Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 193035, 19 pages
doi:10.1155/2009/193035

Research Article

Fixed Points and Stability of a Generalized Quadratic Functional Equation

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Received 26 November 2008; Revised 24 January 2009; Accepted 11 February 2009

Recommended by Patricia J. Y. Wong

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the generalized quadratic functional equation $f(rx+sy) = r^2 f(x) + s^2 f(y) + (rs/2)[f(x+y) - f(x-y)]$ in Banach modules, where r, s are nonzero rational numbers with $r^2 + s^2 \neq 1$.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms: *let* $(G_1, *)$ *be a group and let* $(G_2, \diamond, d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\epsilon > 0$, *does there exist* $\delta(\epsilon) > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality*

$$d\big(h(x * y), h(x) \diamond h(y)\big) < \delta \tag{1.1}$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d\big(h(x), H(x)\big) < \epsilon \tag{1.2}$$

for all $x \in G_1$?

Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. *Assume that* $f : X \to Y$ *satisfies*

$$\big\| f(x+y) - f(x) - f(y) \big\| \le \varepsilon \tag{1.3}$$

for some $\varepsilon \geq 0$ *and all* $x, y \in X$. *Then there exists a unique additive mapping* $T : X \to Y$ *such that*

$$\|f(x) - T(x)\| \leq \varepsilon \tag{1.4}$$

for all $x \in X$.

Aoki [3] and Th. M. Rassias [4] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.1 (Th. M. Rassias [4]). *Let* $f : E \to E'$ *be a mapping from a normed vector space* E *into a Banach space* E' *subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.5}$$

for all $x, y \in E$, *where* ϵ *and* p *are constants with* $\epsilon > 0$ *and* $p < 1$. *Then the limit*

$$L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{1.6}$$

exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.7}$$

for all $x \in E$. *If* $p < 0$, *then the inequality* (1.5) *holds for* $x, y \neq 0$ *and* (1.7) *for* $x \neq 0$. *Also, if for each* $x \in E$ *the mapping* $t \mapsto f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then* L *is* $\mathbb{R}$- *linear.*

Theorem 1.2 (J. M. Rassias [5–7]). *Let* X *be a real normed linear space and let* Y *be a real Banach space. Assume that* $f : X \to Y$ *is a mapping for which there exist constants* $\theta \geq 0$ *and* $p, q \in \mathbb{R}$ *such that* $r = p + q \neq 1$ *and* f *satisfies the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\|x\|^p\|y\|^q \tag{1.8}$$

for all $x, y \in X$. *Then there exists a unique additive mapping* $L : X \to Y$ *satisfying*

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|}\|x\|^r \tag{1.9}$$

for all $x \in X$. *If, in addition,* $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then* L *is linear.*

In 1994, a generalization of Theorems 1.1 and 1.2 was obtained by Găvruţa [8], who replaced the bounds $\varepsilon(\|x\|^p + \|y\|^p)$ and $\theta\|x\|^p\|y\|^q$ by a general control function $\varphi(x, y)$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.10}$$

is called a *quadratic functional equation*. Quadratic functional equations were used to characterize inner product spaces [9–11]. In particular, every solution of the quadratic equation (1.10) is said to be a *quadratic mapping*. It is well known that a mapping f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive mapping B such that $f(x) = B(x,x)$ for all x (see [9, 12]). The biadditive mapping B is given by

$$B(x,y) = \frac{1}{4}\left[f(x+y) - f(x-y)\right]. \tag{1.11}$$

The generalized Hyers-Ulam stability problem for the quadratic functional equation (1.10) was proved by Skof for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space (see [13]). Cholewa [14] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. J. M. Rassias [15] and Czerwik [16], proved the stability of the quadratic functional equation (1.10). Grabiec [17] has generalized these results mentioned above. J. M. Rassias [18] introduced and investigated the stability problem of Ulam for the Euler-Lagrange quadratic mappings:

$$f(a_1x_1 + a_2x_2) + f(a_2x_1 - a_1x_2) = \left(a_1^2 + a_2^2\right)\left[f(x_1) + f(x_2)\right]. \tag{1.12}$$

In addition, J. M. Rassias [19] generalized the Euler-Lagrange quadratic mapping (1.12) and investigated its stability problem. The Euler-Lagrange quadratic mapping (1.12) has provided a lot of influence in the development of general Euler-Lagrange quadratic equations (mappings) which is now known as Euler-Lagrange-Rassias quadratic functional equations (mappings).

Jun and Lee [20] proved the generalized Hyers-Ulam stability of a pexiderized quadratic equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8, 20–47]). We also refer the readers to the books [48–51].

