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Research Article

Hybrid Approximate Proximal Point Algorithms for Variational Inequalities in Banach Spaces

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Let C be a nonempty closed convex subset of a Banach space E with the dual E^*, let $T : C \to E^*$ be a continuous mapping, and let $S : C \to C$ be a relatively nonexpansive mapping. In this paper, by employing the notion of generalized projection operator we study the variational inequality (for short, $\mathrm{VI}(T - f, C)$): find $x \in C$ such that $\langle y - x, Tx - f \rangle \geq 0$ for all $y \in C$, where $f \in E^*$ is a given element. By combining the approximate proximal point scheme both with the modified Ishikawa iteration and with the modified Halpern iteration for relatively nonexpansive mappings, respectively, we propose two modified versions of the approximate proximal point scheme L. C. Ceng and J. C. Yao (2008) for finding approximate solutions of the $\mathrm{VI}(T - f, C)$. Moreover, it is proven that these iterative algorithms converge strongly to the same solution of the $\mathrm{VI}(T - f, C)$, which is also a fixed point of S.

1. Introduction

Let E be a real Banach space with the dual E^*. As usually, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. In particular, if E is a real Hilbert space, then $\langle \cdot, \cdot \rangle$ denotes its inner product. Let C be a nonempty closed convex subset of E and $T : C \to E^*$ be a mapping. Given $f \in E^*$, let us consider the following variational inequality problem (for short, $VI(T - f, C)$): find an element $x \in C$ such that

$$\langle y - x, Tx - f \rangle \geq 0 \quad \forall y \in C. \tag{1.1}$$

Suppose that the $\mathrm{VI}(T - f, C)$ (1.1) has a (unique) solution $x^* \in C$. For any $x_0 \in C$, define the following successive sequence in a uniformly convex and uniformly smooth

Banach space E:

$$x_{n+1} = \Pi_C\Big(J^{-1}\big(Jx_n - \lambda(Tx_n - f)\big)\Big), \quad n = 1, 2, \ldots, \tag{1.2}$$

where $J : E \to E^*$ is the normalized duality mapping on E and $\Pi_C : E \to C$ is the generalized projection operator which assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ with respect to $y \in C$. In [1, Theorem 8.2], Alber proved that the above sequence converges strongly to the solution x^*, that is, $\|x_n - x^*\| \to 0$ as $n \to \infty$, if the following conditions hold:

(i) $T : E \to E^*$ is uniformly monotone, that is,

$$\langle x_1 - x_2, Tx_1 - Tx_2\rangle \geq \psi(\|x_1 - x_2\|) \quad \forall x_1, x_2 \in E, \tag{1.3}$$

where $\psi(t)$ is a continuous strictly increasing function for all $t \geq 0$ with $\psi(0) = 0$;

(ii) $T : E \to E^*$ has φ arbitrary growth, that is,

$$\|Tx - f\| \leq \varphi(\|x - x^*\|) \quad \forall x \in E, \tag{1.4}$$

where $\varphi(t)$ is a continuous nondecreasing function for all $t \geq 0$ with $\varphi(0) \geq 0$. Note that solution methods for the problem (1.1) has also been studied in [2–10].

Let C be a nonempty closed convex subset of a real Banach space E with the dual E^*. Assume that $T : C \to E^*$ is a continuous mapping on C and $S : C \to C$ is a relatively nonexpansive mapping such that $F(S) \neq \emptyset$. The purpose of this paper is to introduce and study two new iterative algorithms (1.5) and (1.6) in a uniformly convex and uniformly smooth Banach space E.

Algorithm 1.1.

$$\begin{gathered}
x_0 \in C \text{ chosen arbitrarily}, \\
\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n - \lambda_n(T\tilde{x}_n - f)\rangle \leq 0, \quad \forall y \in C, \\
z_n = J^{-1}\big(\beta_n J\tilde{x}_n + (1 - \beta_n)JS\tilde{x}_n\big), \\
y_n = J^{-1}(\alpha_n J\tilde{x}_n + (1 - \alpha_n)JSz_n), \\
C_n = \big\{v \in C : \phi(v, y_n) \leq \alpha_n\phi(v, \tilde{x}_n) + (1 - \alpha_n)\phi(v, z_n)\big\}, \\
Q_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{gathered} \tag{1.5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$, $\{\lambda_n\}_{n=0}^{\infty}$ is a bounded sequence in $(0, \infty)$, and $\tilde{x}_n$ is assumed to exist for each x_n, $n = 0, 1, 2, \ldots$.

Algorithm 1.2.

$$
\begin{gathered}
x_0 \in E \text{ chosen arbitrarily,} \\
\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n - \lambda_n (T\tilde{x}_n - f) \rangle \le 0, \quad \forall y \in C, \\
y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) JS\tilde{x}_n), \\
C_n = \{ v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, \tilde{x}_n) \}, \\
Q_n = \{ v \in C : \langle v - x_n, Jx_0 - Jx_n \rangle \le 0 \}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{gathered}
\tag{1.6}
$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$, $\{\lambda_n\}_{n=0}^{\infty}$ is a bounded sequence in $(0,\infty)$, and $\tilde{x}_n$ is assumed to exist for each x_n, $n = 0, 1, 2, \ldots$.

