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Research Article

On the Stability of a New Pexider-Type Functional Equation

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We establish the generalized Hyers-Ulam stability of a Pexider-type functonal equation $f_1(x+y+z)+f_2(x-y)+f_3(x-z)-f_4(x-y-z)-f_5(x+y)-f_6(x+z)=0$, which is mixed of a quadratic and an additive functional equations. Also, we obtain its general solution from the stability results.

1. Introduction

In 1940, Ulam [1] raised the following question. Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, Hyers [2] proved that if $f : V \to X$ is a mapping satisfying

$$\|f(x+y)-f(x)-f(y)\| \le \delta \tag{1.1}$$

for all $x, y \in V$, where V and X are Banach spaces and δ is a given positive number, then there exists a unique additive mapping $T : V \to X$ such that

$$\|f(x)-T(x)\| \le \delta \tag{1.2}$$

for all $x \in V$. In 1978, Rassias [3] gave a significant generalization of Hyers' result. Rassias [4] during the 27th International Symposium on Functional Equations, that took place in Bielsko-Biala, Poland, in 1990, asked the question whether such a theorem can also be proved for a more general setting. Gadja [5] following Rassias's approach [3] gave an affirmative solution to the question. Recently, Găvruţa [6] obtained a further generalization of Rassias' theorem, the so-called generalized Hyers-Ulam-Rassias stability (see also [4, 7–10]). Jun et al. [11–13] also obtained the Hyers-Ulam-Rassias stability of the Pexider equation of $f(x+y) = g(x) + h(y)$. Quadratic functional equation was used to characterize inner product spaces [14]. Several other functional equations were also to characterize inner product spaces. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right). \tag{1.3}$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.4}$$

is related to a symmetric biadditive function [14]. It is natural that each equation is called a quadratic functional equation. A stability problem for the quadratic functional equation was proved by Skof [15] for a function $f : V \to X$, where V is a normed space and X a Banach space. Cholewa [16] noticed that the theorem of Skof is still true if the relevant domain V is replaced by an Abelian group. Czerwik [17] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Jun and Lee [13, 18–22] proved the Hyers-Ulam-Rassias stability of the Pexiderized quadratic equation

$$f(x+y) + g(x-y) = 2h(x) + 2k(y). \tag{1.5}$$

Now, we introduce the following new Pexider type functional equation:

$$f_1(x+y+z) + f_2(x-y) + f_3(z-x) - f_4(x-y-z) - f_5(x+y) - f_6(x+z) = 0, \tag{1.6}$$

which is mixed of a quadratic and an additive functional equations. In this paper, we establish the generalized Hyers-Ulam-Rassias stability for (1.6) on the punctured domain $V \setminus \{0\}$ and obtain its general solution from the stability results. Throughout this paper, let V and X be a normed space and a Banach space, respectively. For convenience, we employ the operators as follows: for a given function $\varphi : V\setminus\{0\}\times V\setminus\{0\}\times V\setminus\{0\}\to[0,\infty)$, let $\varphi', \varphi_e, \varphi_e' : (V\setminus\{0\})^3\to[0,\infty)$, $M, M', M_e, M_e' : V \setminus \{0\}\to[0,\infty)$ be functions defined by

$$\begin{aligned}
\varphi'(x,y,z) &:= \frac{1}{2}\left[\varphi(x,y,z) + \varphi(-x,y,z)\right],\\
\varphi_e(x,y,z) &:= \frac{1}{2}\left[\varphi(x,y,z) + \varphi(-x,-y,-z)\right],\\
\varphi_e'(x,y,z) &:= \frac{1}{4}\left[\varphi(x,y,z) + \varphi(-x,y,z) + \varphi(-x,-y,-z) + \varphi(x,-y,-z)\right],\\
M(x) &:= \varphi'\left(\frac{x}{2},\frac{3x}{2},-\frac{x}{2}\right) + 2\varphi'\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2}\right) + \varphi'\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right),\\
M'(x) &:= \varphi'\left(\frac{x}{2},\frac{x}{2},-\frac{3x}{2}\right) + 2\varphi'\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2}\right) + \varphi'\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right),\\
M(x) &:= \varphi'\left(\frac{x}{2},\frac{3x}{2},-\frac{x}{2}\right) + 2\varphi'\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2}\right) + \varphi'\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right),\\
M_e'(x) &:= \varphi_e'\left(\frac{x}{2},\frac{x}{2},-\frac{3x}{2}\right) + 2\varphi_e'\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2}\right) + \varphi_e'\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right)
\end{aligned} \tag{1.7}$$

for all $x, y, z \in V \setminus \{0\}$.