Let E be a set. A function $d : E \times E \to [0,\infty]$ is called a *generalized metric* on E if d satisfies

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x, y \in E$,

(iii) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz.

Theorem 1.3 (see [52]). *Let (E,d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in E$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.13}$$

for all nonnegative integers n or there exists a nonnegative integer n_0 such that

(1) $d\left(J^n x, J^{n+1} x\right) < \infty$ *for all $n \ge n_0$,*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J,*

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in E : d(J^{n_0}x, y) < \infty\}$,

(4) $d(y, y^*) \leq (1/(1-L))d(y, Jy)$ *for all* $y \in Y$.

Throughout this paper, we assume that r, s are nonzero rational numbers with $r^2 + s^2 \neq 1$, and that A is a unital Banach algebra with unit e, norm $|\cdot|$, and $A_1 := \{a \in A : |a| = 1\}$. Assume that X is a normed left A-module and Y is a (unit linked) Banach left A-module. A quadratic mapping $T : X \to Y$ is called *A-quadratic* if $T(ax) = a^2T(x)$ for all $a \in A$ and all $x \in X$.

In this paper, we investigate an A-quadratic mapping associated with the generalized quadratic functional equation

$$f(rx + sy) = r^2 f(x) + s^2 f(y) + \frac{rs}{2}[f(x + y) - f(x - y)], \tag{1.14}$$

and using the fixed point method (see [24, 25, 38, 53–55]), we prove the generalized Hyers-Ulam stability of A-quadratic mappings in Banach A-modules associated with the functional equation (1.14). In 1996, Isac and Th. M. Rassias [56] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

For convenience, we use the following abbreviation for a given $a \in A$ and a mapping $f : X \to Y$:

$$D_a f(x, y) := f(rax + sy) - r^2a^2 f(x) - s^2 f(y) - \frac{rs}{2}[f(ax + y) - f(ax - y)] \tag{1.15}$$

for all $x, y \in X$.

2. Fixed Points and Stability of the Generalized Quadratic Functional Equation (1.14)

Proposition 2.1. *A mapping* $f : X \to Y$ *satisfies*

$$D_1 f(x, y) = 0 \tag{2.1}$$

for all $x, y \in X$ *if and only if* f *is quadratic.*

Proof. Let f satisfy (2.1). Since $r^2 + s^2 \neq 1$, letting $x = y = 0$ in (2.1), we get $f(0) = 0$. Letting $y = 0$ in (2.1), we get

$$f(rx) = r^2 f(x) \tag{2.2}$$

for all $x \in X$. It follows from (2.1) that $D_1 f(x, y) + D_1 f(x, -y) = 0$ for all $x, y \in X$. Hence

$$f(rx + sy) + f(rx - sy) = 2r^2 f(x) + s^2[f(y) + f(-y)] \tag{2.3}$$

for all $x, y \in X$. We decompose f into the even part and the odd part by putting

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \qquad f_o(x) = \frac{f(x) - f(-x)}{2} \tag{2.4}$$

for all $x \in X$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in X$. It is easy to show that the mappings f_e and f_o satisfy (2.2) and (2.3). Thus we have

$$f_e(rx + sy) + f_e(rx - sy) = 2r^2 f_e(x) + 2s^2 f_e(y), \tag{2.5}$$

$$f_o(rx + sy) + f_o(rx - sy) = 2r^2 f_o(x) \tag{2.6}$$

for all $x, y \in X$. Letting $x = 0$ in (2.5), we get

$$f_e(sy) = s^2 f_e(y) \tag{2.7}$$

for all $y \in X$. It follows from (2.2), (2.5), and (2.7) that

$$f_e(rx + sy) + f_e(rx - sy) = 2f_e(rx) + 2f_e(sy) \tag{2.8}$$

for all $x, y \in X$. Therefore,

$$f_e(x + y) + f_e(x - y) = 2f_e(x) + 2f_e(y) \tag{2.9}$$

for all $x, y \in X$. So f_e is quadratic. We claim that $f_o \equiv 0$. For this, it follows from (2.2) and (2.6) that

$$f_o(rx + sy) + f_o(rx - sy) = 2f_o(rx) \tag{2.10}$$

for all $x, y \in X$. So

$$f_o(x + y) + f_o(x - y) = 2f_o(x) \tag{2.11}$$

for all $x, y \in X$. Letting $y = x$ in (2.11), we get $f_o(2x) = 2f_o(x)$ for all $x \in X$. So it follows from (2.11) that

$$f_o(x + y) + f_o(x - y) = f_o(2x) \tag{2.12}$$

for all $x, y \in X$. Replacing x by $(x + y)/2$ and y by $(x - y)/2$ in (2.12), we infer that f_o is additive. To complete the proof we have two cases.

