Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 357349, 5 pages
doi:10.1155/2009/357349

Research Article

A Note on Hölder Type Inequality for the Fermionic p-Adic Invariant q-Integral

Lee-Chae Jang

Department of Mathematics and Computer Science, KonKuk University, Chungju 380-701, South Korea

Correspondence should be addressed to Lee-Chae Jang, leechae.jang@kku.ac.kr

Received 11 February 2009; Accepted 22 April 2009

Recommended by Kunquan Lan

The purpose of this paper is to find Hölder type inequality for the fermionic p-adic invariant q-integral which was defined by Kim (2008).

1. Introduction

Let p be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{Q}$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the rational number field, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. For a fixed positive integer d with $(p,d)=1$, let

$$\begin{gathered} X = X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\,\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \ \left(\operatorname{mod} dp^N\right)\right\}, \end{gathered} \tag{1.1}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$ (cf. [1–24]).

Let $\mathbb{N}$ be the set of natural numbers. In this paper we assume that $q \in \mathbb{C}_p$, with $|1-q|_p < p^{-1/(p-1)}$, which implies that $q^x = \exp(x \log q)$ for $|p|_p \le 1$. We also use the notations

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \tag{1.2}$$

for all $x \in \mathbb{Z}_p$. For any positive integer N, the distribution is defined by

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{\left[dp^N\right]_q}. \tag{1.3}$$

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x,y) = (f(x) - f(y))/(x-y)$ have a limit $l = f'(a)$ as $(x,y) \to (a,a)$ (cf. [1–24]).

For $f \in UD(\mathbb{Z}_p)$, the above distribution μ_q yields the bosonic p-adic invariant q-integral as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty}\frac{1}{\left[p^N\right]_q}\sum_{x=0}^{p^N-1} f(x)q^x, \tag{1.4}$$

representing the p-adic q-analogue of the Riemann integral for f. In the sense of fermionic, let us define the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty}\frac{1}{\left[p^N\right]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.5}$$

for $f \in UD(\mathbb{Z}_p)$ (see [16]). Now, we consider the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ as

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x). \tag{1.6}$$

From (1.5) we note that

$$I_{-1}(f) + I_{-1}(f) = 2f(0), \tag{1.7}$$

where $f_1(x) = f(x+1)$ (see [16]).

We also introduce the classical Hölder inequality for the Lebesgue integral in [25].

Theorem 1.1. *Let $m, m' \in \mathbb{Q}$ with $1/m + 1/m' = 1$. If $f \in L^m$ and $g \in L^{m'}$, then $f \cdot g \in L^1$ and*

$$\int |fg|dx \le \|f\|_m \|g\|_{m'} \tag{1.8}$$

where $f \in L^m \Leftrightarrow \int |f|^m dx < \infty$ and $g \in L^{m'} \Leftrightarrow \int |g|^{m'} dx < \infty$ and $\|f\|_m = \{\int |f|^m dx\}^{1/m}$.

The purpose of this paper is to find Hölder type inequality for the fermionic p-adic invariant q-integral I_{-1}.

2. Hölder Type Inequality for Fermionic p-Adic Invariant q-Integrals

In order to investigate the Hölder type inequality for I_{-1}, we introduce the new concept of the inequality as follows.

Definition 2.1. For $f, g \in UD(\mathbb{Z}_p)$, we define the inequality on $UD(\mathbb{Z}_p)$ (resp., $\mathbb{C}_p$) as follows. For $f, g \in UD(\mathbb{Z}_p)$ (resp., $x, y \in \mathbb{C}_p$), $f \leq_p g$(resp., $x \leq_p y$) if and only if $|f|_p \leq |g|_p$ (resp., $|x|_p \leq |y|_p$).

Let $m, m' \in \mathbb{Q}$ with $1/m + 1/m' = 1$. By substituting $f(x) = q^x$ and $g(x) = e^{xt}$ into (1.3), we obtain the following equation:

$$\int_{\mathbb{Z}_p} f(x)g(x)\mu_{-1}(x) = \int_{\mathbb{Z}_p} (qe^t)^x d\mu_{-1}(x) = \frac{2}{qe^t+1}, \tag{2.1}$$

$$\int_{\mathbb{Z}_p} f(x)^m \mu_{-1}(x) = \int_{\mathbb{Z}_p} q^{mx} d\mu_{-1}(x) = \frac{2}{q^m+1}, \tag{2.2}$$

$$\int_{\mathbb{Z}_p} g(x)^{m'} \mu_{-1}(x) = \int_{\mathbb{Z}_p} e^{m'xt} d\mu_{-1}(x) = \frac{2}{e^{m't}+1}. \tag{2.3}$$

