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Research Article

Inclusion Properties for Certain Classes of Meromorphic Functions Associated with a Family of Linear Operators

Nak Eun Cho

Department of Applied Mathematics, Pukyong National University, Pusan 608-737, South Korea

Correspondence should be addressed to Nak Eun Cho, necho@pknu.ac.kr

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The purpose of the present paper is to investigate some inclusion properties of certain classes of meromorphic functions associated with a family of linear operators, which are defined by means of the Hadamard product (or convolution). Some invariant properties under convolution are also considered for the classes presented here. The results presented here include several previous known results as their special cases.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with the usual normalization $f(0) = f'(0) - 1 = 0$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$ such that $f(z) = g(w(z))$.

Let $\mathcal{N}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\operatorname{Re}\{\phi(z)\} > 0$ for $z \in \mathbb{U}$. We denote by $\mathcal{S}^*$ and $\mathcal{K}$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are starlike and convex, respectively.

Let $\mathcal{M}$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured open unit disk $\mathbb{D} = \mathbb{U} \setminus \{0\}$. For $0 \leq \eta,\ \beta < 1$, we denote by $\mathcal{MS}(\eta)$, $\mathcal{MK}(\eta)$ and $\mathcal{MC}(\eta, \beta)$ the subclasses of $\mathcal{M}$ consisting of all meromorphic functions which are, respectively, starlike of order η, convex of order η and colse-to-convex of order β and type η in $\mathbb{U}$ (see, for details, [1, 2]).

Making use of the principle of subordination between analytic functions, we introduce the subclasses $\mathcal{MS}(\eta,\phi)$, $\mathcal{MK}(\eta,\phi)$ and $\mathcal{MC}(\eta,\beta;\phi,\psi)$ of the class $\mathcal{M}$ for $0 \le \eta,\beta < 1$ and $\phi,\psi \in \mathcal{N}$, which are defined by

$$\begin{aligned}
\mathcal{MS}(\eta;\phi) &:= \left\{ f \in \mathcal{M} : \frac{1}{1-\eta}\left(-\frac{zf'(z)}{f(z)} - \eta\right) \prec \phi(z) \text{ in } \mathbb{U} \right\}, \\
\mathcal{MK}(\eta;\phi) &:= \left\{ f \in \mathcal{M} : \frac{1}{1-\eta}\left(-\left\{1 + \frac{zf''(z)}{f'(z)}\right\} - \eta\right) \prec \phi(z) \text{ in } \mathbb{U} \right\}, \\
\mathcal{MC}(\eta,\beta;\phi,\psi) &:= \left\{ f \in \mathcal{M} : \exists g \in \mathcal{MS}(\eta;\ \phi) \text{ s.t. } \frac{1}{1-\beta}\left(-\frac{zf'(z)}{g(z)} - \beta\right) \prec \psi(z) \text{ in } \mathbb{U} \right\}.
\end{aligned} \tag{1.2}$$

We note that the classes mentioned above are the familiar classes which have been used widely on the space of analytic and univalent functions in $\mathbb{U}$ (see [3–5]) and for special choices for the functions ϕ and ψ involved in these definitions, we can obtain the well-known subclasses of $\mathcal{M}$. For examples, we have

$$\begin{aligned}
\mathcal{MS}\left(\eta;\frac{1+z}{1-z}\right) &= \mathcal{MS}(\eta), \\
\mathcal{MK}\left(\eta;\frac{1+z}{1-z}\right) &= \mathcal{MK}(\eta), \\
\mathcal{MC}\left(\eta,\beta;\frac{1+z}{1-z},\frac{1+z}{1-z}\right) &= \mathcal{MC}(\eta,\beta).
\end{aligned} \tag{1.3}$$

Now we define the function $\phi(a,c;z)$ by

$$\phi(a,c;z) := \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(a)_{k+1}}{(c)_{k+1}} z^k, \tag{1.4}$$

$$\left(z \in \mathbb{U};\ a \in \mathbb{R};\ c \in \mathbb{R} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{-1,-2,\ldots\}\right), \tag{1.5}$$

where $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\nu)_k := \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1, & \text{if } k = 0,\ \nu \in \mathbb{C} \setminus \{0\}, \\ \nu(\nu+1)\cdots(\nu+k-1), & \text{if } k \in \mathbb{N} := \{1,2,\ldots\},\ \nu \in \mathbb{C}. \end{cases} \tag{1.6}$$

Let $f \in \mathcal{M}$. Denote by $L(a,c) : \mathcal{M} \to \mathcal{M}$ the operator defined by

$$L(a,c)f(z) = \phi(a,c;z) * f(z) \quad (z \in \mathbb{D}), \tag{1.7}$$

where the symbol $(*)$ stands for the Hadamard product (or convolution). The operator $L(a,c)$ was introduced and studied by Liu and Srivastava [6]. Further, we remark in passing that this operator $L(a,c)$ is closely related to the Carlson-Shaffer operator [7] defined on the space of analytic and univalent functions in $\mathbb{U}$.