Case 1 ($r = 1$). Since f_o is additive and satisfies (2.1), letting $x = 0$ and replacing f_o by f in (2.1), we get $s^2 f_o(y) = 0$ for all $y \in X$. Since $s \neq 0$, we get $f_o \equiv 0$.

Case 2 $(r \neq 1)$. Since f_o is additive and satisfies (2.2), we have $(r^2 - r)f_o(x) = 0$ for all $x \in X$. Since $r \neq 0, 1$, we get $f_o \equiv 0$.

Hence $f = f_e$ and this proves that f is quadratic.

Conversely, let f be quadratic. Then there exists a unique symmetric biadditive mapping $B : X \times X \to Y$ such that $f(x) = B(x, x)$ for all $x \in X$ and

$$B(x,y) = \frac{1}{4}\left[f(x+y) - f(x-y)\right] \tag{2.13}$$

for all $x, y \in X$ (see [9, 12]). Hence

$$\begin{aligned} f(rx+sy) &= B(rx+sy, rx+sy) \\ &= r^2 B(x,x) + s^2 B(y,y) + 2rsB(x,y) \\ &= r^2 f(x) + s^2 f(y) + \frac{rs}{2}\left[f(x+y) - f(x-y)\right] \end{aligned} \tag{2.14}$$

for all $x, y \in X$. Hence f satisfies (2.1). □

Corollary 2.2. *Let $f : X \to Y$ be a mapping satisfying*

$$D_a f(x,y) = 0 \tag{2.15}$$

for all $x, y \in X$ and all $a \in A_1$. If for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then f is A-quadratic.

Proof. Let $a = e$. By Proposition 2.1, f is quadratic. Thus f is $\mathbb{Q}$-quadratic. Let $\alpha \in \mathbb{R}$ and let $\{r_n\}_n$ be a sequence of rational numbers such that $\lim_{n\to\infty} r_n = \alpha$. Since f is $\mathbb{Q}$-quadratic and the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x \in X$, we have

$$f(\alpha x) = \lim_{n\to\infty} f(r_n x) = \lim_{n\to\infty} r_n^2 f(x) = \alpha^2 f(x) \tag{2.16}$$

for all $x \in X$. So f is $\mathbb{R}$-quadratic. Letting $y = 0$ in (2.15), we get

$$f(ax) = a^2 f(x) \tag{2.17}$$

for all $x \in X$ and all $a \in A_1$. It is clear that (2.17) is also true for $a = 0$. For each element $a \in A$ $(a \neq 0)$, $a = |a| \cdot (a/|a|)$. Since f is $\mathbb{R}$-quadratic and $f(bx) = b^2 f(x)$ for all $x \in X$ and all $b \in A_1$, we have

$$f(ax) = f\left(|a| \cdot \frac{a}{|a|}x\right) = |a|^2 f\left(\frac{a}{|a|}x\right) = |a|^2 \cdot \frac{a^2}{|a|^2} \cdot f(x) = a^2 f(x) \tag{2.18}$$

for all $x \in X$ and all $a \in A$ $(a \neq 0)$. So the $\mathbb{R}$-quadratic mapping $f : X \to Y$ is also A-quadratic. This completes the proof. □

Now we prove the generalized Hyers-Ulam stability of A-quadratic mappings in Banach A-modules.

Theorem 2.3. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that*

$$\|D_a f(x,y)\| \leq \varphi(x,y) \tag{2.19}$$

for all $x, y \in X$ and all $a \in A_1$. Let $0 < L < 1$ be a constant such that $r^2\varphi(x,y) \leq L\varphi(rx,ry)$ for all $x, y \in X$. If for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique A-quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \leq \frac{L}{r^2(1-L)}\varphi(x,0) \tag{2.20}$$

for all $x \in X$.

Proof. It follows from $r^2\varphi(x,y) \leq L\varphi(rx,ry)$ that

$$\lim_{n\to\infty} r^{2n}\varphi\left(\frac{x}{r^n},\frac{y}{r^n}\right) = 0 \tag{2.21}$$

for all $x, y \in X$.