2. Generalized Hyers-Ulam-Rassias stability

We need the following lemma to prove our main results.

Lemma 2.1. *Let a be a positive real number. Let $\Phi : V \setminus \{0\} \to [0, \infty)$ be a map such that*

$$\widetilde{\Phi}(x) := \sum_{l=0}^{\infty} \frac{1}{a^{l+1}} \Phi(2^l x) < \infty \quad \forall\, x \in V \setminus \{0\} \tag{2.1}$$

or

$$\widetilde{\Phi}(x) := \sum_{l=0}^{\infty} a^l \Phi\left(\frac{x}{2^{l+1}}\right) < \infty \quad \forall\, x \in V \setminus \{0\}. \tag{2.2}$$

Suppose that the function $f : V \to X$ satisfies the inequality

$$\left\| f(x) - \frac{f(2x)}{a} \right\| \le \frac{\Phi(x)}{a} \tag{2.3}$$

for all $x \in V \setminus \{0\}$ and $f(0) = 0$. Then, there exists exactly one function $F : V \to X$ satisfying

$$\| f(x) - F(x) \| \le \widetilde{\Phi}(x) \quad \forall\, x \in V \setminus \{0\}, \qquad aF(x) = F(2x) \quad \forall\, x \in V. \tag{2.4}$$

Proof. First we assume that Φ satisfies

$$\sum_{l=0}^{\infty} \frac{\Phi(2^l x)}{a^{l+1}} < \infty \tag{2.5}$$

for all $x \in V \setminus \{0\}$. Replacing x by $2^n x$ and dividing it by a^n in (2.3), we have

$$\left\| \frac{f(2^n x)}{a^n} - \frac{f(2^{n+1} x)}{a^{n+1}} \right\| \le \frac{\Phi(2^n x)}{a^{n+1}} \tag{2.6}$$

for all $n \in \mathbb{N}$ and $x \in V \setminus \{0\}$. Induction argument implies that

$$\left\| f(x) - \frac{f(2^n x)}{a^n} \right\| \le \sum_{s=0}^{n-1} \frac{\Phi(2^s x)}{a^{s+1}} \tag{2.7}$$

for all $n \in \mathbb{N}$ and $x \in V \setminus \{0\}$. Hence

$$\left\| \frac{f(2^n x)}{a^n} - \frac{f(2^m x)}{a^m} \right\| \le \sum_{s=n}^{m-1} \frac{\Phi(2^s x)}{a^{s+1}} \tag{2.8}$$

for all positive integers $m > n$ and $x \in V \setminus \{0\}$. This shows that $\{f(2^n x)/a^n\}$ is a Cauchy sequence for $x \in V \setminus \{0\}$ and thus converges. Therefore, we can define $F : V \to X$ such that

$$F(x) = \begin{cases} \lim_{n\to\infty} \dfrac{f(2^n x)}{a^n}, & \text{if } x \ne 0, \\ 0, & \text{if } x = 0 \end{cases} \tag{2.9}$$

for all $x \in V$. From (2.7) and the definition of F, we obtain

$$\|f(x) - F(x)\| \leq \widetilde{\Phi}(x), \qquad aF(x) = F(2x) \tag{2.10}$$

for all $x \in V \setminus \{0\}$. Now, let $F' : V \setminus \{0\} \to X$ be another mapping satisfying the above inequality and equality. Then, it follows that

$$\begin{aligned} \|F(x) - F'(x)\| &\leq \left\| \frac{f(2^m x)}{a^m} - \frac{F(2^m x)}{a^m} \right\| + \left\| \frac{f(2^m x)}{a^m} - \frac{F'(2^m x)}{a^m} \right\| \\ &\leq \frac{\widetilde{\Phi}(2^m x)}{a^m} \end{aligned} \tag{2.11}$$

which tends to zero by the definition of $\widetilde{\Phi}$ as $m \to \infty$ for all $x \in V$. So we can conclude that $F(x) = F'(x)$ for all $x \in V$. This proves the uniqueness of F.