Corresponding to the function $\phi(a,c;z)$, let $\phi^{\dagger}(a,c;z)$ be defined such that

$$\phi(a,c;z) * \phi_{\lambda}(a,c;z) = \frac{1}{z(1-z)^{\lambda}} \quad (\lambda > 0). \tag{1.8}$$

Analogous to $L(a,c)$, we now introduce a linear operator $\mathcal{L}_{\lambda}(a,c)$ on $\mathcal{M}$ as follows:

$$\mathcal{L}_{\lambda}(a,c)f(z)c = \phi_{\lambda}(a,c;z) * f(z), \tag{1.9}$$

$$\left(a,c \in \mathbb{R} \setminus \mathbb{Z}_0^-;\ \lambda > 0;\ z \in \mathbb{U};\ f \in \mathcal{M}\right). \tag{1.10}$$

We note that

$$\mathcal{L}_1(2,1)f(z) = f(z), \qquad \mathcal{L}_1(1,1)f(z) = zf'(z) + 2f(z). \tag{1.11}$$

We note that the operator $\mathcal{L}_{\lambda}(a,c)$ is motivated essentially to the integral operator for analytic functions defined by Choi et al. [3], which extends the Noor integral operator studied by K. I. Noor and M. A. Noor [8] (also, see [9–13]).

Next, by using the operator $\mathcal{L}_{\lambda}(a,c)$, we introduce the following classes of meromprphic functions for $\phi, \psi \in \mathcal{N}$, $a,c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > 0$ and $0 \le \eta,\ \beta < 1$:

$$\begin{aligned}
\mathcal{MS}_{a,c}^{\lambda}(\eta;\phi) &:= \left\{ f \in \mathcal{M} : \mathcal{L}_{\lambda}(a,c)f \in \mathcal{MS}(\eta;\phi) \right\}, \\
\mathcal{MK}_{a,c}^{\lambda}(\eta;\phi) &:= \left\{ f \in \mathcal{M} : \mathcal{L}_{\lambda}(a,c)f \in \mathcal{MK}(\eta;\phi) \right\}, \\
\mathcal{MC}_{a,c}^{\lambda}(\eta,\beta;\phi,\psi) &:= \left\{ f \in \mathcal{M} : \mathcal{L}_{\lambda}(a,c)f \in \mathcal{MC}(\eta,\beta;\phi,\psi) \right\}.
\end{aligned} \tag{1.12}$$

We also note that

$$f(z) \in \mathcal{MK}_{a,c}^{\lambda}(\eta;\phi) \Longleftrightarrow -zf'(z) \in \mathcal{MS}_{a,c}^{\lambda}(\eta;\phi). \tag{1.13}$$

In particular, we set

$$\begin{aligned}
\mathcal{MS}_{a,c}^{\lambda}\left(\eta;\frac{1+Az}{1+Bz}\right) &= \mathcal{MS}_{a,c}^{\lambda}(\eta;A,B) \quad (-1 < B < A \le 1), \\
\mathcal{MK}_{a,c}^{\lambda}\left(\eta;\frac{1+Az}{1+Bz}\right) &= \mathcal{MK}_{a,c}^{\lambda}(\eta;A,B) \quad (-1 < B < A \le 1).
\end{aligned} \tag{1.14}$$

In this paper, we investigate several inclusion properties of the classes $\mathcal{MS}_{a,c}^{\lambda}(\eta;\phi)$, $\mathcal{MK}_{a,c}^{\lambda}(\eta;\phi)$, and $\mathcal{MC}_{a,c}^{\lambda}(\eta;\phi)$ associated with the operator $\mathcal{L}_{\lambda}(a,c)$ defined by (1.9).Some invariant properties under convolution are also considered for the classes mentioned above. Furthermore, relevant connections of the results presented here with those obtained in earlier works are pointed out.

2. Inclusion Properties Involving the Operator $\mathcal{L}_\lambda(a,c)$

The following lemmas will be required in our investigation.

Lemma 2.1. *Let* $\phi_{\lambda_i}(a,c;z)$, $\phi_\lambda(a_i,c;z)$, *and* $\phi_\lambda(a,c_i;z)$ $(i=1,2)$ *be defined by* (1.9). *Then for* $\lambda_i>0$, $a_i,c_i\in\mathbb{R}\setminus\mathbb{Z}_0^-$ $(i=1,2)$,

$$\phi_{\lambda_1}(a,c;z)=\phi_{\lambda_2}(a,c;z)*f_{\lambda_1,\lambda_2}(z), \tag{2.1}$$

$$\phi_\lambda(a_2,c;z)=\phi_\lambda(a_1,c;z)*f_{a_1,a_2}(z), \tag{2.2}$$

$$\phi_\lambda(a,c_1;z)=\phi_\lambda(a,c_2;z)*f_{c_1,c_2}(z), \tag{2.3}$$

where

$$f_{s,t}(z)=\frac{1}{z}+\sum_{k=0}^{\infty}\frac{(s)_{k+1}}{(t)_{k+1}}z^k \quad (z\in\mathbb{D}). \tag{2.4}$$