Letting $y = 0$ in (2.19), we get

$$\|f(rax) - r^2a^2f(x)\| \leq \varphi(x,0) \tag{2.22}$$

for all $x \in X$ and all $a \in A_1$. Hence

$$\left\|f(ax) - r^2a^2f\left(\frac{x}{r}\right)\right\| \leq \varphi\left(\frac{x}{r},0\right) \leq \frac{L}{r^2}\varphi(x,0) \tag{2.23}$$

for all $x \in X$ and all $a \in A_1$. Let $E := \{g : X \to Y \mid g(0) = 0\}$. We introduce a generalized metric on E as follows:

$$d(g,h) := \inf\{C \in [0,\infty] : \|g(x) - h(x)\| \leq C\varphi(x,0)\ \forall x \in X\}. \tag{2.24}$$

It is easy to show that (E,d) is a generalized complete metric space [24].

Now we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = r^2 g\left(\frac{x}{r}\right), \quad \forall g \in E,\ x \in X. \tag{2.25}$$

Let $g, h \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le C$. From the definition of d, we have

$$\|g(x) - h(x)\| \le C\varphi(x, 0) \tag{2.26}$$

for all $x \in X$. By the assumption and the last inequality, we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = r^2 \left\| g\left(\frac{x}{r}\right) - h\left(\frac{x}{r}\right) \right\| \le r^2 C\varphi\left(\frac{x}{r}, 0\right) \le CL\varphi(x, 0) \tag{2.27}$$

for all $x \in X$. So

$$d(\Lambda g, \Lambda h) \le Ld(g, h) \tag{2.28}$$

for any $g, h \in E$. It follows from (2.23) (by letting $a = e$) that $d(\Lambda f, f) \le L/r^2$. According to Theorem 1.3, the sequence $\{\Lambda^n f\}$ converges to a fixed point Q of Λ, that is,

$$Q : X \longmapsto Y, \qquad Q(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} r^{2n} f\left(\frac{x}{r^n}\right), \tag{2.29}$$

and $Q(rx) = r^2 Q(x)$ for all $x \in X$. Also Q is the unique fixed point of Λ in the set $E^* = \{g \in E : d(f, g) < \infty\}$ and

$$d(Q, f) \le \frac{1}{1-L} d(\Lambda f, f) \le \frac{L}{r^2(1-L)}, \tag{2.30}$$

that is, the inequality (2.20) holds true for all $x \in X$. It follows from the definition of Q, (2.19), and (2.21) that

$$\|D_a Q(x, y)\| = \lim_{n\to\infty} r^{2n} \left\| D_a f\left(\frac{x}{r^n}, \frac{y}{r^n}\right) \right\| \le \lim_{n\to\infty} r^{2n} \varphi\left(\frac{x}{r^n}, \frac{y}{r^n}\right) = 0 \tag{2.31}$$

for all $x, y \in X$ and all $a \in A_1$. By Proposition 2.1 (by letting $a = e$), the mapping Q is quadratic. Let $L : Y \to \mathbb{R}$ be a continuous linear functional. For any $x \in X$, we consider the mapping $\psi_x : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_x(t) := L[Q(tx)]. \tag{2.32}$$

Since Q is quadratic and L is linear,

$$\begin{aligned} \psi_x(u+v) + \psi_x(u-v) &= L[Q(ux+vx) + Q(ux-vx)] \\ &= L[2Q(ux) + 2Q(vx)] \\ &= 2\psi_x(u) + 2\psi_x(v) \end{aligned} \tag{2.33}$$

for all $u, v \in \mathbb{R}$. So ψ_x is quadratic. Also ψ_x is measurable since it is the pointwise limit of the sequence

$$\psi_{n,x}(t) := r^{2n} L\left[f\left(\frac{tx}{r^n}\right)\right]. \tag{2.34}$$

It follows from [48, Corollary 10.2] that $\psi_x(t) = t^2\psi_x(1)$ for all $t \in \mathbb{R}$. Then

$$L[Q(tx)] = \psi_x(t) = t^2\psi_x(1) = t^2 L[Q(x)] = L[t^2 Q(x)] \tag{2.35}$$

for all $t \in \mathbb{R}$. Hence $Q(tx) = t^2 Q(x)$ for all $t \in \mathbb{R}$ and all $x \in X$. By Corollary 2.2, the mapping Q is A-quadratic. □

Corollary 2.4. *Let $p > 0$ and θ be nonnegative real numbers such that $r^2 < |r|^p$ and let $f : X \to Y$ be a mapping satisfying the inequality*

$$\|D_a f(x, y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{2.36}$$

for all $x, y \in X$ and all $a \in A_1$. If for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique A-quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{\theta}{|r|^p - r^2}\|x\|^p \tag{2.37}$$

for all $x \in X$.

