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Research Article

Bargmann-Type Inequality for Half-Linear Differential Operators

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We consider the perturbed half-linear Euler differential equation $(\Phi(x'))' + [\gamma/t^p + c(t)]\Phi(x) = 0$, $\Phi(x) := |x|^{p-2}x$, $p > 1$, with the subcritical coefficient $\gamma < \gamma_p := ((p-1)/p)^p$. We establish a Bargmann-type necessary condition for the existence of a nontrivial solution of this equation with at least $(n+1)$ zero points in $(0,\infty)$.

1. Introduction

The classical Bargmann inequality [1] originates from the nonrelativistic quantum mechanics and gives an upper bound for the number of bound states produced by a radially symmetric potential in the two-body system. In the subsequent papers, various proofs and reformulations of this inequality have been presented, we refer to [2, Chapter XIII], and to [3–5] for some details.

In the language of singular differential operators, Bargmann's inequality concerns the one-dimensional Schrödinger operator

$$\tau(y) := y'' + \left[\frac{\gamma}{t^2} + c(t)\right]y, \quad \gamma < \frac{1}{4}, \ t \in (0,\infty). \tag{1.1}$$

It states that if the Friedrichs realization of τ has at least n negative eigenvalues below theessential spectrum (what is equivalent to the existence of a nontrivial solution of the equation $\tau(y) = 0$ having at least $(n+1)$ zeros in $(0,\infty)$), then

$$\int_0^\infty tc_+(t)dt > n\sqrt{1-4\gamma}, \tag{1.2}$$

where $c_+(t) = \max\{c(t), 0\}$.

This inequality can be seen as follows. The Euler differential equation

$$x'' + \frac{\gamma}{t^2}x = 0 \tag{1.3}$$

with the subcritical coefficient $\gamma < 1/4$ is disconjugate in $(0,\infty)$, that is, any nontrivial solution of (1.3) has at most one zero in this interval. Hence, if the equation $\tau(y) = 0$, with τ given by (1.1), has a solution with at least $(n+1)$ positive zeros, the perturbation function c must be "sufficiently positive" in view of the Sturmian comparison theorem. Inequality (1.2) specifies exactly what "sufficient positiveness" means.

In this paper, we treat a similar problem in the scope of the theory of half-linear differential equations:

$$\left(r(t)\Phi(x')\right)' + c(t)\Phi(x) = 0, \qquad \Phi(x) := |x|^{p-2}x, \quad p > 1. \tag{1.4}$$

In physical sciences, there are known phenomena which can be described by differential equations with the so-called p-Laplacian $\Delta_p u :=$ div $(\|\nabla u\|^{p-2}\nabla u)$, see, for example, [6]. If the potential in such an equation is radially symmetric, this equation can be reduced to a half-linear equation of the form (1.4).

There are many results of the linear oscillation theory, which concern the Sturm-Liouville differential equation:

$$\left(r(t)x'\right)' + c(t)x = 0, \tag{1.5}$$

which has been extended to (1.4). In particular, the linear Sturmian theory holds almost verbatim for (1.4), see, for example, [7, 8]. We will recall elements of the half-linear oscillation theory in the next section. Our main result concerns the perturbed half-linear Euler differential equation

$$\left(\Phi(x')\right) + \left[\frac{\gamma}{t^p} + c(t)\right]\Phi(x) = 0, \quad t \in (0,\infty), \tag{1.6}$$

where c is a continuous function, and shows that if γ is the so-called subcritical coefficient, that is, $\gamma < \gamma_p := (p/(p-1))^p$, and there exists a solution of (1.6) with at least $(n+1)$ zeros in $(0,\infty)$, then the integral $\int_0^\infty t^{p-1}c_+(t)dt$ satisfies an inequality which reduces to (1.2) in the linear case $p = 2$.

2. Preliminaries

In this short section, we present some elements of the half-linear oscillation theory which we need in the proof of our main result. As we have mentioned in the previous section, the linear and half-linear oscillation theories are in many aspects very similar, so (1.4) can be classified as oscillatory or nonoscillatory as in the linear case.