Next we assume that Φ satisfies

$$\sum_{l=0}^{\infty} a^l \Phi\left(\frac{x}{2^{l+1}}\right) < \infty \tag{2.12}$$

for all $x \in V \setminus \{0\}$. Replacing x by $2^{-n-1}x$ and multiplying it by a^{n+1} in (2.3), we have

$$\left\|a^n f(2^{-n}x) - a^{n+1} f(2^{-n-1}x)\right\| \leq a^n \Phi(2^{-n-1}x) \tag{2.13}$$

for all $n \in \mathbb{N}$ and $x \in V \setminus \{0\}$. Induction argument implies that

$$\|f(x) - a^n f(2^{-n}x)\| \leq \sum_{s=0}^{n-1} a^s \Phi(2^{-s-1}x) \tag{2.14}$$

for all $n \in \mathbb{N}$ and $x \in V \setminus \{0\}$. Hence

$$\|a^n f(2^{-n}x) - a^m f(2^{-m}x)\| \leq \sum_{s=n}^{m-1} a^s \Phi(2^{-s-1}x) \tag{2.15}$$

for all positive integers $m > n$ and $x \in V \setminus \{0\}$. This shows that $\{a^n f(2^{-n}x)\}$ is a Cauchy sequence for $x \in V \setminus \{0\}$ and thus converges. Therefore we can define $F : V \to X$ such that

$$F(x) = \begin{cases} \lim\limits_{n\to\infty} a^n f(2^{-n}x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases} \tag{2.16}$$

for all $x \in V$. From (2.14) and the definition of F, we obtain

$$\|f(x) - F(x)\| \leq \widetilde{\Phi}(x), \qquad aF(x) = F(2x) \tag{2.17}$$

for all $x \in V \setminus \{0\}$.

The uniqueness of F is proved similarly as the first case. This completes the proof. □

We establish the stability results for the even functions in Theorems 2.2 and 2.3.

Theorem 2.2. *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \times V \setminus \{0\} \to [0,\infty)$ be a function such that*

$$\widetilde{\varphi}(x,y,z) := \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} \varphi(2^l x, 2^l y, 2^l z) < \infty \tag{a}$$

holds for all $x, y, z \in V \setminus \{0\}$. If the even functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy the inequality

$$\left\| f_1(x+y+z) + f_2(x-y) + f_3(x-z) - f_4(x-y-z) - f_5(x+y) - f_6(x+z) \right\| \le \varphi(x,y,z) \tag{2.18}$$

for all $x, y, z \in V \setminus \{0\}$, then there exists exactly one quadratic function $Q : V \to X$ satisfying the inequalities

$$\begin{aligned} \left\| f_1(x) - f_1(0) - Q(x) \right\| &\le \frac{1}{2}\left[\varphi'\left(\frac{x}{2}, \frac{3}{2}x, -x\right) + \varphi'\left(\frac{x}{2}, \frac{x}{2}, -x\right)\right] \\ &\quad + \frac{1}{2}\widetilde{M}(2x) + \widetilde{M}(x) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, -\frac{x}{2}\right) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{2}\right), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \left\| f_2(x) - f_2(0) - Q(x) \right\| &\le \widetilde{M}(x) + \varphi'\left(\frac{x}{4}, \frac{3x}{4}, -\frac{x}{4}\right) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right), \\ \left\| f_3(x) - f_3(0) - Q(x) \right\| &\le \widetilde{M}'(x) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, -\frac{3x}{4}\right) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right), \\ \left\| f_4(x) - f_4(0) - Q(x) \right\| &\le \frac{1}{2}\left[\varphi'\left(\frac{x}{2}, \frac{3}{2}x, -x\right) + \varphi'\left(\frac{x}{2}, \frac{x}{2}, -x\right)\right] \\ &\quad + \frac{1}{2}\widetilde{M}(2x) + \widetilde{M}(x) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, -\frac{x}{2}\right) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{2}\right), \\ \left\| f_5(x) - f_5(0) - Q(x) \right\| &\le \widetilde{M}(x) + \varphi'\left(\frac{x}{4}, \frac{3x}{4}, -\frac{x}{4}\right) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right), \\ \left\| f_6(x) - f_6(0) - Q(x) \right\| &\le \widetilde{M}'(x) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, -\frac{3x}{4}\right) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \end{aligned} \tag{2.20}$$