Proof. From (1.8), we know that

$$\phi_\lambda(a,c;z)=\frac{1}{z}+\sum_{k=0}^{\infty}\frac{(c)_{k+1}(\lambda)_{k+1}}{(a)_{k+1}(1)_{k+1}}z^k \quad (z\in\mathbb{D}). \tag{2.5}$$

Therefore (2.1), (2.2) and (2.3) follow from (2.5) immediately. □

Lemma 2.2 (see [14, pages 60–61]). *Let* $t\ge s>0$. *If* $t\ge 2$ *or* $s+t\ge 3$, *then the function* $z^2f_{s,t}(z)$ *belongs to the class* $\mathcal{K}$, *where* $f_{s,t}$ *is defined by* (2.4).

Lemma 2.3 (see[15]). *Let* $f\in\mathcal{K}$ *and* $g\in\mathcal{S}^*$. *Then for every analytic function* h *in* $\mathbb{U}$,

$$\frac{(f*hg)(\mathbb{U})}{(f*g)(\mathbb{U})}\subset\overline{\mathrm{co}}h(\mathbb{U}), \tag{2.6}$$

where $\overline{\mathrm{co}}h(\mathbb{U})$ *denote the closed convex hull of* $h(\mathbb{U})$.

At first, the inclusion relationship involving the class $\mathcal{MS}^{\lambda}_{a,c}(\eta;\phi)$ *is contained in Theorem 2.4.*

Theorem 2.4. *Let* $\lambda_2\ge\lambda_1>0$, $a,c\in\mathbb{R}\setminus\mathbb{Z}_0^-$ *and* $\phi\in\mathcal{N}$ *with* $\mathrm{Re}\{\phi(z)\}<(2-\eta)/(1-\eta)$ $(0\le\eta<1)$. *If* $\lambda_2\ge 2$ *or* $\lambda_1+\lambda_2\ge 3$, *then*

$$\mathcal{MS}^{\lambda_2}_{a,c}(\eta;\phi)\subset\mathcal{MS}^{\lambda_1}_{a,c}(\eta;\phi). \tag{2.7}$$

Proof. Let $f \in \mathcal{MS}_{a,c}^{\lambda_2}(\eta;\phi)$. From the definition of $\mathcal{MS}_{a,c}^{\lambda_2}(\eta;\phi)$, we have

$$\frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{L}_{\lambda_2}(a,c)f(z)\right)'}{\mathcal{L}_{\lambda_2}(a,c)f(z)}-\eta\right)=\phi(w(z)) \quad (z \in \mathbb{U}), \tag{2.8}$$

$$\frac{z\left(z^2\mathcal{L}_{\lambda_2}(a,c)f(z)\right)'}{z^2\mathcal{L}_{\lambda_2}(a,c)f(z)}=2-(1-\eta)\phi(w(z))-\eta \prec \frac{1+z}{1-z} \quad (z \in \mathbb{U}), \tag{2.9}$$

where w is analytic in $\mathbb{U}$ with $|w(z)| < 1$ $(z \in \mathbb{U})$ and $w(0) = 0 = \phi(0) - 1$. By using (1.9), (2.1) and (2.8), we get

$$\begin{aligned}
-\frac{z\left(\mathcal{L}_{\lambda_1}(a,c)f(z)\right)'}{\mathcal{L}_{\lambda_1}(a,c)f(z)} &= -\frac{z\left(\phi_{\lambda_1}(a,c;z)*f(z)\right)'}{\phi_{\lambda_1}(a,c;z)*f(z)}\\
&= -\frac{z\left(\phi_{\lambda_2}(a,c;z)*f_{\lambda_1,\lambda_2}(z)*f(z)\right)'}{\phi_{\lambda_2}(a,c;z)*f_{\lambda_1,\lambda_2}(z)*f(z)}\\
&= \frac{f_{\lambda_1,\lambda_2}(z)*\left[-z\left(\mathcal{L}_{\lambda_2}(a,c)f(z)\right)'\right]}{f_{\lambda_1,\lambda_2}(z)*\mathcal{L}_{\lambda_2}(a,c)f(z)}\\
&= \frac{f_{\lambda_1,\lambda_2}(z)*\left[(1-\eta)\phi(w(z))+\eta\right]\mathcal{L}_{\lambda_2}(a,c)f(z)}{f_{\lambda_1,\lambda_2}(z)*\mathcal{L}_{\lambda_2}(a,c)f(z)}.
\end{aligned} \tag{2.10}$$