If x is a solution of (1.4) such that $x(t) \neq 0$ is some interval I, then $w := r\Phi(x'/x)$ is a solution of the Riccati-type differential equation

$$w' + c(t) + (p-1)r^{1-q}|w|^q = 0, \quad q := \frac{p}{p-1}. \tag{2.1}$$

If (1.4) is nonoscillatory, that is, (2.1) possesses a solution which exists on some interval $[T, \infty)$, among all such solutions of (2.1), there exists the *minimal* one $\tilde{w}$, minimal in the sense that any other solution w of (2.1) which exists on some interval $[t_w, \infty)$ satisfies $w(t) > \tilde{w}(t)$ in this interval, see [9, 10] for details.

In our treatment, the so-called half-linear Euler differential equation

$$(\Phi(x'))' + \frac{\gamma}{t^p}\Phi(x) = 0 \tag{2.2}$$

appears. If we look for a solution of this equation in the form $x(t) = t^\lambda$, then λ is a root of the algebraic equation

$$|\lambda|^p - \Phi(\lambda) + \frac{\gamma}{p-1} = 0. \tag{2.3}$$

By a simple calculation (see, e.g., [8, Section 1.3]), one finds that (2.3) has a real root if and only if γ is less than or equal to the so-called critical constant $\gamma_p := ((p-1)/p)^p$, and hence (2.2) is nonoscillatory if and only if $\gamma \leq \gamma_p$. In this case, the associated Riccati equation is of the form

$$w' + \frac{\gamma}{t^p} + (p-1)|w|^q = 0, \tag{2.4}$$

and its minimal solution is $\tilde{w}(t) = \Phi(\lambda_1)t^{1-p}$, where λ_1 is the smaller of (the two real) roots of (2.3). If $v(t) = t^{p-1}w$, then v is a solution of the equation

$$v' = \frac{p-1}{t} - \frac{p-1}{t}|v|^q - \frac{\gamma}{t}, \tag{2.5}$$

and $\tilde{v}(t) \equiv \Phi(\lambda_1)$ is the minimal solution of this equation. A detailed study of half-linear Euler equation and of its perturbations can be found in [11].

3. Bargmann's Type Inequality

In this section, we present our main results, the half-linear version of Bargmann's inequality. We are motivated by the work in [4] where a short proof of this inequality based on the Riccati technique is presented. Here we show that this method, properly modified, can also be applied to (1.6).

Theorem 3.1. *Suppose that* (1.6) *with* $\gamma < \gamma_p = ((p-1)/p)^p$ *has a nontrivial solution with at least* $(n+1)$ *zeros in* $(0,\infty)$. *Then*

$$\int_0^\infty t^{p-1} c_+(t)dt > nk(\gamma, q), \tag{3.1}$$

where $k(\gamma, q)$ *is the absolute value of the difference of the real roots of*

$$F_\gamma(\lambda) := |\lambda|^q - \lambda + (q-1)\gamma = 0 \tag{3.2}$$

and $q = p/(p-1)$ *is the conjugate number to* p. *Moreover, the constant* $k(\gamma, q)$ *is strict in the sense that for every* $\varepsilon > 0$, *there exists a continuous function* c *such that* (1.6) *possesses a solution with* $(n+1)$ *zeros in* $(0,\infty)$ *and*

$$\int_0^\infty t^{p-1} c_+(t)dt \le nk(\gamma, q) + \varepsilon. \tag{3.3}$$

Proof. Let x be a solution of (1.6) with $(n+1)$ zeros in $(0,\infty)$, denote these zeros by $t_0 < t_1 < \cdots < t_n$, and let $v(t) = t^{p-1}\Phi(x'/x)$. Then by a direct computation we see that v is a solution of the Riccati-type differential equation

$$\begin{aligned} v' &= \frac{p-1}{t} v - \frac{\gamma}{t} - (p-1)|v|^q - t^{p-1}c(t) \\ &= -(p-1)F_\gamma(v) - t^{p-1}c(t), \quad t \in (t_i, t_{i+1}), \ i = 0, \ldots, n-1, \end{aligned} \tag{3.4}$$