From (2.1), (2.2), and (2.3), we derive

$$\begin{aligned}
\frac{\int_{\mathbb{Z}_p} f(x)g(x)d\mu_{-1}(x)}{\left\{\int_{\mathbb{Z}_p} f(x)^m d\mu_{-1}\right\}^{1/m}\left\{\int_{\mathbb{Z}_p} g(x)^{m'} d\mu_{-1}\right\}^{1/m'}} &= \frac{(e^{mt}+1)^{1/m}\left(q^{m'}+1\right)^{1/m'}}{qe^t+1} \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n} \binom{\frac{1}{m}}{l} e^{lmt} \binom{\frac{1}{m'}}{n-l} q^{(n-l)m'} \frac{1}{qe^t+1} \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n} \binom{\frac{1}{m}}{l} \binom{\frac{1}{m'}}{n-l} q^{(n-l)m'} \frac{e^{lmt}}{qe^t+1}.
\end{aligned} \tag{2.4}$$

We remark that the nth Frobenius-Euler numbers $H_n(q)$ and the nth Frobenius-Euler polynomials $H_n(q, x)$ attached to algebraic number $q(\neq 1)$ may be defined by the exponential generating functions (see [16]):

$$\frac{1-q}{e^t-q} = \sum_{n=0}^{\infty} H_n(q)\frac{t^n}{n!}, \tag{2.5}$$

$$\frac{1-q}{e^t-q}e^{xt} = \sum_{n=0}^{\infty} H_n(q, x)\frac{t^n}{n!}. \tag{2.6}$$

Then, it is easy to see that

$$\frac{[2]_q e^{mlt}}{qe^x+1}=\sum_{k=0}^{\infty}H_n\left(-q^{-1},ml\right)\frac{t^k}{k!}. \tag{2.7}$$

From (2.4) and (2.7), we have the following theorem.

Theorem 2.2. *Let $m, m' \in \mathbb{Q}$ with $1/m+1/m'=1$. If one takes $f(x)=q^x$ and $g(x)=e^{xt}$, then one has*

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}f(x)g(x)d\mu_{-1}(x)}{\left\{\int_{\mathbb{Z}_p}f(x)^m d\mu_{-1}\right\}^{1/m}\left\{\int_{\mathbb{Z}_p}g(x)^{m'}d\mu_{-1}\right\}^{1/m'}}\\&\quad=\frac{1}{[2]_q}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{\frac{1}{m}}{l}\binom{\frac{1}{m'}}{n-l}q^{(n-l)m'}\sum_{k=0}^{\infty}H_k\left(-q^{-1},ml\right)\frac{t^k}{k!}.\end{aligned} \tag{2.8}$$

We note that for $m, m', k, l \in \mathbb{Q}$ with $1/m+1/m'=1$,

$$\max\left\{\left|\frac{1}{[2]_q}\right|_p,\left|\binom{\frac{1}{m}}{l}\right|_p,\left|\binom{\frac{1}{m'}}{n-l}\right|_p,\left|q^{m'(l-1)}\right|_p,\left|\frac{1}{k!}\right|_p\right\}\le 1, \tag{2.9}$$

By Theorem 2.2 and (2.7) and the definition of p-adic norm, it is easy to see that

$$\left|\frac{\int_{\mathbb{Z}_p}f(x)g(x)d\mu_{-1}(x)}{\left\{\int_{\mathbb{Z}_p}f(x)^m d\mu_{-1}\right\}^{1/m}\left\{\int_{\mathbb{Z}_p}g(x)^{m'}d\mu_{-1}\right\}^{1/m'}}\right|_p\le\max\left\{\left|H_k(-q^{-1},ml\right|_p\right\}, \tag{2.10}$$

for all $m, m', k, l \in \mathbb{Q}$ with $1/m+1/m'=1$. We note that $M=\max\{|H_k(-q^{-1},ml)|_p\}$ lies in $(0,\infty)$. Thus by Definition 2.1 and (2.10), we obtain the following Hölder type inequality theorem for fermionic p-adic invariant q-integrals.

Theorem 2.3. *Let $m, m' \in \mathbb{Q}$ with $1/m+1/m'=1$ and $M=\max\{|H_k(-q^{-1},ml)|_p\}$. If one takes $f(x)=q^x$ and $g(x)=e^{xt}$, then one has*

$$\int_{\mathbb{Z}_p}f(x)g(x)d\mu_{-1}(x)\le_p M\left\{\int_{\mathbb{Z}_p}f(x)^m d\mu_{-1}\right\}^{1/m}\left\{\int_{\mathbb{Z}_p}g(x)^{m'}d\mu_{-1}\right\}^{1/m'}. \tag{2.11}$$

Acknowledgment

This paper was supported by the KOSEF 2009-0073396.