for all $x \in V \setminus \{0\}$, where

$$\widetilde{M}(x) := \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} M(2^l x), \qquad \widetilde{M}'(x) := \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} M'(2^l x) \tag{2.21}$$

for all $x \in V \setminus \{0\}$. Moreover, the function Q is given by

$$Q(x) = \lim_{n\to\infty} \frac{f_k(2^n x)}{4^n} \tag{2.22}$$

for all $x \in V$ and for $k = 1, 2, 3, 4, 5, 6$.

Proof. Replace x by $-x$ in (2.18) to obtain

$$\left\| f_1(x-y-z) + f_2(x+y) + f_3(x+z) - f_4(x+y+z) - f_5(x-y) - f_6(x-z) \right\| \le \varphi(-x,y,z) \tag{2.23}$$

for all $x, y, z \in V \setminus \{0\}$. From (2.18) and (2.23), we get

$$\begin{aligned} &\| (f_1+f_4)(x+y+z) + (f_2+f_5)(x-y) + (f_3+f_6)(x-z) \\ &\quad -(f_1+f_4)(x-y-z) - (f_2+f_5)(x+y) - (f_3+f_6)(x+z) \| \le \varphi(-x,y,z) + \varphi(x,y,z) \end{aligned} \tag{2.24}$$

for all $x, y, z \in V \setminus \{0\}$. Let the functions $F, G, H : V \to X$ be defined by

$$\begin{aligned} F(x) &= \frac{1}{2}\left[f_1(x) + f_4(x) - f_1(0) - f_4(0)\right], \\ G(x) &= \frac{1}{2}\left[f_2(x) + f_5(x) - f_2(0) - f_5(0)\right], \\ H(x) &= \frac{1}{2}\left[f_3(x) + f_6(x) - f_3(0) - f_6(0)\right] \end{aligned} \tag{2.25}$$

for all $x, y, z \in V$. Then, it follows from (2.24) that

$$\left\| F(x+y+z) + G(x-y) + H(x-z) - F(x-y-z) - G(x+y) - H(x+z) \right\| \le \varphi'(x,y,z) \tag{2.26}$$

for all $x, y, z \in V \setminus \{0\}$, where $\varphi'(x,y,z) = (1/2)[\varphi(x,y,z) + \varphi(-x,y,z)]$. Replace y and z by x and $-x$ in (2.26) to get

$$\left\| H(2x) - G(2x) \right\| \le \varphi'(x,x,-x) \tag{2.27}$$

for all $x \in V \setminus \{0\}$.

Replacing y, z by x in (2.26) and using (2.27), we get

$$\left\| F(3x) - F(x) - 2G(2x) \right\| \le \varphi'(x,x,x) + \varphi'(x,x,-x) \tag{2.28}$$

for all $x \in V \setminus \{0\}$. Replacing x, y, z by x, $3x$, $-x$ in (2.26) and using (2.27), one obtains

$$\left\| F(3x) - F(x) - G(4x) + 2G(2x) \right\| \le \varphi'(x,3x,-x) + \varphi'(x,x,-x) \tag{2.29}$$

for all $x \in V \setminus \{0\}$. From (2.28) and the above inequality, we have

$$\left\| G(4x) - 4G(2x) \right\| \le \varphi'(x,3x,-x) + 2\varphi'(x,x,-x) + \varphi'(x,x,x) \tag{2.30}$$

for all $x \in V \setminus \{0\}$. Replacing x by $x/2$ and dividing it by 4 in the above inequality, we get

$$\left\| G(x) - \frac{G(2x)}{4} \right\| \le \frac{M(x)}{4} \tag{2.31}$$

for all $x \in V \setminus \{0\}$. By Lemma 2.1, there exists $\lim_{n\to\infty}(G(2^n x)/4^n)$ for all $x \in V$ satisfying

$$\left\| G(x) - \lim_{n\to\infty} \frac{G(2^n x)}{4^n} \right\| \leq \widetilde{M}(x) \tag{2.32}$$

for all $x \in V \setminus \{0\}$, where

$$\widetilde{M}(x) := \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} M(2^l x). \tag{2.33}$$