In this paper, strong convergence results on these two iterative algorithms are established; that is, under appropriate conditions, both the sequence $\{x_n\}$ generated by algorithm (1.5) and the sequence $\{x_n\}$ generated by algorithm (1.6) converge strongly to the same point $\Pi_{F(S)} x_0$, which is a solution of the VI$(T - f, C)$. Our results represent the improvement, generalization, and development of the previously known results in the literature including Li [8], Zeng and Yao [9], Ceng and Yao [10], and Qin and Su [11].

Notation 1. $\rightharpoonup$ stands for weak convergence and $\rightarrow$ for strong convergence.

2. Preliminaries

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$
Jx = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \tag{2.1}
$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E is smooth, then J is single-valued and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We will still denote the single-valued duality mapping by J.

Recall that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and hence, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined as in [1, 12] by

$$
\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \forall x, y \in E. \tag{2.2}
$$

It is clear that in a Hilbert space H, (2.2) reduces to $\phi(x, y) = \|x - y\|^2$, for all $x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.3}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [13]). In a Hilbert space, $\Pi_C = P_C$.

From [1], in uniformly convex and uniformly smooth Banach spaces, we have

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \quad \forall x, y \in E. \tag{2.4}$$

Let C be a closed convex subset of E, and let S be a mapping from C into itself. A point p in C is called an asymptotically fixed point of S [14] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $Sx_n - x_n \to 0$. The set of asymptotical fixed points of S will be denoted by $\widehat{F}(S)$. A mapping S from C into itself is called relatively nonexpansive [15–17] if $\widehat{F}(S) = F(S)$ and $\phi(p, Sx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(S)$.

A Banach space E is called strictly convex if $\|(x + y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $x_n - y_n \to 0$ for any two sequences $\{x_n\}, \{y_n\} \subset E$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be a unit sphere of E. Then the Banach space E is called smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. Recall also that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. A Banach space is said to have the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, whenever $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, we have $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [18, 19] for more details.

Remark 2.1 ([11]). If E is a reflexive, strictly convex, and smooth Banach space, then for any $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2.4), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|y\|^2$. From the definition of J, we have $Jx = Jy$. Therefore, we have $x = y$; see [18, 19] for more details.

We need the following lemmas and proposition for the proof of our main results.

Lemma 2.2 (Kamimura and Takahashi [20]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.3 (Alber [1]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx_0 - Jx \rangle \geq 0 \quad \forall z \in C. \tag{2.6}$$

Lemma 2.4 (Alber [1]). *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x) \quad \forall y \in C. \tag{2.7}$$

Lemma 2.5 (Matsushita and Takahashi [21]). *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let S be a relatively nonexpansive mapping from C into itself. Then $F(S)$ is closed and convex.*

Lemma 2.6 (Chang [7]). *Let E be a smooth Banach space. Then the following inequality holds*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x + y)\rangle \quad \forall x, y \in E. \tag{2.8}$$

3. Main Results

Now we are in a position to prove the main theorems of this paper.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $T : C \to E^*$ be a continuous mapping and, let $S : C \to C$ be a relatively nonexpansive mapping such that $F(S) \ne \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ and $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence in $(0,\infty)$ such that $\limsup_{n\to\infty}\alpha_n < 1$, $\beta_n \to 1$ and $\lambda_n \to \lambda \in (0,\infty)$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the following algorithm:*

$$\begin{gathered}
x_0 \in C \text{ chosen arbitrarily}, \\
\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n - \lambda_n(T\tilde{x}_n - f)\rangle \le 0, \quad \forall y \in C, \\
z_n = J^{-1}(\beta_n J\tilde{x}_n + (1-\beta_n)JS\tilde{x}_n), \\
y_n = J^{-1}(\alpha_n J\tilde{x}_n + (1-\alpha_n)JSz_n), \\
C_n = \{v \in C : \phi(v, y_n) \le \alpha_n\phi(v, \tilde{x}_n) + (1-\alpha_n)\phi(v, z_n)\}, \\
Q_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{gathered} \tag{3.1}$$

where $\tilde{x}_n$ is assumed to exist for each x_n, $n = 0, 1, 2, \ldots$. If S is uniformly continuous and $\lim_{n\to\infty}\langle x_n - \tilde{x}_n, T\tilde{x}_n - f\rangle = 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(S)}x_0$, which is a solution of the $VI(T - f, C)$ (1.1).

Proof. First of all, let us show that C_n and Q_n are closed and convex for each $n \ge 0$. Indeed, from the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \ge 0$. We claim that C_n is convex. For any $v_1, v_2 \in C_n$ and any $t \in (0,1)$, put $v = tv_1 + (1-t)v_2$. It is sufficient to show that $v \in C_n$. Note that the inequality

$$\phi(v, y_n) \le \alpha_n\phi(v, \tilde{x}_n) + (1-\alpha_n)\phi(v, z_n) \tag{3.2}$$

is equivalent to the one

$$2\alpha_n\langle v, J\tilde{x}_n\rangle + 2(1-\alpha_n)\langle v, Jz_n\rangle - 2\langle v, Jy_n\rangle \le \alpha_n\|\tilde{x}_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|y_n\|^2. \tag{3.3}$$