References

[1] M. Cenkci, Y. Simsek, and V. Kurt, "Further remarks on multiple p-adic q-L-function of two variables," *Advanced Studies in Contemporary Mathematics*, vol. 14, no. 1, pp. 49–68, 2007.

[2] L.-C. Jang, "A new q-analogue of Bernoulli polynomials associated with p-adic q-integrals," *Abstract and Applied Analysis*, vol. 2008, Article ID 295307, 6 pages, 2008.

[3] L.-C. Jang, S.-D. Kim, D.-W. Park, and Y.-S. Ro, "A note on Euler number and polynomials," *Journal of Inequalities and Applications*, vol. 2006, Article ID 34602, 5 pages, 2006.

[4] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[5] T. Kim, "0 q-integrals associated with multiple Changhee q-Bernoulli polynomials," *Russian Journal of Mathematical Physics*, vol. 10, no. 1, pp. 91–98, 2003.

[6] T. Kim, "On Euler-Barnes multiple zeta functions," *Russian Journal of Mathematical Physics*, vol. 10, no. 3, pp. 261–267, 2003.

[7] T. Kim, "Analytic continuation of multiple q-zeta functions and their values at negative integers," *Russian Journal of Mathematical Physics*, vol. 11, no. 1, pp. 71–76, 2004.

[8] T. Kim, "Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 186–196, 2005.

[9] T. Kim, "Multiple p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 13, no. 2, pp. 151–157, 2006.

[10] T. Kim, "q-generalized Euler numbers and polynomials," *Russian Journal of Mathematical Physics*, vol. 13, no. 3, pp. 293–298, 2006.

[11] T. Kim, "Lebesgue-Radon-Nikodým theorem with respect to q-Volkenborn distribution on μ_q," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 266–271, 2007.

[12] T. Kim, "q-extension of the Euler formula and trigonometric functions," *Russian Journal of Mathematical Physics*, vol. 14, no. 3, pp. 275–278, 2007.

[13] T. Kim, "q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients," *Russian Journal of Mathematical Physics*, vol. 15, no. 1, pp. 51–57, 2008.

[14] T. Kim, "An invariant p-adic q-integral on $\mathbb{Z}_p$," *Applied Mathematics Letters*, vol. 21, no. 2, pp. 105–108, 2008.

[15] T. Kim, "An identity of thesymmetry for the Frobenius-Euler polynomials associated with the Fermionic p-adic invariant q-integrals on $\mathbb{Z}_p$," to appear in *Rocky Mountain Journal of Mathematics*, http://arxiv.org/abs/0804.4605.

[16] T. Kim, "Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials," *Journal of Difference Equations and Applications*, vol. 14, no. 12, pp. 1267–1277, 2008.

[17] T. Kim, J. Y. Choi, and J. Y. Sug, "Extended q-Euler numbers and polynomials associated with fermionic p-adic q-integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 14, no. 2, pp. 160–163, 2007.

[18] T. Kim, M.-S. Kim, L.-C. Jang, and S.-H. Rim, "New q-Euler numbers and polynomials associated with p-adic q-integrals," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 2, pp. 243–252, 2007.

[19] T. Kim and Y. Simsek, "Analytic continuation of the multiple Daehee q-L-functions associated with Daehee numbers," *Russian Journal of Mathematical Physics*, vol. 15, no. 1, pp. 58–65, 2008.

[20] H. Ozden, Y. Simsek, S.-H. Rim, and I. N. Cangul, "A note on p-adic q-Euler measure," *Advanced Studies in Contemporary Mathematics*, vol. 14, no. 2, pp. 233–239, 2007.

[21] S.-H. Rim and T. Kim, "A note on p-adic Euler measure on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 13, no. 3, pp. 358–361, 2006.

[22] Y. Simsek, "On p-adic twisted q-L-functions related to generalized twisted Bernoulli numbers," *Russian Journal of Mathematical Physics*, vol. 13, no. 3, pp. 340–348, 2006.

[23] Y. Simsek, "Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions," *Advanced Studies in Contemporary Mathematics*, vol. 16, no. 2, pp. 251–278, 2008.

[24] H. M. Srivastava, T. Kim, and Y. Simsek, "q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 241–268, 2005.

[25] H. L. Royden, *Real Analysis*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1998.