By the similar method in obtaining inequality (2.32), we get

$$\left\| H(x) - \lim_{n\to\infty} \frac{H(2^n x)}{4^n} \right\| \leq \widetilde{M}'(x) \tag{2.34}$$

for all $x \in V \setminus \{0\}$, where

$$\widetilde{M}'(x) := \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} M'(2^l x). \tag{2.35}$$

From (2.27), we have

$$\lim_{n\to\infty} \frac{G(2^n x)}{4^n} = \lim_{n\to\infty} \frac{H(2^n x)}{4^n} \tag{2.36}$$

for all $x \in V$. From (2.36), we can define a map $Q : V \to X$ by

$$Q(x) = \lim_{n\to\infty} \frac{G(2^n x)}{4^n} \tag{2.37}$$

for all $x \in V$. It follows from (2.26), (2.32), and (2.37) that

$$\begin{aligned}
\|F(x) - Q(x)\| &\leq \frac{1}{2} \left\| F(x) + G(x) + H\left(\frac{3}{2}x\right) - G(2x) - H\left(\frac{x}{2}\right) \right\| + \frac{1}{2}\|G(x) - Q(x)\| \\
&\quad + \left\| F(x) + G(x) + H\left(\frac{1}{2}x\right) - H\left(\frac{3}{2}x\right) \right\| + \frac{1}{2}\|G(2x) - Q(2x)\| \\
&\leq \frac{1}{2}\varphi'\left(\frac{x}{2}, \frac{3}{2}x, -x\right) + \frac{1}{2}\widetilde{M}(2x) + \frac{1}{2}\varphi'\left(\frac{x}{2}, \frac{x}{2}, -x\right) + \widetilde{M}(x)
\end{aligned} \tag{2.38}$$

for all $x \in V \setminus \{0\}$. Replacing x by $2^n x$, dividing it by 4^n in the above inequality and taking the limit in the resulted inequality as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{F(2^n x)}{4^n} = Q(x) \tag{2.39}$$

for all $x \in V$. Using (2.26), (2.36), (2.37), and (2.39), we obtain

$$Q(x+y+z) + Q(x-y) + Q(z-x) - Q(x-y-z) - Q(x+y) - Q(x+z) = 0 \tag{2.40}$$

for all $x, y, z \in V \setminus \{0\}$. Replacing x and z by $x/2$ in (2.40) and using the fact $Q(0) = 0$, we have

$$Q(x+y) + Q\left(\frac{x}{2} - y\right) - Q(-y) - Q\left(\frac{x}{2} + y\right) - Q(x) = 0 \tag{2.41}$$

for all $x, y \in V$. Replace x and z by $x/2$ and $-x/2$ in (2.40) to have

$$Q(y) + Q\left(\frac{x}{2} - y\right) + Q(x) - Q(x-y) - Q\left(\frac{x}{2} + y\right) = 0 \tag{2.42}$$

for all $x, y \in V$. Subtracting (2.41) from (2.42) and using the evenness of Q, we lead to

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = 0, \tag{2.43}$$

for all $x, y \in V$.

On the other hand, it follows from (2.18) and (2.23) that

$$\begin{aligned} &\|(f_1 - f_4)(x+y+z) + (f_2 - f_5)(x-y) + (f_3 - f_6)(x-z) \\ &\quad + (f_1 - f_4)(x-y-z) + (f_2 - f_5)(x+y) + (f_3 - f_6)(-x+z)\| \le \varphi(-x, y, z) + \varphi(x, y, z) \end{aligned} \tag{2.44}$$

for all $x, y, z \in V \setminus \{0\}$. Let the functions $F', G', H' : V \to X$ be defined by

$$F'(x) = \frac{1}{2}[f_1(x) - f_4(x)], \qquad G'(x) = \frac{1}{2}[f_2(x) - f_5(x)], \qquad H'(x) = \frac{1}{2}[f_3(x) - f_6(x)] \tag{2.45}$$

for all $x, y, z \in V$.