Observe that there hold the following:

$$\phi(v, y_n) = \|v\|^2 - 2\langle v, Jy_n\rangle + \|y_n\|^2, \qquad \phi(v, \tilde{x}_n) = \|v\|^2 - 2\langle v, J\tilde{x}_n\rangle + \|\tilde{x}_n\|^2, \tag{3.4}$$

and $\phi(v, z_n) = \|v\|^2 - 2\langle v, Jz_n\rangle + \|z_n\|^2$. Thus, we have

$$\begin{aligned}
&2\alpha_n\langle v, J\tilde{x}_n\rangle + 2(1-\alpha_n)\langle v, Jz_n\rangle - 2\langle v, Jy_n\rangle \\
&\quad = 2\alpha_n\langle tv_1 + (1-t)v_2, J\tilde{x}_n\rangle + 2(1-\alpha_n)\langle tv_1 + (1-t)v_2, Jz_n\rangle \\
&\qquad - 2\langle tv_1 + (1-t)v_2, Jy_n\rangle \\
&\quad = 2t\alpha_n\langle v_1, J\tilde{x}_n\rangle + 2(1-t)\alpha_n\langle v_2, J\tilde{x}_n\rangle \\
&\qquad + 2(1-\alpha_n)t\langle v_1, Jz_n\rangle + 2(1-\alpha_n)(1-t)\langle v_2, Jz_n\rangle \\
&\qquad - 2t\langle v_1, Jy_n\rangle - 2(1-t)\langle v_2, Jy_n\rangle \\
&\quad \le \alpha_n\|\tilde{x}_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|y_n\|^2.
\end{aligned} \tag{3.5}$$

This implies that $v \in C_n$. So, C_n is convex. Next let us show that $F(S) \subset C_n$ for all n. Indeed, we have for all $w \in F(S)$

$$\begin{aligned}
\phi(w, y_n) &= \phi\Big(w, J^{-1}(\alpha_n J\tilde{x}_n + (1-\alpha_n)JSz_n)\Big) \\
&= \|w\|^2 - 2\langle w, \alpha_n J\tilde{x}_n + (1-\alpha_n)JSz_n\rangle + \|\alpha_n J\tilde{x}_n + (1-\alpha_n)JSz_n\|^2 \\
&\le \|w\|^2 - 2\alpha_n\langle w, J\tilde{x}_n\rangle - 2(1-\alpha_n)\langle w, JSz_n\rangle + \alpha_n\|\tilde{x}_n\|^2 + (1-\alpha_n)\|Sz_n\|^2 \\
&\le \alpha_n\phi(w, \tilde{x}_n) + (1-\alpha_n)\phi(w, Sz_n) \\
&\le \alpha_n\phi(w, \tilde{x}_n) + (1-\alpha_n)\phi(w, z_n).
\end{aligned} \tag{3.6}$$

So $w \in C_n$ for all $n \ge 0$. Next let us show that

$$F(S) \subset Q_n \quad \forall n \ge 0. \tag{3.7}$$

We prove this by induction. For $n = 0$, we have $F(S) \subset C = Q_0$. Assume that $F(S) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 2.3, we have

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \ge 0, \quad \forall z \in C_n \cap Q_n. \tag{3.8}$$

As $F(S) \subset C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in F(S)$. This together with the definition of Q_{n+1} implies that $F(S) \subset Q_{n+1}$. Hence (3.7) holds for all $n \ge 0$. This implies that $\{x_n\}$ is well defined.

On the other hand, it follows from the definition of Q_n that $x_n = \Pi_{Q_n} x_0$. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \quad \forall n \geq 0. \tag{3.9}$$

Thus $\{\phi(x_n, x_0)\}$ is nondecreasing. Also from $x_n = \Pi_{Q_n} x_0$ and Lemma 2.4, it follows that

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0) \tag{3.10}$$

for each $w \in F(S) \subset Q_n$ for each $n \geq 0$. Consequently, $\{\phi(x_n, x_0)\}$ is bounded. Moreover, according to the inequality

$$(\|x_n\| - \|x_0\|)^2 \leq \phi(x_n, x_0) \leq (\|x_n\| + \|x_0\|)^2, \tag{3.11}$$

we conclude that $\{x_n\}$ is bounded and so is $\{Sx_n\}$. Indeed, since S is relatively nonexpansive, we derive for each $p \in F(S)$

$$\phi(p, Sx_n) \leq \phi(p, x_n) \leq (\|p\| + \|x_n\|)^2 \quad \forall n \geq 0, \tag{3.12}$$

and hence $\{\phi(p, Sx_n)\}$ is bounded. Again from $(\|p\| - \|Sx_n\|)^2 \leq \phi(p, Sx_n)$, we know that $\{Sx_n\}$ is also bounded.

On account of the boundedness and nondecreasing property of $\{\phi(x_n, x_0)\}$, we deduce that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. From Lemma 2.4, we derive

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{Q_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \end{aligned} \tag{3.13}$$

for all $n \geq 0$. This implies that $\phi(x_{n+1}, x_n) \to 0$. So it follows from Lemma 2.2 that $x_{n+1} - x_n \to 0$. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, from the definition of C_n, we also have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, \tilde{x}_n) + (1 - \alpha_n)\phi(x_{n+1}, z_n). \tag{3.14}$$