Therefore by using (2.8), we obtain

$$\begin{aligned}
&\frac{1}{1-\eta}\left(-\frac{z\left(\mathcal{L}_{\lambda_1}(a,c)f(z)\right)'}{\mathcal{L}_{\lambda_1}(a,c)f(z)}-\eta\right)\\
&= \frac{1}{1-\eta}\left(\frac{f_{\lambda_1,\lambda_2}(z)*\left[(1-\eta)\phi(w(z))+\eta\right]\mathcal{L}_{\lambda_2}(a,c)f(z)}{f_{\lambda_1,\lambda_2}(z)*\mathcal{L}_{\lambda_2}(a,c)f(z)}-\eta\right)\\
&= \frac{1}{1-\eta}\left(\frac{z^2f_{\lambda_1,\lambda_2}(z)*\left[(1-\eta)\phi(w(z))+\eta\right]z^2\mathcal{L}_{\lambda_2}(a,c)f(z)}{z^2f_{\lambda_1,\lambda_2}(z)*z^2\mathcal{L}_{\lambda_2}(a,c)f(z)}-\eta\right).
\end{aligned} \tag{2.11}$$

It follows from (2.9) and Lemma 2.2 that $z^2\mathcal{L}_{\lambda_2}(a,c)f(z) \in \mathcal{S}^*$ and $z^2 f_{\lambda_1,\lambda_2} \in \mathcal{K}$, respectively. Let us put $s(w(z)) := (1-\eta)\phi(w(z)) + \eta$. Then by applying Lemma 2.3 to (2.10), we obtain

$$\frac{\left\{z^2f_{\lambda_1,\lambda_2}*s(w(z))z^2\mathcal{L}_{\lambda_2}(a,c)f\right\}}{\left\{z^2f_{\lambda_1,\lambda_2}*z^2\mathcal{L}_{\lambda_2}(a,c)f\right\}}(\mathbb{U}) \subset \overline{\text{co}}s(w(\mathbb{U})) \subset s(\mathbb{U}), \tag{2.12}$$

since s is convex univalent. Therefore from the definition of subordination and (2.12), we have

$$\frac{1}{1-\eta}\left(-\frac{z(\mathcal{L}_{\lambda_1}(a,c)f(z))'}{\mathcal{L}_{\lambda_1}(a,c)f(z)}-\eta\right)\prec\phi(z)\quad(z\in\mathbb{U}),\tag{2.13}$$

or, equivalently, $f\in\mathcal{MS}_{a,c}^{\lambda_1}(\phi)$, which completes the proof of Theorem 2.4.

By using (1.13), (2.2) and (2.3), we have the following Theorem 2.5 and Theorem 2.6. □

Theorem 2.5. *Let $\lambda>0$, $a_2\geq a_1>0$, $c\in\mathbb{R}\setminus\mathbb{Z}_0^-$ and $\phi\in\mathcal{N}$ with $\operatorname{Re}\{\phi(z)\}<(2-\eta)/(1-\eta)$ $(0\leq\eta<1)$. If $a_2\geq 2$ or $a_1+a_2\geq 3$, then*

$$\mathcal{MS}_{a_1,c}^{\lambda}(\eta;\phi)\subset\mathcal{MS}_{a_2,c}^{\lambda}(\eta;\phi).\tag{2.14}$$

Theorem 2.6. *Let $\lambda>0$, $a\in\mathbb{R}\setminus\mathbb{Z}_0^-$, $c_2\geq c_1>0$ and $\phi\in\mathcal{N}$ with $\operatorname{Re}\{\phi(z)\}<(2-\eta)/(1-\eta)$ $(0\leq\eta<1)$. If $c_2\geq 2$ or $c_1+c_2\geq 3$, then*

$$\mathcal{MS}_{a,c_2}^{\lambda}(\eta;\phi)\subset\mathcal{MS}_{a,c_1}^{\lambda}(\eta;\phi).\tag{2.15}$$

Next, we prove the inclusion theorem involving the class $\mathcal{MK}_{a,c}^{\lambda}(\eta;\phi)$.

Theorem 2.7. *Let $\lambda_2\geq\lambda_1>0$, $a,c\in\mathbb{R}\setminus\mathbb{Z}_0^-$ and $\phi\in\mathcal{N}$ with $\operatorname{Re}\{\phi(z)\}<(2-\eta)/(1-\eta)$ $(0\leq\eta<1)$. If $\lambda_2\geq 2$ or $\lambda_1+\lambda_2\geq 3$, then*

$$\mathcal{MK}_{a,c}^{\lambda_2}(\eta;\phi)\subset\mathcal{MK}_{a,c}^{\lambda_1}(\eta;\phi).\tag{2.16}$$