From (2.44), we have

$$\|F'(x+y+z) + G'(x-y) + H'(x-z) + F'(x-y-z) + G'(x+y) + H'(x+z)\| \le \varphi'(x, y, z) \tag{2.46}$$

for all $x, y, z \in V \setminus \{0\}$. Replace y, z by x in (2.46) to get

$$\|F'(3x) + F'(x) + G'(2x) + G'(0) + H'(2x) + H'(0)\| \le \varphi'(x, x, x) \tag{2.47}$$

for all $x, y, z \in V \setminus \{0\}$. Replace x, y, z by x, $3x$, $-x$ in (2.46) to get

$$\|F'(3x) + F'(x) + G'(2x) + G'(4x) + H'(2x) + H'(0)\| \le \varphi'(x, 3x, -x) \tag{2.48}$$

for all $x, y, z \in V \setminus \{0\}$. From (2.47) and the above inequality, we have

$$\|G'(4x) - G'(0)\| \le \varphi'(x, 3x, -x) + \varphi'(x, x, x) \tag{2.49}$$

for all $x \in V \setminus \{0\}$.

Replace x, y, z by x, x, $-3x$ in (2.46) to get

$$\|F'(3x) + F'(x) + G'(2x) + G'(0) + H'(2x) + H'(4x)\| \le \varphi'(x, x, -3x) \tag{2.50}$$

for all $x, y, z \in V \setminus \{0\}$. From (2.47) and the above inequality, we get

$$\left\|H'(4x) - H'(0)\right\| \leq \varphi'(x, x, -3x) + \varphi'(x, x, x) \tag{2.51}$$

for all $x \in V \setminus \{0\}$. It follows from (2.46) that

$$\begin{aligned}\left\|F'(4x) - F'(0)\right\| &\leq \left\|F'(0) + G'(0) + H'(3x) + F'(2x) + G'(2x) + H'(x)\right\| \\ &\quad + \left\|F'(4x) + G'(0) + H'(x) + F'(2x) + G'(2x) + H'(3x)\right\| \\ &\leq \varphi'(x, x, -2x) + \varphi'(x, x, 2x)\end{aligned} \tag{2.52}$$

for all $x \in V \setminus \{0\}$. By the definitions of F, G, H, F', G', H', we have

$$\begin{aligned} f_1(x) - f_1(0) - Q(x) &= F(x) + F'(x) - F'(0) - Q(x), \\ f_2(x) - f_2(0) - Q(x) &= G(x) + G'(x) - F'(0) - Q(x), \\ f_3(x) - f_3(0) - Q(x) &= H(x) + H'(x) - H'(0) - Q(x), \\ f_4(x) - f_4(0) - Q(x) &= F(x) - F'(x) + F'(0) - Q(x), \\ f_5(x) - f_5(0) - Q(x) &= G(x) - G'(x) + G'(0) - Q(x), \\ f_6(x) - f_6(0) - Q(x) &= H(x) - H'(x) + H'(0) - Q(x) \end{aligned} \tag{2.53}$$

for all $x \in V \setminus \{0\}$. Hence by using (2.32), (2.34), (2.36), (2.37), (2.38), (2.49), (2.51), and (2.52), the inequalities in (2.19) can be shown. The uniqueness of Q follows from Lemma 2.1. □

Theorem 2.3. *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{l=0}^{\infty} 4^l \varphi\left(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}, \frac{z}{2^{l+1}}\right) < \infty \tag{a$'$}$$

holds for all $x, y, z \in V \setminus \{0\}$. If the even functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy inequality (2.18) *for all $x, y, z \in V \setminus \{0\}$, then there exists exactly one quadratic function $Q : V \to X$ satisfying inequalities* (2.19) *for all $x \in V \setminus \{0\}$, where*

$$\widetilde{M}(x) := \sum_{l=0}^{\infty} 4^l M\left(\frac{x}{2^{l+1}}\right), \qquad \widetilde{M}'(x) := \sum_{l=0}^{\infty} 4^l M'\left(\frac{x}{2^{l+1}}\right). \tag{2.54}$$

Moreover, the function Q is given by

$$Q(x) = \lim_{n \to \infty} 4^n \left(f_k(2^{-n}x) - f_k(0)\right) \tag{2.55}$$

for all $x \in V$ and for $k = 1, 2, 3, 4, 5, 6$.