$$v(t_i-) = -\infty, \qquad v(t_i+) = \infty. \tag{3.5}$$

Let $\lambda_1 < \lambda_2$ be the roots of (3.2). Such pair of roots exists and it is unique since the function $F_\gamma(\lambda)$ is convex, $F_\gamma(\pm\infty) = \infty$, $F'_\gamma(1/\Phi(q)) = 0$, and $F_\gamma(1/\Phi(q)) = (\gamma - \gamma_p)/(p-1) < 0$. According to (3.5), there exist $\xi_i, \eta_i \in (t_i, t_{i+1})$ such that $v(\xi_i) = \lambda_2$, $v(\eta_i) = \lambda_1$, and $\lambda_1 < v(t) < \lambda_2$ for $t \in (\xi_i, \eta_i)$, which means that $F_\gamma(v(t)) < 0$ for $t \in (\xi_i, \eta_i)$. Then, we have

$$\begin{aligned} \int_0^\infty t^{p-1} c_+(t)dt &\ge \sum_{i=0}^n \int_{\xi_i}^{\eta_i} t^{p-1} c_+(t)dt \ge \sum_{i=0}^n \int_{\xi_i}^{\eta_i} t^{p-1} c(t)dt \\ &= \sum_{i=1}^n \int_{\xi_i}^{\eta_i} \left[-v'(t) - (p-1)F_\gamma(v(t))\right]dt > \sum_{i=1}^n v(t)\Big|_{\eta_i}^{\xi_i} \\ &= \sum_{i=1}^n \left[v(\xi_i) - v(\eta_i)\right] = n(\lambda_2 - \lambda_1) = nk(\gamma, q). \end{aligned} \tag{3.6}$$

Now we prove that the constant $k(\gamma,q)$ is exact. Let $\varepsilon > 0$ be arbitrary and α_i, β_i, T_i be sequences of positive real numbers constructed in the following way. Let $t_0 \in (0,\infty)$ be arbitrary and consider the differential equation

$$(\Phi(x'))' + \frac{\gamma}{t^p}\Phi(x) = 0. \tag{3.7}$$

Denote by x_0 its nontrivial solution satisfying $x_0(t_0) = 0$, $x_0'(t_0) = 1$ (such solution exists and it is unique, see, e.g., [8, Section 1.1]) and let $v_0 := t^{p-1}\Phi(x_0'/x_0)$. Since $\lim_{t\to\infty} v_0(t) = v_2$, see [8, page 39], there exists $T_1 > t_0$ such that $v_0(T_1)$.

Now, let

$$\alpha_1 := \frac{\gamma_p - \gamma}{T_1}, \qquad \beta_1 := \frac{\varepsilon T_1}{4n(\gamma_p - \gamma)}, \tag{3.8}$$

and define for $t \in [T_1, T_1 + \beta_1]$ the function

$$\widehat{c}_1(t) := \frac{1}{\beta_1 t^{p-1}}\left[k(\gamma,q) + \frac{\varepsilon}{4n} + \alpha_1\right]. \tag{3.9}$$

Consider the solution v of the equation

$$v' = -(p-1)\frac{|v|^q}{t} + (p-1)\frac{v}{t} - \frac{\gamma}{t} - t^{p-1}\widehat{c}_1(t), \quad t \in [T_1, T_1 + \beta_1], \tag{3.10}$$

given by the initial conditions $v(T_1) = v_0(T_1)$. Then for $t \in [T_1, T_1 + \beta_1]$

$$\begin{aligned} v' &= -\frac{p-1}{t}\left[|v|^q - v + \frac{\gamma_p}{p-1}\right] + \frac{\gamma_p - \gamma}{t} - t^{p-1}\widehat{c}_1(t) \\ &\le \frac{\gamma_p - \gamma}{t} - \frac{1}{\beta_i}\left(k(\gamma,q) + \frac{\varepsilon}{4n}\right) - \frac{\gamma_p - \gamma}{T_1} \\ &\le -\frac{1}{\beta_i}\left(k(\gamma,q) + \frac{\varepsilon}{4n}\right). \end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned} v(T_1 + \beta_1) &= v(T_1) + \int_{T_1}^{T_1+\beta_1} v'(t)dt < v_2 + \frac{\varepsilon}{4n} - \left(k(\gamma,q) + \frac{\varepsilon}{4n}\right) \\ &= v_2 - (v_2 - v_1) = v_1. \end{aligned} \tag{3.12}$$