Proof. Applying (1.13) and Theorem 2.4, we observe that

$$\begin{aligned}
f(z)\in\mathcal{MK}_{a,c}^{\lambda_2}(\eta;\phi)&\Longleftrightarrow\mathcal{L}_{\lambda_2}(a,c)f(z)\in\mathcal{MK}(\eta;\phi)\\
&\Longleftrightarrow -z(\mathcal{L}_{\lambda_2}(a,c)f(z))'\in\mathcal{MS}(\eta;\phi)\\
&\Longleftrightarrow\mathcal{L}_{\lambda_2}(a,c)(-zf'(z))\in\mathcal{MS}(\eta;\phi)\\
&\Longleftrightarrow -zf'(z)\in\mathcal{MS}_{a,c}^{\lambda_2}(\eta;\phi)\\
&\Longrightarrow -zf'(z)\in\mathcal{MS}_{a,c}^{\lambda_1}(\eta;\phi)\\
&\Longleftrightarrow\mathcal{L}_{\lambda_1}(a,c)(-zf'(z))\in\mathcal{MS}(\eta;\phi)\\
&\Longleftrightarrow -z(\mathcal{L}_{\lambda_1}(a,c)f(z))'\in\mathcal{MS}(\eta;\phi)\\
&\Longleftrightarrow\mathcal{L}_{\lambda_1}(a,c)f(z)\in\mathcal{MK}(\eta;\phi)\\
&\Longleftrightarrow f(z)\in\mathcal{MK}_{a,c}^{\lambda_1}(\eta;\phi),
\end{aligned}\tag{2.17}$$

which evidently proves Theorem 2.7.

By using a similar method as in the proof of Theorem 2.7, we obtain the following two theorems. □

Theorem 2.8. *Let* $\lambda > 0$, $a_2 \geq a_1 > 0$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$ *and* $\phi \in \mathcal{N}$ *with* $\operatorname{Re}\{\phi(z)\} < (2-\eta)/(1-\eta)$ $(0 \leq \eta < 1)$. *If* $a_2 \geq 2$ *or* $a_1 + a_2 \geq 3$, *then*

$$\mathcal{MK}^{\lambda}_{a_1,c}(\eta;\phi) \subset \mathcal{MK}^{\lambda}_{a_2,c}(\eta;\phi). \tag{2.18}$$

Theorem 2.9. *Let* $\lambda > 0$, $a \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $c_2 \geq c_1 > 0$ *and* $\phi \in \mathcal{N}$ *with* $\operatorname{Re}\{\phi(z)\} < (2-\eta)/(1-\eta)$ $(0 \leq \eta < 1)$. *If* $c_2 \geq 2$ *or* $c_1 + c_2 \geq 3$, *then*

$$\mathcal{MK}^{\lambda}_{a,c_2}(\eta;\phi) \subset \mathcal{MK}^{\lambda}_{a,c_1}(\eta;\phi). \tag{2.19}$$

Taking $\phi(z) = (1+Az)/(1+Bz)$ $(-1 < B < A \leq 1;\ z \in \mathbb{U})$ *in Theorems 2.4–2.9, we have the following corollaries below.*

Corollary 2.10. *Let* $(1+A)(1-\eta) < (2-\eta)(1+B)$ $(-1 < B < A \leq 1;\ 0 \leq \eta < 1)$ *and* $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$. *If* $\lambda_2 \geq \lambda_1 > 0$ *and* $\lambda_2 \geq \min\{2, 3-\lambda_1\}$, *and* $a_2 \geq a_1 > 0$ *and* $a_2 \geq \min\{2, 3-a_1\}$, *then*

$$\begin{gathered}\mathcal{MS}^{\lambda_2}_{a_1,c}[\eta;A,B] \subset \mathcal{MS}^{\lambda_1}_{a_1,c}[\eta;A,B] \subset \mathcal{MS}^{\lambda_1}_{a_2,c}[\eta;A,B],\\ \mathcal{MK}^{\lambda_2}_{a_1,c}[\eta;A,B] \subset \mathcal{MK}^{\lambda_1}_{a_1,c}[\eta;A,B] \subset \mathcal{MK}^{\lambda_1}_{a_2,c}[\eta;A,B].\end{gathered} \tag{2.20}$$

Corollary 2.11. *Let* $(1+A)(1-\eta) < (2-\eta)(1+B)$ $(-1 < B < A \leq 1;\ 0 \leq \eta < 1)$ *and* $\lambda > 0$. *If* $a_2 \geq a_1 > 0$ *and* $a_2 \geq \min\{2, 3-a_1\}$, *and* $c_2 \geq c_1 > 0$ *and* $c_2 \geq \min\{2, 3-c_1\}$, *then*

$$\begin{gathered}\mathcal{MS}^{\lambda}_{a_1,c_2}[\eta;A,B] \subset \mathcal{MS}^{\lambda}_{a_1,c_1}[\eta;A,B] \subset \mathcal{MS}^{\lambda}_{a_2,c_1}[\eta;A,B],\\ \mathcal{MK}^{\lambda}_{a_1,c_2}[\eta;A,B] \subset \mathcal{MK}^{\lambda}_{a_1,c_1}[\eta;A,B] \subset \mathcal{MK}^{\lambda}_{a_2,c_1}[\eta;A,B].\end{gathered} \tag{2.21}$$