Observe that

$$\begin{aligned} \phi(x_{n+1}, z_n) &= \phi\left(x_{n+1}, J^{-1}(\beta_n J\tilde{x}_n + (1-\beta_n) JS\tilde{x}_n)\right) \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n J\tilde{x}_n + (1-\beta_n) JS\tilde{x}_n\rangle + \|\beta_n J\tilde{x}_n + (1-\beta_n) JS\tilde{x}_n\|^2 \\ &\leq \|x_{n+1}\|^2 - 2\beta_n\langle x_{n+1}, J\tilde{x}_n\rangle - 2(1-\beta_n)\langle x_{n+1}, JS\tilde{x}_n\rangle \\ &\quad + \beta_n\|\tilde{x}_n\|^2 + (1-\beta_n)\|S\tilde{x}_n\|^2 \\ &= \beta_n\phi(x_{n+1}, \tilde{x}_n) + (1-\beta_n)\phi(x_{n+1}, S\tilde{x}_n). \end{aligned} \tag{3.15}$$

On the other hand, since from (3.1) we have for each $n \geq 0$

$$\left\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n - \lambda_n\left(T\tilde{x}_n - f\right)\right\rangle \leq 0, \quad \forall y \in C, \tag{3.16}$$

utilizing Lemma 2.3 we obtain $\tilde{x}_n = \Pi_C(J^{-1}(Jx_n - \lambda_n(T\tilde{x}_n - f)))$. Thus, in terms of Lemmas 2.4 and 2.6 we conclude that

$$\begin{aligned}
\phi(x_n, \tilde{x}_n) &= \phi\Big(x_n, \Pi_C\Big(J^{-1}\left(Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right)\Big)\Big) \\
&\leq \phi\Big(x_n, J^{-1}\left(Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right)\Big) \\
&\quad - \phi\Big(\Pi_C\Big(J^{-1}\left(Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right)\Big), J^{-1}\left(Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right)\Big) \\
&= \phi\Big(x_n, J^{-1}\left(Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right)\Big) - \phi\Big(\tilde{x}_n, J^{-1}\left(Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right)\Big) \\
&= \|x_n\|^2 - 2\left\langle x_n, Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right\rangle + \|Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\|^2 \\
&\quad - \Big[\|\tilde{x}_n\|^2 - 2\left\langle \tilde{x}_n, Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right\rangle + \|Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\|^2\Big] \\
&= \|x_n\|^2 - 2\left\langle x_n, Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right\rangle - \|\tilde{x}_n\|^2 + 2\left\langle \tilde{x}_n, Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right\rangle \\
&= \|x_n\|^2 - \|\tilde{x}_n\|^2 + 2\left\langle \tilde{x}_n - x_n, Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right\rangle \\
&= \|\tilde{x}_n + x_n - \tilde{x}_n\|^2 - \|\tilde{x}_n\|^2 + 2\left\langle \tilde{x}_n - x_n, Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right\rangle \\
&\leq \|\tilde{x}_n\|^2 + 2\langle x_n - \tilde{x}_n, Jx_n\rangle - \|\tilde{x}_n\|^2 + 2\left\langle \tilde{x}_n - x_n, Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right\rangle \\
&= 2\langle x_n - \tilde{x}_n, Jx_n\rangle + 2\left\langle \tilde{x}_n - x_n, Jx_n - \lambda_n\left(T\tilde{x}_n - f\right)\right\rangle \\
&= 2\lambda_n\langle x_n - \tilde{x}_n, T\tilde{x}_n - f\rangle.
\end{aligned} \tag{3.17}$$

Since $\lambda_n \to \lambda \in (0, \infty)$ and $\lim_{n\to\infty}\langle x_n - \tilde{x}_n, T\tilde{x}_n - f\rangle = 0$, we obtain $\phi(x_n, \tilde{x}_n) \to 0$. Thus by Lemma 2.2 we have $x_n - \tilde{x}_n \to 0$. From $\|\tilde{x}_n\| \leq \|\tilde{x}_n - x_n\| + \|x_n\|$, it follows that $\{\tilde{x}_n\}$ is bounded. At the same time, observe that

$$\begin{aligned}
\phi(x_{n+1}, \tilde{x}_n) - \phi(x_n, \tilde{x}_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, J\tilde{x}_n\rangle + \|\tilde{x}_n\|^2 \\
&\quad - \Big[\|x_n\|^2 - 2\langle x_n, J\tilde{x}_n\rangle + \|\tilde{x}_n\|^2\Big] \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, J\tilde{x}_n\rangle - \|x_n\|^2 + 2\langle x_n, J\tilde{x}_n\rangle \\
&= (\|x_{n+1}\| - \|x_n\|)(\|x_{n+1}\| + \|x_n\|) + 2\langle x_n - x_{n+1}, J\tilde{x}_n\rangle \\
&\leq \|x_{n+1} - x_n\|(\|x_{n+1}\| + \|x_n\|) + 2\|x_n - x_{n+1}\|\|\tilde{x}_n\| \\
&= \|x_{n+1} - x_n\|(\|x_{n+1}\| + \|x_n\| + 2\|\tilde{x}_n\|),
\end{aligned} \tag{3.18}$$

and hence

$$\phi(x_{n+1}, \tilde{x}_n) \leq \phi(x_n, \tilde{x}_n) + \|x_{n+1} - x_n\|(\|x_{n+1}\| + \|x_n\| + 2\|\tilde{x}_n\|). \tag{3.19}$$

From $\phi(x_n, \tilde{x}_n) \to 0, x_{n+1} - x_n \to 0$, and the boundedness of $\{x_n\}$ and $\{\tilde{x}_n\}$, we derive $\phi(x_{n+1}, \tilde{x}_n) \to 0$. Note that $S : C \to C$ is uniformly continuous. Hence $S\tilde{x}_n - Sx_n \to 0$ by virtue of $\tilde{x}_n - x_n \to 0$. Since

$$\|S\tilde{x}_n\| \leq \|S\tilde{x}_n - Sx_n\| + \|Sx_n\|, \tag{3.20}$$

it is known that $\{S\tilde{x}_n\}$ is bounded. Consequently, from (3.15), $\phi(x_{n+1}, \tilde{x}_n) \to 0$ and $\beta_n \to 1$, it follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, z_n) = 0. \tag{3.21}$$