Now consider again (3.7) and the associated Riccati-type differential equation

$$v' = -\frac{\gamma}{t^p} + (p-1)v - (p-1)|v|^q \tag{3.13}$$

(which is related to (3.7) by the substitution $v = t^{p-1}\Phi(x'/x)$). This equation has a constant solution $v = v_1$ and this solution is the minimal one (see the end of Section 2). This means that any solution of (3.13) which starts with the initial condition $v(T_1 + \beta_1) < v_1$ blows down to $-\infty$ at a finite time $t_1 > T_1 + \beta_1$, which is a zero point of the associated solution x of (3.7). Now, let

$$\widetilde{c}_1(t) = \begin{cases} 0, & t \in [t_0, T_1], \\ \widehat{c}_1(t), & t \in \left[T_1, T_1 + \beta_1\right], \\ 0, & t \in \left[T_1 + \beta_1, t_1\right]. \end{cases} \tag{3.14}$$

In summary, we have constructed a solution of the equation

$$\left(\Phi(x')\right)' + \left[\frac{\gamma}{t^p} + \widetilde{c}_1(t)\right]\Phi(x) = 0 \tag{3.15}$$

for which $x(t_0) = 0 = x(t_1)$ and

$$\begin{aligned} \int_{t_0}^{t_1} t^{p-1}\widetilde{c}_1(t)dt &= \int_{T_1}^{T_1+\beta_1} t^{p-1}\widehat{c}_1(t)dt \\ &= k(\gamma, q) + \frac{\varepsilon}{4n} + \alpha_1\beta_1 \\ &= k(\gamma, q) + \frac{\varepsilon}{4n} + \frac{\varepsilon}{4n} \\ &= k(\gamma, q) + \frac{\varepsilon}{2n}. \end{aligned} \tag{3.16}$$

The construction of $T_i, \beta_i, \alpha_i, \widehat{c}_i(t)$ and $\widetilde{c}_i(t)$, $i = 2, \ldots, n$, is now analogical. As a result we obtain the function $\widetilde{c} : (0, \infty) \to [0, \infty)$ defined as $\widetilde{c}(t) = 0$ for $t \in (0, t_0]$ and $t \in [t_n, \infty)$, and $\widetilde{c}(t) = \widetilde{c}_i(t)$ for $t \in [t_{i-1}, t_i]$, for which

$$\int_0^\infty t^{p-1}\widetilde{c}(t)dt = nk(\gamma, q) + \frac{\varepsilon}{2}, \tag{3.17}$$

and the equation

$$\left(\Phi(x')\right)' + \left[\frac{\gamma}{t^p} + \widetilde{c}(t)\right]\Phi(x) = 0 \tag{3.18}$$

has a solution with zeros at $t = t_i$, $i = 0, \ldots, n$.

Finally, we change the discontinuous function $\widetilde{c}(t)$ to a continuous one $c(t) \geq \widetilde{c}(t)$ such that $\int_{t_0}^{t_n} t^{p-1}[c(t) - \widetilde{c}(t)]dt < \varepsilon/2$. Such a modification is an easy technical construction which can be described explicitly, but for us is only important its existence. According to the Sturmian comparison theorem, the equation $(\Phi(x'))' + [\gamma/t^p + c(t)]\Phi(x) = 0$ possesses a nontrivial solution with at least $(n+1)$ zeros and

$$\int_0^\infty t^{p-1} c(t)\,dt \le nk(\gamma, q) + \varepsilon, \tag{3.19}$$

which we needed to prove. □

Remark 3.2. If $p = 2$, then $F_\gamma(\lambda) = \lambda^2 - \lambda + \gamma$ and the roots of (3.2) are

$$\lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\gamma}\right). \tag{3.20}$$

Hence, $k(\gamma, 2) = |\lambda_1 - \lambda_2| = \sqrt{1 - 4\gamma}$ and (3.1) reduces to (1.2).

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