Corollary 2.12. *Let* $(1+A)(1-\eta) < (2-\eta)(1+B)$ $(-1 < B < A \leq 1;\ 0 \leq \eta < 1)$ *and* $a \in \mathbb{R} \setminus \mathbb{Z}_0^-$. *If* $\lambda_2 \geq \lambda_1 > 0$ *and* $\lambda_2 \geq \min\{2, 3-\lambda_1\}$, *and* $c_2 \geq c_1 > 0$ *and* $c_2 \geq \min\{2, 3-c_1\}$, *then*

$$\begin{gathered}\mathcal{MS}^{\lambda_2}_{a,c_2}[\eta;A,B] \subset \mathcal{MS}^{\lambda_2}_{a,c_1}[\eta;A,B] \subset \mathcal{MS}^{\lambda_1}_{a,c_1}[\eta;A,B],\\ \mathcal{MK}^{\lambda_2}_{a,c_2}[\eta;A,B] \subset \mathcal{MK}^{\lambda_2}_{a,c_1}[\eta;A,B] \subset \mathcal{MK}^{\lambda_1}_{a,c_1}[\eta;A,B].\end{gathered} \tag{2.22}$$

To prove theorems below, we need the following lemma.

Lemma 2.13. *Let* $\phi \in \mathcal{N}$ *with* $\operatorname{Re}\{\phi(z)\} < (2-\eta)/(1-\eta)$ $(0 \leq \eta < 1)$. *If* $f \in \mathcal{M}$ *with* $z^2 f(z) \in \mathcal{K}$ *and* $q \in \mathcal{MS}(\eta;\phi)$, *then* $f * q \in \mathcal{MS}(\eta;\phi)$.

Proof. Let $q \in \mathcal{MS}(\eta;\phi)$. Then

$$-zq'(z) = \left[(1-\eta)\phi(w(z)) + \eta\right]q(z) \quad (z \in \mathbb{U}), \tag{2.23}$$

where w is an analytic function in $\mathbb{U}$ with $|w(z)| < 1 \quad (z \in \mathbb{U})$ and $w(0) = 0$. Thus we have

$$\begin{aligned}
&\frac{1}{1-\eta}\left(-\frac{z(f(z) * q(z))'}{f(z) * q(z)} - \eta\right) \\
&\quad = \frac{1}{1-\eta}\left(\frac{f(z) * [-zq'(z)]}{f(z) * q(z)} - \eta\right) \\
&\quad = \frac{1}{1-\eta}\left(\frac{f(z) * [(1-\eta)\phi(w(z)) + \eta]q(z)}{f(z) * q(z)} - \eta\right) \quad (z \in \mathbb{D}).
\end{aligned} \tag{2.24}$$

By using the similar arguments to those used in the proof of Theorem 2.4, we conclude that (2.24) is subordinated to ϕ in $\mathbb{U}$ and so $f * q \in \mathcal{MS}(\eta;\phi)$.

Finally, we give the inclusion properties involving the class $\mathcal{MC}_{a,c}^{\lambda}(\eta,\beta;\phi,\psi)$. □

Theorem 2.14. *Let* $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$ *and* $\phi, \psi \in \mathcal{N}$ *with* $\operatorname{Re}\{\phi(z)\} < (2-\eta)/(1-\eta)$ $(0 \le \eta < 1)$. *If* $\lambda_2 \ge \lambda_1 > 0$ *and* $\lambda_2 \ge \min\{2, 3-\lambda_1\}$, *and* $a_2 \ge a_1 > 0$ *and* $a_2 \ge \min\{2, 3-a_1\}$, *then*

$$\mathcal{MC}_{a_1,c}^{\lambda_2}(\eta,\beta;\phi,\psi) \subset \mathcal{MC}_{a_1,c}^{\lambda_1}(\eta,\beta;\phi,\psi) \subset \mathcal{MC}_{a_2,c}^{\lambda_1}(\eta,\beta;\phi,\psi). \tag{2.25}$$

Proof. We begin by proving that

$$\mathcal{MC}_{a_1,c}^{\lambda_2}(\eta,\beta;\phi,\psi) \subset \mathcal{MC}_{a_1,c}^{\lambda_1}(\eta,\beta;\phi,\psi). \tag{2.26}$$

Let $f \in \mathcal{MC}_{a_1,c}^{\lambda_2}(\eta,\beta;\phi,\psi)$. Then there exists a function $q_2 \in \mathcal{MS}(\eta;\phi)$ such that

$$\frac{1}{1-\beta}\left(-\frac{z(\mathcal{L}_{\lambda_2}(a_1,c)f(z))'}{q_2(z)} - \beta\right) \prec \psi(z) \quad (0 \le \beta < 1;\ z \in \mathbb{U}). \tag{2.27}$$