Proof. Letting $a = e$ and $x = y = 0$ in (2.36), we get $f(0) = 0$. Now, the proof follows from Theorem 2.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{2.38}$$

for all $x, y \in X$. Then we can choose $L = |r|^{2-p}$ and we get the desired result. □

Remark 2.5. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\Phi : X^2 \to [0, \infty)$ such that

$$\|D_a f(x, y)\| \le \Phi(x, y) \tag{2.39}$$

for all $x, y \in X$ and all $a \in A_1$. Let $0 < L < 1$ be a constant such that $\Phi(rx, ry) \le r^2 L\Phi(x, y)$ for all $x, y \in X$. By a similar method to the proof of Theorem 2.3, one can show that if for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique A-quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \le \frac{1}{r^2(1 - L)}\Phi(x, 0) \tag{2.40}$$

for all $x \in X$.

For the case $\Phi(x,y) := \delta + \theta(\|x\|^p + \|y\|^p)$ (where δ, θ are nonnegative real numbers and $p > 0$ with $1 < |r|^p < r^2$), there exists a unique A-quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \le \frac{\delta}{r^2 - |r|^p} + \frac{\theta}{r^2 - |r|^p}\|x\|^p \tag{2.41}$$

for all $x \in X$.

Corollary 2.6. *Let $p, q > 0$ and let θ be nonnegative real numbers such that $r^2 \neq |r|^{p+q}$ and let $f : X \to Y$ be a mapping satisfying the inequality*

$$\|D_a f(x,y)\| \le \theta \|x\|^p \|y\|^q \tag{2.42}$$

for all $x, y \in X$ and all $a \in A_1$. If for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then f is A-quadratic.

Theorem 2.7. *Let $f : X \to Y$ be an even mapping for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (2.19) *and*

$$\lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{2.43}$$

for all $x, y \in X$ and all $a \in A_1$. Let $0 < L < 1$ be a constant such that the mapping

$$x \longmapsto \phi(x) := \varphi\left(\frac{x}{r}, \frac{x}{s}\right) + \varphi\left(\frac{x}{r}, \frac{-x}{s}\right) + 2\varphi\left(\frac{x}{r}, 0\right) + 2\varphi\left(0, \frac{x}{s}\right) \tag{2.44}$$

satisfying $4\phi(x) \le L\phi(2x)$ for all $x \in X$. If for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique A-quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \le \frac{L}{4(1-L)}\phi(x) \tag{2.45}$$

for all $x \in X$.

Proof. Since $\varphi(0,0) = 0$, it follows from (2.19) that $f(0) = 0$ and

$$\begin{aligned} &\|D_a f(x,y) + D_a f(x,-y) - 2D_a f(x,0) - 2D_a f(0,y)\| \\ &\quad \le \varphi(x,y) + \varphi(x,-y) + 2\varphi(x,0) + 2\varphi(0,y) \end{aligned} \tag{2.46}$$

for all $x, y \in X$ and all $a \in A_1$. Therefore,

$$\begin{aligned} &\|f(rax + sy) + f(rax - sy) - 2f(rax) - 2f(sy)\| \\ &\quad \le \varphi(x,y) + \varphi(x,-y) + 2\varphi(x,0) + 2\varphi(0,y) \end{aligned} \tag{2.47}$$

for all $x, y \in X$ and all $a \in A_1$. Letting $a = e$ and replacing x by x/r and y by y/s in (2.47), we get

$$\left\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \right\| \leq \Phi(x,y) \tag{2.48}$$

for all $x, y \in X$, where

$$\Phi(x,y) := \varphi\left(\frac{x}{r}, \frac{y}{s}\right) + \varphi\left(\frac{x}{r}, \frac{-y}{s}\right) + 2\varphi\left(\frac{x}{r}, 0\right) + 2\varphi\left(0, \frac{y}{s}\right). \tag{2.49}$$

Letting $y = x$ in (2.48), we get

$$\left\| f(2x) - 4f(x) \right\| \leq \phi(x) \tag{2.50}$$

for all $x \in X$. Hence

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq \phi\left(\frac{x}{2}\right) \leq \frac{L}{4}\phi(x) \tag{2.51}$$

for all $x \in X$. Let $E := \{g : X \to Y \mid g(0) = 0\}$. We introduce a generalized metric on E as follows:

$$d(g,h) := \inf\left\{C \in [0,\infty] : \left\| g(x) - h(x) \right\| \leq C\phi(x) \ \forall x \in X\right\}. \tag{2.52}$$

Now we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = 4g\left(\frac{x}{2}\right), \quad \forall g \in E, \ x \in X. \tag{2.53}$$

Similar to the proof of Theorem 2.3, we deduce that the sequence $\{\Lambda^n f\}$ converges to a fixed point Q of Λ which is A-quadratic. Also Q is the unique fixed point of Λ in the set $E^* = \{g \in E : d(f,g) < \infty\}$ and satisfies (2.45). □

Corollary 2.8. *Let $p > 2$ and let θ be nonnegative real numbers and let $f : X \to Y$ be an even mapping satisfying the inequality* (2.36) *for all $x, y \in X$ and all $a \in A_1$. If for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique A-quadratic mapping $Q : X \to Y$ such that*

$$\left\| f(x) - Q(x) \right\| \leq \frac{4\theta\left(|r|^p + |s|^p\right)}{\left(2^p - 4\right)|rs|^p}\|x\|^p \tag{2.54}$$

for all $x \in X$.