Further, it follows from (3.14), $\phi(x_{n+1}, \tilde{x}_n) \to 0$ and $\phi(x_{n+1}, z_n) \to 0$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{3.22}$$

Utilizing Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - \tilde{x}_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{3.23}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - J\tilde{x}_n\| = 0. \tag{3.24}$$

Furthermore, we have

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|. \tag{3.25}$$

It follows from $x_{n+1} - x_n \to 0$ and $x_{n+1} - z_n \to 0$ that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.26}$$

Noticing that

$$\begin{aligned}
\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n J\tilde{x}_n + (1-\alpha_n)JSz_n)\| \\
&= \|\alpha_n(Jx_{n+1} - J\tilde{x}_n) + (1-\alpha_n)(Jx_{n+1} - JSz_n)\| \\
&= \|(1-\alpha_n)(Jx_{n+1} - JSz_n) - \alpha_n(J\tilde{x}_n - Jx_{n+1})\| \\
&\geq (1-\alpha_n)\|Jx_{n+1} - JSz_n\| - \alpha_n\|J\tilde{x}_n - Jx_{n+1}\|,
\end{aligned} \tag{3.27}$$

we have

$$\|Jx_{n+1} - JSz_n\| \leq \frac{1}{1-\alpha_n}\left(\|Jx_{n+1} - Jy_n\| + \alpha_n\|J\tilde{x}_n - Jx_{n+1}\|\right). \tag{3.28}$$

From (3.24) and $\limsup_{n\to\infty}\alpha_n < 1$, we obtain

$$\lim_{n\to\infty}\|Jx_{n+1} - JSz_n\| = 0. \tag{3.29}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - Sz_n\| = 0. \tag{3.30}$$

Observe that

$$\|x_n - Sx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sz_n\| + \|Sz_n - Sx_n\|. \tag{3.31}$$

Since S is uniformly continuous, it follows from (3.26), (3.30) and $x_{n+1} - x_n \to 0$ that $x_n - Sx_n \to 0$.

Finally, let us show that $\{x_n\}$ converges strongly to $\Pi_{F(S)}x_0$, which is a solution of the VI$(T-f, C)$ (1.1). Indeed, assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x} \in E$. Then $\tilde{x} \in \widehat{F}(S) = F(S)$. Next let us show that $\tilde{x} = \Pi_{F(S)}x_0$ and convergence is strong. Put $\overline{x} = \Pi_{F(S)}x_0$. From $x_{n+1} = \Pi_{C_n\cap Q_n}x_0$ and $\overline{x} \in F(S) \subset C_n \cap Q_n$, we have $\phi(x_{n+1}, x_0) \leq \phi(\overline{x}, x_0)$. Now from weakly lower semicontinuity of the norm, we derive

$$\begin{aligned}
\phi(\tilde{x}, x_0) &= \|\tilde{x}\|^2 - 2\langle \tilde{x}, Jx_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{i\to\infty}\left(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0\rangle + \|x_0\|^2\right) \\
&= \liminf_{i\to\infty}\phi(x_{n_i}, x_0) \\
&\leq \limsup_{i\to\infty}\phi(x_{n_i}, x_0) \\
&\leq \phi(\overline{x}, x_0).
\end{aligned} \tag{3.32}$$

It follows from the definition of $\Pi_{F(S)}x_0$ that $\tilde{x} = \overline{x}$ and hence

$$\lim_{i\to\infty}\phi(x_{n_i}, x_0) = \phi(\overline{x}, x_0). \tag{3.33}$$

So we have $\lim_{i\to\infty}\|x_{n_i}\| = \|\overline{x}\|$. Utilizing the Kadec-Klee property of E, we conclude that $\{x_{n_i}\}$ converges strongly to $\Pi_{F(S)}x_0$. Since $\{x_{n_i}\}$ is an arbitrarily weakly convergent subsequence of $\{x_n\}$, we know that $\{x_n\}$ converges strongly to $\overline{x} = \Pi_{F(S)}x_0$. Now observe that from (3.1) we have for each $n \geq 0$

$$\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n - \lambda_n(T\tilde{x}_n - f)\rangle \leq 0, \quad \forall y \in C. \tag{3.34}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, from $x_n - \tilde{x}_n \to 0$ we infer that $Jx_n - J\tilde{x}_n \to 0$. Noticing that $x_n \to \overline{x}$ and $T : C \to E^*$ is a continuous mapping, we obtain that $\tilde{x}_n \to \overline{x}$ and $T\tilde{x}_n \to T\overline{x}$. Therefore, from $\lambda_n \to \lambda \in (0, \infty)$, it follows that