From (2.27), we obtain

$$-z(\mathcal{L}_{\lambda_2}(a_1,c)f(z))' = ((1-\beta)\psi(w(z)) + \beta)q_2(z), \tag{2.28}$$

where w is an analytic function in $\mathbb{U}$ with $|w(z)| < 1$ $(z \in \mathbb{U})$ and $w(0) = 0$. By virtue of (2.3), Lemmas 2.2 and 2.13, we see that $f_{\lambda_1,\lambda_2}(z) * q_2(z) \equiv q_1(z)$ belongs to $\mathcal{MS}(\eta;\phi)$. Then, making use of (2.1), we have

$$\begin{aligned}
&\frac{1}{1-\beta}\left(-\frac{z(\mathcal{L}_{\lambda_1}(a_1,c)f(z))'}{q_1(z)}-\beta\right)\\
&\quad=\frac{1}{1-\beta}\left(\frac{f_{\lambda_1,\lambda_2}(z) * \left[-z(\mathcal{L}_{\lambda_2}(a_1,c)f(z))'\right]}{f_{\lambda_1,\lambda_2}(z) * q_2(z)}-\beta\right)\\
&\quad=\frac{1}{1-\beta}\left(\frac{f_{\lambda_1,\lambda_2}(z) * \left[(1-\beta)\psi(w(z))+\beta\right]q_2(z)}{f_{\lambda_1,\lambda_2}(z) * q_2(z)}-\beta\right)\\
&\quad=\frac{1}{1-\beta}\left(\frac{z^2 f_{\lambda_1,\lambda_2}(z) * \left[(1-\beta)\psi(w(z))+\beta\right]z^2 q_2(z)}{z^2 f_{\lambda_1,\lambda_2}(z) * z^2 q_2(z)}-\beta\right)\\
&\quad\prec \psi(z) \quad (z \in \mathbb{U}).
\end{aligned} \tag{2.29}$$

Therefore we prove that $f \in \mathcal{MC}^{\lambda_1}_{a_1,c}(\eta,\beta;\phi,\psi)$.

For the second part, by using arguments similar to those detailed above with (2.2), we obtain

$$\mathcal{MC}^{\lambda_1}_{a_1,c}(\eta,\beta;\phi,\psi) \subset \mathcal{MC}^{\lambda_1}_{a_2,c}(\eta,\beta;\phi,\psi) \tag{2.30}$$

Thus the proof of Theorem 2.14 is completed.

The following results can be obtained by using the same techniques as in the proof of Theorem 2.14 and so we omit the detailed proofs involved. □

Theorem 2.15. *Let* $\lambda > 0$ *and* $\phi,\psi \in \mathcal{N}$ *with* $\mathrm{Re}\{\phi(z)\} < (2-\eta)/(1-\eta)$ $(0 \le \eta < 1)$. *If* $a_2 \ge a_1 > 0$ *and* $a_2 \ge \min\{2, 3-a_1\}$, *and* $c_2 \ge c_1 > 0$ *and* $c_2 \ge \min\{2, 3-c_1\}$, *then*

$$\mathcal{MC}^{\lambda}_{a_1,c_2}(\eta,\beta;\phi,\psi) \subset \mathcal{MC}^{\lambda}_{a_1,c_1}(\eta,\beta;\phi,\psi) \subset \mathcal{MC}^{\lambda}_{a_2,c_1}(\eta,\beta;\phi,\psi). \tag{2.31}$$

Theorem 2.16. *Let* $a \in \mathbb{R} \setminus \mathbb{Z}_0^-$ *and* $\phi,\psi \in \mathcal{N}$ *with* $\mathrm{Re}\{\phi(z)\} < (2-\eta)/(1-\eta)$ $(0 \le \eta < 1)$. *If* $\lambda_2 \ge \lambda_1 > 0$ *and* $\lambda_2 \ge \min\{2, 3-\lambda_1\}$, *and* $c_2 \ge c_1 > 0$ *and* $c_2 \ge \min\{2, 3-c_1\}$, *then*

$$\mathcal{MC}^{\lambda_2}_{a,c_2}(\eta,\beta;\phi,\psi) \subset \mathcal{MC}^{\lambda_2}_{a,c_1}(\eta,\beta;\phi,\psi) \subset \mathcal{MC}^{\lambda_1}_{a,c_1}(\eta,\beta;\phi,\psi). \tag{2.32}$$

Remark 2.17. For $a = \lambda + 1$ $(\lambda > -1)$ and $c = 1$, Theorems 2.4, 2.5, 2.7, 2.8, and 2.14 reduce to the corresponding results obtained by Cho and Noor [16].

3. Inclusion Properties Involving Various Operators

The next theorem shows that the classes $\mathcal{MS}^{\lambda}_{a,c}(\eta;\phi)$, $\mathcal{MK}^{\lambda}_{a,c}(\eta;\phi)$ and $\mathcal{MC}^{\lambda}_{a,c}(\eta,\beta;\phi,\psi)$ are invariant under convolution with convex functions.

Theorem 3.1. *Let* $\lambda > 0$, $a > 0, c \in \mathbb{R}\setminus\mathbb{Z}_0^-$, $\phi, \varphi \in \mathcal{N}$ *with* $\mathrm{Re}\{\phi(z)\} < (2-\eta)/(1-\eta)$ $(0 \le \eta < 1)$ *and let* $g \in \mathcal{M}$ *with* $z^2 g(z) \in \mathcal{K}$. *Then*

(i) $f \in \mathcal{MS}_{a,c}^{\lambda}(\eta;\phi) \Rightarrow g*f \in \mathcal{MS}_{a,c}^{\lambda}(\eta;\phi)$,

(ii) $f \in \mathcal{MK}_{a,c}^{\lambda}(\eta;\phi) \Rightarrow g*f \in \mathcal{MK}_{a,c}^{\lambda}(\eta;\phi)$,

(iii) $f \in \mathcal{MC}_{a,c}^{\lambda}(\eta,\beta;\phi,\varphi) \Rightarrow g*f \in \mathcal{MC}_{a,c}^{\lambda}(\eta,\beta;\phi,\varphi)$.