Proof. Letting $a = e$ and $x = y = 0$ in (2.36), we get $f(0) = 0$. Now the proof follows from Theorem 2.7 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.55}$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Remark 2.9. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ for which there exists a function $\Phi : X^2 \to [0, \infty)$ such that

$$\lim_{n\to\infty} \frac{1}{4^n}\Phi(2^n x, 2^n y) = 0, \qquad \left\|D_a f(x, y)\right\| \leq \Phi(x, y) \tag{2.56}$$

for all $x, y \in X$ and all $a \in A_1$. Let $0 < L < 1$ be a constant such that the mapping

$$x \longmapsto \phi(x) := \Phi\left(\frac{x}{r}, \frac{x}{s}\right) + \Phi\left(\frac{x}{r}, \frac{-x}{s}\right) + 2\Phi\left(\frac{x}{r}, 0\right) + 2\Phi\left(0, \frac{x}{s}\right) \tag{2.57}$$

satisfying $\phi(2x) \leq 4L\phi(x)$ for all $x \in X$. By a similar method to the proof of Theorem 2.7, one can show that if for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique A-quadratic mapping $Q : X \to Y$ satisfying

$$\left\|f(x) - Q(x)\right\| \leq \frac{1}{4(1 - L)}\phi(x) \tag{2.58}$$

for all $x \in X$.

For the case $\Phi(x, y) := \delta + \theta(\|x\|^p + \|y\|^p)$ (where δ, θ are nonnegative real numbers and $0 < p < 2$), there exists a unique A-quadratic mapping $Q : X \to Y$ satisfying

$$\left\|f(x) - Q(x)\right\| \leq \frac{6\delta}{4 - 2^p} + \frac{4\theta\left(|r|^p + |s|^p\right)}{(4 - 2^p)|rs|^p}\|x\|^p \tag{2.59}$$

for all $x \in X$.

Corollary 2.10. *Let $p, q > 0$ and let θ be nonnegative real numbers such that $p + q \neq 2$ and let $f : X \to Y$ be an even mapping satisfying the inequality* (2.42) *for all $x, y \in X$ and all $a \in A_1$. If for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then f is A-quadratic.*

We may omit the evenness of the mapping f in Theorem 2.7.

Theorem 2.11. *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (2.19) *and* (2.43) *for all $x, y \in X$ and all $a \in A_1$. Let $0 < L < 1$ be a constant such that the mapping*

$$x \longmapsto \phi(x) := \varphi\left(\frac{x}{r}, \frac{x}{s}\right) + \varphi\left(\frac{x}{r}, \frac{-x}{s}\right) + 2\varphi\left(\frac{x}{r}, 0\right) + 2\varphi\left(0, \frac{x}{s}\right) \tag{2.60}$$

satisfying $4\phi(x) \le L\phi(2x)$ *for all* $x \in X$. *If for each* $x \in X$ *the mapping* $t \mapsto f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then there exists a unique* A*-quadratic mapping* $Q : X \to Y$ *satisfying*

$$\|f(x) - Q(x)\| \le \frac{L(4-3L)}{8(1-L)(2-L)}[\phi(x) + \phi(-x)] \tag{2.61}$$

for all $x \in X$.

Proof. Since $\varphi(0,0) = 0$, it follows from (2.19) that $f(0) = 0$. We decompose f into the even part f_e and the odd part f_o. It follows from (2.19) that

$$\begin{aligned} \|D_a f_e(x,y)\| &\le \frac{1}{2}[\varphi(x,y) + \varphi(-x,-y)], \\ \|D_a f_o(x,y)\| &\le \frac{1}{2}[\varphi(x,y) + \varphi(-x,-y)] \end{aligned} \tag{2.62}$$

for all $x, y \in X$ and all $a \in A_1$. By Theorem 2.7, there exists a unique A-quadratic mapping $Q : X \to Y$ satisfying

$$\|f_e(x) - Q(x)\| \le \frac{L}{8(1-L)}[\phi(x) + \phi(-x)] \tag{2.63}$$

for all $x \in X$. We get from (2.62) that

$$\|D_a f_o(x,y) + D_a f_o(x,-y) - 2D_a f_o(x,0)\| \le \Psi(x,y) \tag{2.64}$$

for all $x, y \in X$ and all $a \in A_1$, where

$$\Psi(x,y) := \frac{1}{2}[\varphi(x,y) + \varphi(-x,-y) + \varphi(x,-y) + \varphi(-x,y) + 2\varphi(x,0) + 2\varphi(-x,0)]. \tag{2.65}$$