$$\begin{aligned}
&\left|\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n - \lambda_n(T\tilde{x}_n - f)\rangle - \langle y - \overline{x}, -\lambda(T\overline{x} - f)\rangle\right| \\
&\quad = \left|\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n\rangle + \langle y - \tilde{x}_n, -\lambda_n(T\tilde{x}_n - f)\rangle - \langle y - \overline{x}, -\lambda(T\overline{x} - f)\rangle\right| \\
&\quad \leq \left|\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n\rangle\right| + \left|\langle y - \tilde{x}_n, -\lambda_n(T\tilde{x}_n - f)\rangle - \langle y - \overline{x}, -\lambda(T\overline{x} - f)\rangle\right| \\
&\quad = \left|\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n\rangle\right| + \big|\langle y - \tilde{x}_n - (y - \overline{x}), -\lambda_n(T\tilde{x}_n - f)\rangle \\
&\qquad + \langle y - \overline{x}, -\lambda_n(T\tilde{x}_n - f) - (-\lambda(T\overline{x} - f))\rangle\big| \\
&\quad \leq \|y - \tilde{x}_n\| \|Jx_n - J\tilde{x}_n\| + \|\tilde{x}_n - \overline{x}\| \|\lambda_n(T\tilde{x}_n - f)\| \\
&\qquad + \|y - \overline{x}\| \|\lambda_n(T\tilde{x}_n - f) - \lambda(T\overline{x} - f)\| \\
&\quad \leq \|y - \tilde{x}_n\| \|Jx_n - J\tilde{x}_n\| + \|\tilde{x}_n - \overline{x}\| \|\lambda_n(T\tilde{x}_n - f)\| \\
&\qquad + \|y - \overline{x}\| \left[\lambda_n \|T\tilde{x}_n - T\overline{x}\| + |\lambda_n - \lambda| \|T\overline{x} - f\|\right] \longrightarrow 0 \quad (n \longrightarrow \infty),
\end{aligned} \tag{3.35}$$

that is,

$$\lim_{n \to \infty} \langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n - \lambda_n(T\tilde{x}_n - f)\rangle = \langle y - \overline{x}, -\lambda(T\overline{x} - f)\rangle. \tag{3.36}$$

Letting $n \to \infty$ we conclude from (3.34) that

$$\langle y - \overline{x}, -\lambda(T\overline{x} - f)\rangle \leq 0 \quad \forall y \in C, \tag{3.37}$$

and hence

$$\langle y - \overline{x}, T\overline{x} - f\rangle \geq 0 \quad \forall y \in C. \tag{3.38}$$

This shows that $\overline{x} = \Pi_{F(S)} x_0$ is a solution of the VI$(T - f, C)$ (1.1). This completes the proof. □

Corollary 3.2 ([11, Theorem 2.1]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, and let $S : C \to C$ be a relatively nonexpansive mapping such that $F(S) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ such that* $\limsup_{n\to\infty}\alpha_n < 1$ *and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{gathered}
x_0 \in C \text{ chosen arbitrarily}, \\
z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S x_n), \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S z_n), \\
C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\
Q_n = \{v \in C : \langle v - x_n, J x_0 - J x_n \rangle \le 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0,
\end{gathered} \tag{3.39}$$

where J is the single-valued duality mapping on E. If S is uniformly continuous, then $\{x_n\}$ converges strongly to $\Pi_{F(S)} x_0$.

Proof. In Theorem 3.1, we know from (3.1) and Lemma 2.3 that

$$\langle y - \tilde{x}_n, J x_n - J\tilde{x}_n - \lambda_n (T\tilde{x}_n - f)\rangle \le 0, \quad \forall y \in C, \tag{3.40}$$

is equivalent to $\tilde{x}_n = \Pi_C(J^{-1}(Jx_n - \lambda_n(T\tilde{x}_n - f)))$. Now, put $Tx = f$ for all $x \in C$. Then we have

$$\begin{aligned}
\tilde{x}_n &= \Pi_C\Big(J^{-1}(Jx_n - (T\tilde{x}_n - f))\Big) \\
&= \Pi_C\Big(J^{-1}(Jx_n - (f - f))\Big) \\
&= x_n,
\end{aligned} \tag{3.41}$$

for all n. Thus algorithm (3.1) reduces to algorithm (3.39). By Theorem 3.1 we obtain the desired result. □

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $T : C \to E^*$ be a continuous mapping, and let $S : C \to C$ be a relatively nonexpansive mapping such that $F(S) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\{\lambda_n\}_{n=0}^{\infty} \subset (0,\infty)$ satisfies $\lim_{n\to\infty}\lambda_n = \lambda > 0$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the following* *algorithm:*

$$\begin{gathered}
x_0 \in C \text{ chosen arbitrarily}, \\
\left\langle y - \widetilde{x}_n, Jx_n - J\widetilde{x}_n - \lambda_n \left(T\widetilde{x}_n - f\right)\right\rangle \leq 0, \quad \forall y \in C, \\
y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n) JS\widetilde{x}_n), \\
C_n = \left\{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, \widetilde{x}_n)\right\}, \\
Q_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{gathered} \tag{3.42}$$

where $\widetilde{x}_n$ *is assumed to exist for each* x_n, $n = 0, 1, 2, \ldots$ *If* S *is uniformly continuous and* $\lim_{n\to\infty}\langle x_n - \widetilde{x}_n, T\widetilde{x}_n - f\rangle = 0$, *then* $\{x_n\}$ *converges strongly to* $\Pi_{F(S)}x_0$, *which is a solution of the* $VI(T - f, C)$ (1.1).