Proof. (i) Let $f \in \mathcal{MS}_{a,c}^{\lambda}(\eta;\phi)$. Then we have

$$\frac{1}{1-\eta}\left(-\frac{z(\mathcal{L}_\lambda(a,c)(g*f)(z))'}{\mathcal{L}_\lambda(a,c)(g*f)(z)}-\eta\right)=\frac{1}{1-\eta}\left(\frac{g(z)*\left[-z(\mathcal{L}_\lambda(a,c)f(z))'\right]}{g(z)*\mathcal{L}_\lambda(a,c)f(z)}-\eta\right). \tag{3.1}$$

By using the same techniques as in the proof of Theorem 2.4, we obtain (i).

(ii) Let $f \in \mathcal{MK}_{a,c}^{\lambda}(\phi)$. Then, by (1.13), $-zf'(z) \in \mathcal{MS}_{a,c}(\eta;\phi)$ and hence from (i), $g(z)*[-zf'(z)] \in \mathcal{MS}_{a,c}^{\lambda}(\eta;\phi)$. Since

$$g(z)*[-zf'(z)] = -z(g*f)'(z), \tag{3.2}$$

we have (ii) applying (1.13) once again.

(iii) Let $f \in \mathcal{MC}_{a,c}^{\lambda}(\eta,\beta;\phi,\varphi)$. Then there exists a function $q \in \mathcal{MS}(\eta;\phi)$ such that

$$-z(\mathcal{L}_\lambda(a,c)f(z))' = [(1-\beta)\varphi(w(z))+\beta]q(z) \quad (0 \le \beta < 1;\ z \in \mathbb{U}), \tag{3.3}$$

where w is an analytic function in $\mathbb{U}$ with $|w(z)| < 1$ $(z \in \mathbb{U})$ and $w(0) = 0$. From Lemma 2.13, we have that $g*q \in \mathcal{MS}(\eta;\phi)$. Since

$$\begin{aligned}
&\frac{1}{1-\beta}\left(-\frac{z(\mathcal{L}_\lambda(a,c)(g*f)(z))'}{(g*q)(z)}-\beta\right)\\
&\quad=\frac{1}{1-\beta}\left(\frac{g(z)*\left[-z(\mathcal{L}_\lambda(a,c)f(z))'\right]}{g(z)*q(z)}-\beta\right)\\
&\quad=\frac{1}{1-\beta}\left(\frac{z^2g(z)*[(1-\beta)\varphi(w(z))+\beta]z^2q(z)}{z^2g(z)*z^2q(z)}-\beta\right)\\
&\quad\prec \varphi(z) \quad (z\in\mathbb{U}),
\end{aligned} \tag{3.4}$$

we obtain (iii).

Now we consider the following operators defined by

$$\Psi_1(z) = \frac{1}{z}\sum_{k=1}^{\infty}\frac{1+c}{k+c}z^{k-1} \quad (\operatorname{Re}\{c\} \geq 0;\ z \in \mathbb{D}),$$
$$\Psi_2(z) = \frac{1}{z^2(1-x)}\log\left[\frac{1-xz}{1-z}\right] \quad \left(\log 1 = 0;\ |x| \leq 1, x \neq 1;\ z \in \mathbb{D}\right). \tag{3.5}$$

It is well known [17] that the operators $z^2\Psi_1$ and $z^2\Psi_2$ are convex univalent in $\mathbb{U}$. Therefore we have the following result, which can be obtained from Theorem 3.1 immediately. □

Corollary 3.2. *Let* $a > 0$, $\lambda > 0$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\phi, \varphi \in \mathcal{N}$ *with* $\operatorname{Re}\{\phi(z)\} < (2-\eta)/(1-\eta)$ $(0 \leq \eta < 1)$ *and let* Ψ_i $(i = 1, 2)$ *be defined by* (3.5)*, respectively. Then*

(i) $f \in \mathcal{MS}_{a,c}^{\lambda}(\eta;\phi) \Rightarrow \Psi_i * f \in \mathcal{MS}_{a,c}^{\lambda}(\eta;\phi)$,

(ii) $f \in \mathcal{MK}_{a,c}^{\lambda}(\eta;\phi) \Rightarrow \Psi_i * f \in \mathcal{MK}_{a,c}^{\lambda}(\eta;\phi)$,

(iii) $f \in \mathcal{MC}_{a,c}^{\lambda}(\eta,\beta;\phi,\varphi) \Rightarrow \Psi_i * f \in \mathcal{MC}_{a,c}^{\lambda}(\eta,\beta;\phi,\varphi)$.

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