Hence

$$\|f_o(x+y) + f_o(x-y) - 2f_o(x)\| \le \Psi\left(\frac{x}{r}, \frac{y}{s}\right) \tag{2.66}$$

for all $x, y \in X$. Letting $y = x$ in (2.66), we get

$$\|f_o(2x) - 2f_o(x)\| \le \Psi\left(\frac{x}{r}, \frac{x}{s}\right) \tag{2.67}$$

for all $x \in X$. Therefore,

$$\left\|2f_o\left(\frac{x}{2}\right) - f_o(x)\right\| \le \frac{1}{2}\left[\phi\left(\frac{x}{2}\right) + \phi\left(\frac{-x}{2}\right)\right] \le \frac{L}{8}[\phi(x) + \phi(-x)] \tag{2.68}$$

for all $x \in X$. Let $E := \{g : X \to Y \mid g(0) = 0\}$. We introduce a generalized metric on E as follows:

$$d(g,h) := \inf\{C \in [0,\infty] : \|g(x) - h(x)\| \le C[\phi(x) + \phi(-x)]\ \forall x \in X\}. \tag{2.69}$$

Now we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = 2g\left(\frac{x}{2}\right), \quad \forall g \in E,\ x \in X. \tag{2.70}$$

Similar to the proof of Theorem 2.3, we deduce that the sequence $\{\Lambda^n f_o\}$ converges to a fixed point T of Λ which is quadratic and

$$d(T, f_o) \le \frac{2}{2-L} d(\Lambda f_o, f_o) \le \frac{2L}{16-8L}. \tag{2.71}$$

Also T is odd since f_o is odd. Therefore, $T \equiv 0$ since T is quadratic too. Now (2.61) follows from (2.63) and (2.71). □

Corollary 2.12. *Let $p > 2$ and let θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying the inequality* (2.36) *for all $x, y \in X$ and all $a \in A_1$. If for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique A-quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{8\theta(2^p - 3)(|r|^p + |s|^p)}{(2^p - 2)(2^p - 4)|rs|^p}\|x\|^p \tag{2.72}$$

for all $x \in X$.

Proof. Letting $a = e$ and $x = y = 0$ in (2.36), we get $f(0) = 0$. Now the proof follows from Theorem 2.11 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{2.73}$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Remark 2.13. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\Phi : X^2 \to [0,\infty)$ such that

$$\lim_{n\to\infty} \frac{1}{2^n}\Phi(2^n x, 2^n y) = 0, \qquad \|D_a f(x,y)\| \le \Phi(x,y) \tag{2.74}$$

for all $x, y \in X$ and all $a \in A_1$. Let $0 < L < 1/2$ be a constant such that the mapping

$$x \longmapsto \phi(x) := \Phi\left(\frac{x}{r}, \frac{x}{s}\right) + \Phi\left(\frac{x}{r}, \frac{-x}{s}\right) + 2\Phi\left(\frac{x}{r}, 0\right) + 2\Phi\left(0, \frac{x}{s}\right) \tag{2.75}$$

satisfying $\phi(2x) \leq 4L\phi(x)$ for all $x \in X$. By a similar method to the proof of Theorem 2.11, one can show that if for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique A-quadratic mapping $Q : X \to Y$ satisfying

$$\begin{aligned} \left\| f_e(x) - Q(x) \right\| &\leq \frac{1}{8(1-L)} \left[\phi(x) + \phi(-x) \right], \\ \left\| f_o(x) \right\| &\leq \frac{1}{4(1-2L)} \left[\phi(x) + \phi(-x) \right] \end{aligned} \tag{2.76}$$

for all $x \in X$. Hence

$$\left\| f(x) - Q(x) \right\| \leq \frac{3-4L}{8(1-L)(1-2L)} \left[\phi(x) + \phi(-x) \right] \tag{2.77}$$

for all $x \in X$.