Proof. We only derive the difference. First, let us show that C_n is closed and convex for each $n \geq 0$. From the definition of C_n, it is obvious that C_n is closed for each $n \geq 0$. We prove that C_n is convex. Similarly to the proof of Theorem 3.1, since

$$\phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, \widetilde{x}_n) \tag{3.43}$$

is equivalent to

$$2\alpha_n\langle v, Jx_0\rangle + 2(1-\alpha_n)\langle v, J\widetilde{x}_n\rangle - 2\langle v, Jy_n\rangle \leq \alpha_n\|x_0\|^2 + (1-\alpha_n)\|\widetilde{x}_n\|^2 - \|y_n\|^2, \tag{3.44}$$

we know that C_n is convex. Next, let us show that $F(S) \subset C_n$ for each $n \geq 0$. Indeed, we have for each $w \in F(S)$

$$\begin{aligned}
\phi(w, y_n) &= \phi\Big(w, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n)\Big) \\
&= \|w\|^2 - 2\langle w, \alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n\rangle + \|\alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n\|^2 \\
&\leq \|w\|^2 - 2\alpha_n\langle w, Jx_0\rangle - 2(1-\alpha_n)\langle w, JS\widetilde{x}_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|S\widetilde{x}_n\|^2 \\
&\leq \alpha_n\phi(w, x_0) + (1-\alpha_n)\phi(w, S\widetilde{x}_n) \\
&\leq \alpha_n\phi(w, x_0) + (1-\alpha_n)\phi(w, \widetilde{x}_n).
\end{aligned} \tag{3.45}$$

So $w \in C_n$ for all $n \geq 0$ and $F(S) \subset C_n$. Similarly to the proof of Theorem 3.1, we also obtain $F(S) \subset Q_n$ for all $n \geq 0$. Consequently, $F(S) \subset C_n \cap Q_n$ for all $n \geq 0$. Therefore, the sequence $\{x_n\}$ generated by (3.42) is well defined. As in the proof of Theorem 3.1, we can obtain $\phi(x_{n+1}, x_n) \to 0$. Since $x_{n+1} = \Pi_{C_n \cap Q_n}x_0 \in C_n$, from the definition of C_n, we also have

$$\phi(x_{n+1}, y_n) \leq \alpha_n\phi(x_{n+1}, x_0) + (1-\alpha_n)\phi(x_{n+1}, \widetilde{x}_n). \tag{3.46}$$

As in the proof of Theorem 3.1, we can deduce from $\lambda_n \to \lambda \in (0, \infty)$ and $\lim_{n\to\infty}\langle x_n - \tilde{x}_n, T\tilde{x}_n - f\rangle = 0$ that $\phi(x_n, \tilde{x}_n) \to 0$ and hence $x - \tilde{x}_n \to 0$ by Lemma 2.2. Further, it follows from $\phi(x_n, \tilde{x}_n) \to 0, x_{n+1} - x_n \to 0$ and the boundedness of $\{x_n\}$ and $\{\tilde{x}_n\}$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, \tilde{x}_n) = 0. \tag{3.47}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, from the definition of C_n, we also have

$$\phi(x_{n+1}, y_n) \le \alpha_n \phi(x_{n+1}, x_0) + (1 - \alpha_n)\phi(x_{n+1}, \tilde{x}_n). \tag{3.48}$$

It follows from (3.47) and $\alpha_n \to 0$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{3.49}$$

Utilizing Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - \tilde{x}_n\| = 0. \tag{3.50}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - J\tilde{x}_n\| = 0. \tag{3.51}$$

Note that

$$\begin{aligned} \|JS\tilde{x}_n - Jy_n\| &= \|JS\tilde{x}_n - (\alpha_n Jx_0 + (1 - \alpha_n) JS\tilde{x}_n)\| \\ &= \alpha_n \|Jx_0 - JS\tilde{x}_n\|. \end{aligned} \tag{3.52}$$

Therefore, from $\alpha_n \to 0$ we have

$$\lim_{n\to\infty} \|JS\tilde{x}_n - Jy_n\| = 0. \tag{3.53}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} \|S\tilde{x}_n - y_n\| = 0. \tag{3.54}$$

It follows that

$$\|x_n - Sx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - S\tilde{x}_n\| + \|S\tilde{x}_n - Sx_n\|. \tag{3.55}$$

Since S is uniformly continuous, it follows from (3.50) and (3.54) that $x_n - Sx_n \to 0$.

Finally, let us show that $\{x_n\}$ converges strongly to $\Pi_{F(S)}x_0$, which is a solution of the VI$(T-f,C)$ (1.1). Indeed, assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x} \in E$. Then $\tilde{x} \in \widehat{F}(S) = F(S)$. Next let us show that $\tilde{x} = \Pi_{F(S)}x_0$ and convergence is strong. Put $\overline{x} = \Pi_{F(S)}x_0$. From $x_{n+1} = \Pi_{C_n \cap Q_n}x_0$ and $\overline{x} \in F(S) \subset C_n \cap Q_n$, we have $\phi(x_{n+1}, x_0) \le \phi(\overline{x}, x_0)$. Now from weakly lower semicontinuity of the norm, we derive

$$
\begin{aligned}
\phi(\tilde{x}, x_0) &= \|\tilde{x}\|^2 - 2\langle \tilde{x}, Jx_0\rangle + \|x_0\|^2 \\
&\le \liminf_{i\to\infty}\left(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0\rangle + \|x_0\|^2\right) \\
&= \liminf_{i\to\infty}\phi(x_{n_i}, x_0) \\
&\le \limsup_{i\to\infty}\phi(x_{n_i}, x_0) \\
&\le \phi(\overline{x}, x_0).
\end{aligned}
\tag{3.56}
$$

