempty subset of a metric space (X, d) and let f, g be self-mappings of M. Assume that* $\mathrm{cl}\, f(M) \subset g(M)$, $\mathrm{cl}\, f(M)$ *is complete, and f, g satisfy the following condition: there exists $h \in [0,1)$ such that*

$$d(fx, fy) \leq h \ \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}, \tag{2.15}$$

for all $x, y \in M$. Then $C(f, g) \neq \varnothing$.

The following result (Theorem 2.19) properly contains [17, Theorem 3.3], [4, Theorems 3.2-3.3], [5, Theorem 2.11], and [14, Theorem 2.2]. The proof is analogous to the proof of Theorem 2.7. In fact, instead of applying Theorem 2.4, we apply Theorem 2.18 to get the conclusion.

Theorem 2.19. *Let M be a nonempty subset of a normed (resp., Banach) space X and let T, f, g be self-mappings of M. Suppose that $F(f) \cap F(g)$ is q-starshaped,* $\mathrm{cl}\, T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ *(resp.,* $\mathrm{wcl}\, T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$*),* $\mathrm{cl}(T(M))$ *is compact (resp.,* $\mathrm{wcl}(T(M))$ *is weakly compact), and T is continuous on M (resp., $I - T$ is demiclosed at 0). If the following condition holds:*

$$\begin{aligned}\|Tx - Ty\| \leq \max\{&\|fx - gy\|, \operatorname{dist}(fx, [q, Tx]), \operatorname{dist}(gy, [q, Ty]),\\ &\operatorname{dist}(gy, [q, Tx]), \operatorname{dist}(fx, [q, Ty])\}\end{aligned} \tag{2.16}$$

for all $x, y \in M$, then $M \cap F(T) \cap F(f) \cap F(g) \neq \varnothing$.

Theorem 2.20. *Let f, g, T be self-mappings of a Banach space X with $u \in F(T) \cap F(f) \cap F(g)$ and $M \in \Im_0$ such that $T(M_u) \subset f(M) \subset M = g(M)$. Suppose that $\|fx - u\| \leq \|x - u\|$, $\|gx - u\| = \|x - u\|$, $\|Tx - u\| \leq \|fx - gu\|$ for all $x \in M$, and* $\mathrm{cl}(f(M_u))$ *is compact. Then one has the following:*

(1) *$P_M(u)$ is nonempty closed and convex,*

(2) *$T(P_M(u)) \subset f(P_M(u)) \subset P_M(u) = g(P_M(u))$,*

(3) *$P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \varnothing$ provided T is continuous, $F(g)$ is q-starshaped,* $\mathrm{cl}(f(F(g))) \subseteq F(g)$*, and the pair (f, g) satisfies the inequality* (2.5) *for all $x, y \in P_M(u)$, $F(f)$ is q-starshaped with $q \in F(f) \cap F(g) \cap P_M(u)$,* $\mathrm{cl}\, T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$*, and the inequality* (2.16) *holds for all $q \in F(f) \cap F(g)$ and $x, y \in P_M(u)$.*

Proof. (1) and (2) follow from [5, 8, Theorem 2.14]. By (2), the compactness of $\mathrm{cl}(f(M_u))$ implies that $\mathrm{cl}(f(P_M(u)))$ and $\mathrm{cl}(T(P_M(u)))$ is compact. Theorem 2.7 implies that $F(f) \cap F(g) \cap P_M(u) \neq \varnothing$. Further, $F(f) \cap F(g)$ is q-starshaped with $q \in F(f) \cap F(g) \cap P_M(u)$. Therefore, the conclusion now follows from Theorem 2.19 applied to $P_M(u)$. □

Remark 2.21. (1) Theorem 2.20 extends [13, Theorem 4.1], [10, Theorem 8], [5, Theorem 2.13], [8, Theorem 2.14], and [14, Theorem 2.11].

(2) Theorems 2.7–2.16 represent very strong variants of the results in [3, 8, 11, 13] in the sense that the commutativity or compatibility of the mappings T and f is replaced by the hypothesis that (T, f) is a Banach operator pair, f needs not be linear or affine, and T needs not be f-nonexpansive.

(3) The Banach operator pairs are different from those of weakly compatible, C_q-commuting and R-subweakly commuting mappings and so our results are different from those in [3, 7, 8, 17]. Consider $M = \mathbb{R}^2$ with the norm $\|(x, y)\| = |x| + |y|$ for all $(x, y) \in M$. Define two self-mappings T and f on M as follows:

$$\begin{aligned} T(x,y) &= \left(x^3 + x - 1, \frac{\sqrt[3]{x^2 + y^3 - 1}}{3}\right), \\ f(x,y) &= \left(x^3 + x - 1, \sqrt[3]{x^2 + y^3 - 1}\right). \end{aligned} \tag{2.17}$$

Then we have the following:

$$\begin{gathered} F(T) = \{(1,0)\}, \qquad F(f) = \{(1,y) : y \in R^1\}, \\ C(T,f) = \{(x,y) : y = \sqrt[3]{1 - x^2},\ x \in R^1\}, \\ T(F(f)) = \{T(1,y) : y \in R^1\} = \left\{\left(1, \frac{y}{3}\right) : y \in R^1\right\} \subseteq \{(1,y) : y \in R^1\} = F(f). \end{gathered} \tag{2.18}$$

Thus (T, f) is a Banach operator pair. It is easy to see that T is (f, θ, L)-weak contraction and T, f do not commute on the set $C(T, f)$, and so are not weakly compatible. Clearly, f is not affine or linear, $F(f)$ is convex and $(1, 0)$ is a common fixed point of T and f.

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