For the case $\Phi(x,y) := \delta + \theta(\|x\|^p + \|y\|^p)$ (where δ, θ are nonnegative real numbers and $0 < p < 1$), there exists a unique A-quadratic mapping $Q : X \to Y$ satisfying

$$\left\| f(x) - Q(x) \right\| \leq \frac{12\delta(3-2^p)}{(2-2^p)(4-2^p)} + \frac{8\theta(3-2^p)(|r|^p + |s|^p)}{(2-2^p)(4-2^p)|rs|^p} \|x\|^p \tag{2.78}$$

for all $x \in X$.

For the case $p = 2$, we have the following counterexample which is a modification of the example of Czerwik [16].

Example 2.14. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) := \begin{cases} \mu x^2 & \text{for } |x| < 1, \\ \mu & \text{for } |x| \geq 1, \end{cases} \tag{2.79}$$

where μ is a positive real number. Consider the function $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} \alpha^{-2n} \phi(\alpha^n x), \tag{2.80}$$

where $\alpha = \sqrt{1 + r^2 + s^2 + |rs|}$. It is clear that f is continuous and bounded by $(\alpha^2/(\alpha^2-1))\mu$ on $\mathbb{R}$. We prove that

$$\left| f(rx+sy) - r^2 f(x) - s^2 f(y) - \frac{rs}{2} \left[f(x+y) - f(x-y) \right] \right| \leq \frac{\alpha^{10}}{\alpha^2 - 1} \mu (x^2 + y^2) \tag{2.81}$$

for all $x, y \in \mathbb{R}$. To see this, if $x^2 + y^2 = 0$ or $x^2 + y^2 \geq \alpha^{-4}$, then

$$\begin{aligned} &\left| f(rx+sy) - r^2 f(x) - s^2 f(y) - \frac{rs}{2}\left[f(x+y) - f(x-y)\right]\right| \\ &\quad \leq \alpha^2 \mu \sum_{n=0}^{\infty} \alpha^{-2n} \leq \frac{\alpha^8}{\alpha^2 - 1}\mu\left(x^2+y^2\right). \end{aligned} \tag{2.82}$$

Now suppose that $x^2 + y^2 < \alpha^{-4}$. Then there exists a nonnegative integer k such that

$$\alpha^{-4(k+2)} \leq x^2 + y^2 < \alpha^{-4(k+1)}. \tag{2.83}$$

Therefore,

$$\alpha^{2k}|x|, \alpha^{2k}|y|, \alpha^{2k}|rx+sy|, \alpha^{2k}|x \pm y| \in (-1, 1). \tag{2.84}$$

Hence

$$\alpha^{2m}|x|, \alpha^{2m}|y|, \alpha^{2m}|rx+sy|, \alpha^{2m}|x \pm y| \in (-1, 1) \tag{2.85}$$

for all $m = 0, 1, \ldots, 2k$. From the definition of f and (2.83), we have

$$\begin{aligned} &\left| f(rx+sy) - r^2 f(x) - s^2 f(y) - \frac{rs}{2}\left[f(x+y) - f(x-y)\right]\right| \\ &\quad \leq \alpha^2 \mu \sum_{n=2k+1}^{\infty} \alpha^{-2n} \leq \frac{\alpha^{10}}{\alpha^2 - 1}\mu\left(x^2+y^2\right). \end{aligned} \tag{2.86}$$

Therefore, f satisfies (2.81). Let $Q : \mathbb{R} \to \mathbb{R}$ be a quadratic function such that

$$\left| f(x) - Q(x) \right| \leq \beta x^2 \tag{2.87}$$

for all $x \in \mathbb{R}$. Then there exists a constant $c \in \mathbb{R}$ such that $Q(x) = cx^2$ for all $x \in \mathbb{R}$ (see [57]). So we have

$$\left| f(x) \right| \leq (\beta + |c|)x^2 \tag{2.88}$$

for all $x \in \mathbb{R}$. Let $m \in \mathbb{N}$ with $m\mu > \beta + |c|$. If $x \in (0, \alpha^{1-m})$, then $\alpha^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. So

$$f(x) \geq \sum_{n=0}^{m-1} \alpha^{-2n} \phi(\alpha^n x) = m\mu x^2 > (\beta + |c|)x^2, \tag{2.89}$$

which contradicts (2.88).

Corollary 2.15. *Let $p, q > 0$ and let θ be nonnegative real numbers such that $p + q > 2$ $(p + q < 1)$ and let $f : X \to Y$ be a mapping satisfying the inequality* (2.42) *for all $x, y \in X$ and all $a \in A_1$. If for each $x \in X$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then f is A-quadratic.*

Acknowledgments

The authors would like to thank the referees for bringing some useful references to their attention. The second author was supported by Korea Research Foundation Grant KRF-2008-313-C00041.

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