It follows from the definition of $\Pi_{F(S)}x_0$ that $\tilde{x} = \overline{x}$ and hence $\lim_{i\to\infty}\phi(x_{n_i}, x_0) = \phi(\overline{x}, x_0)$. So, we have $\lim_{i\to\infty}\|x_{n_i}\| = \|\overline{x}\|$. Utilizing the Kadec-Klee property of E, we conclude that $\{x_{n_i}\}$ converges strongly to $\Pi_{F(S)}x_0$. Since $\{x_{n_i}\}$ is an arbitrarily weakly convergent subsequence of $\{x_n\}$, we know that $\{x_n\}$ converges strongly to $\overline{x} = \Pi_{F(S)}x_0$. Now observe that from (3.1), we have for each $n \ge 0$

$$
\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n - \lambda_n(T\tilde{x}_n - f)\rangle \le 0 \quad \forall y \in C. \tag{3.57}
$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, from $x_n - \tilde{x}_n \to 0$, we infer that $Jx_n - J\tilde{x}_n \to 0$. Noticing that $x_n \to \overline{x}$ and $T : C \to E^*$ is a continuous mapping, we obtain that $\tilde{x}_n \to \overline{x}$ and $T\tilde{x}_n \to T\overline{x}$. Observe that

$$
\begin{aligned}
&\left|\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n - \lambda_n(T\tilde{x}_n - f)\rangle - \langle y - \overline{x}, -\lambda(T\overline{x} - f)\rangle\right| \\
&\quad = \left|\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n\rangle + \langle y - \tilde{x}_n, -\lambda_n(T\tilde{x}_n - f)\rangle - \langle y - \overline{x}, -\lambda(T\overline{x} - f)\rangle\right| \\
&\quad \le \left|\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n\rangle\right| + \left|\langle y - \tilde{x}_n, -\lambda_n(T\tilde{x}_n - f)\rangle - \langle y - \overline{x}, -\lambda(T\overline{x} - f)\rangle\right| \\
&\quad = \left|\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n\rangle\right| + \left|\langle y - \tilde{x}_n - (y - \overline{x}), -\lambda_n(T\tilde{x}_n - f)\rangle\right. \\
&\qquad \left. + \langle y - \overline{x}, -\lambda_n(T\tilde{x}_n - f) - (-\lambda(T\overline{x} - f))\rangle\right| \\
&\quad \le \|y - \tilde{x}_n\|\|Jx_n - J\tilde{x}_n\| + \|\tilde{x}_n - \overline{x}\|\|\lambda_n(T\tilde{x}_n - f)\| \\
&\qquad + \|y - \overline{x}\|\|\lambda_n(T\tilde{x}_n - f) - \lambda(T\overline{x} - f)\| \\
&\quad \le \|y - \tilde{x}_n\|\|Jx_n - J\tilde{x}_n\| + \|\tilde{x}_n - \overline{x}\|\|\lambda_n(T\tilde{x}_n - f)\| \\
&\qquad + \|y - \overline{x}\|\left[\lambda_n\|T\tilde{x}_n - T\overline{x}\| + |\lambda_n - \lambda|\|T\overline{x} - f\|\right],
\end{aligned}
\tag{3.58}
$$

It follows from $\lambda_n \to \lambda \in (0, \infty)$ that

$$
\lim_{n\to\infty}\langle y - \tilde{x}_n, Jx_n - J\tilde{x}_n - \lambda_n(T\tilde{x}_n - f)\rangle = \langle y - \overline{x}, -\lambda(T\overline{x} - f)\rangle. \tag{3.59}
$$

Letting $n \to \infty$ we conclude from (3.34) that

$$\langle y - \overline{x}, T\overline{x} - f \rangle \geq 0 \quad \forall y \in C. \tag{3.60}$$

This shows that $\overline{x} = \Pi_{F(S)} x_0$ is a solution of the VI$(T - f, C)$ (1.1). This completes the proof. □

Corollary 3.4 ([11, Theorem 2.2]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, and let $S : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{gathered} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J S x_n), \\ C_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle v - x_n, J x_0 - J x_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{gathered} \tag{3.61}$$

where J is the single-valued duality mapping on E. If $F(S)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(S)} x_0$.

Proof. In Theorem 3.3, we know from (3.42) and Lemma 2.3 that

$$\langle y - \widetilde{x}_n, J x_n - J\widetilde{x}_n - \lambda_n (T\widetilde{x}_n - f) \rangle \leq 0, \quad \forall y \in C, \tag{3.62}$$

is equivalent to $\widetilde{x}_n = \Pi_C(J^{-1}(Jx_n - \lambda_n(T\widetilde{x}_n - f)))$. Now, put $Tx = f$ for all $x \in C$. Then we have

$$\begin{aligned} \widetilde{x}_n &= \Pi_C\Big(J^{-1}(Jx_n - (T\widetilde{x}_n - f))\Big) \\ &= \Pi_C\Big(J^{-1}(Jx_n - (f - f))\Big) \\ &= x_n, \end{aligned} \tag{3.63}$$

for all n. Thus algorithm (3.42) reduces to algorithm (3.61). Thus under the lack of the uniform continuity of S, it follows from (3.55) that $x_n - Sx_n \to 0$. By the careful analysis of the proof of Theorem 3.3, we can obtain the desired result. □

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