Proof. The proof is similar to that of Theorem 2.2. □

Applying Theorems 2.2 and 2.3, we get the following corollary in the sense of Rassias inequality.

Corollary 2.4. *Let $p \neq 2$ and $\varepsilon > 0$. If the even functions $f_i : V \to X$, $i = 1, 2, \ldots, 6$, satisfy*

$$\begin{aligned}&\left\|f_1(x + y + z) + f_2(x - y) + f_3(x - z) - f_4(x - y - z) - f_5(x + y) - f_6(x + z)\right\| \\ &\quad \leq \varepsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right)\end{aligned} \tag{2.56}$$

for all $x, y, z \in V \setminus \{0\}$.

Then there exist exactly one quadratic function $Q : V \to X$ satisfying

$$\|f_1(x) - f_1(0) - Q(x)\| \le \left[1 + \frac{(3^p + 11)(2^p + 2)}{2\cdot 2^p|2^p - 4|} + \frac{7 + 3^p}{2\cdot 2^p} + \frac{4}{4^p}\right]\cdot\varepsilon\cdot\|x\|^p,$$

$$\|f_4(x) - f_4(0) - Q(x)\| \le \left[1 + \frac{(3^p + 11)(2^p + 2)}{2\cdot 2^p|2^p - 4|} + \frac{7 + 3^p}{2\cdot 2^p} + \frac{4}{4^p}\right]\cdot\varepsilon\cdot\|x\|^p, \tag{2.57}$$

$$\|f_j(x) - f_j(0) - Q(x)\| \le \left[\frac{3^p + 11}{2^p|2^p - 4|} + \frac{3^p + 5}{4^p}\right]\cdot\varepsilon\cdot\|x\|^p$$

for all $x \in V \setminus \{0\}$ and $j = 2, 3, 5, 6$. Moreover, the function Q is given by

$$Q(x) = \begin{cases} \lim_{n\to\infty} \dfrac{f_k(2^n x)}{4^n} & \text{if } p < 2, \\ \lim_{n\to\infty} 4^n\left(f_k(2^{-n}x) - f_k(0)\right) & \text{if } p > 2 \end{cases} \tag{2.58}$$

for all $x \in V \setminus \{0\}$ and $k = 1, 2, 3, 4, 5, 6$

Proof. Apply Theorem 2.2 for $p < 2$ and Theorem 2.3 for $p > 2$. □

We establish Theorems 2.5 and 2.6 for the odd functions.

Theorem 2.5. *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ be a function such that*

$$\widehat{\varphi}(x, y, z) := \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} \varphi(2^l x, 2^l y, 2^l z) < \infty \tag{b}$$

holds for all $x, y, z \in V \setminus \{0\}$. If the odd functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy

$$\|f_1(x + y + z) + f_2(x - y) + f_3(x - z) - f_4(x - y - z) - f_5(x + y) - f_6(x + z)\| \le \varphi(x, y, z) \tag{2.59}$$

for all $x, y, z \in V \setminus \{0\}$, then there exist exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$\begin{aligned}\|f_1(x) - A(x) + A_1(x) + A_2(x)\| &\le \varphi'\left(\frac{x}{4}, \frac{x}{4}, -\frac{x}{2}\right) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{2}\right) \\ &\quad + 2\widehat{M}\left(\frac{x}{2}\right) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, x\right) + \widehat{\varphi}'\left(\frac{x}{2}, -\frac{x}{2}, x\right),\end{aligned} \tag{2.60}$$

$$\|f_2(x) - A(x) - A_1(x)\| \le \widehat{M}(x) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}\right) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right),$$

$$\|f_3(x) - A(x) - A_2(x)\| \le \widehat{M}'(x) + \widehat{\varphi}'\left(\frac{x}{2}, -\frac{x}{2}, \frac{3x}{2}\right) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right),$$

$$\begin{aligned}\|f_4(x) - A(x) - A_1(x) - A_2(x)\| &\le \varphi'\left(\frac{x}{4}, \frac{x}{4}, -\frac{x}{2}\right) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{2}\right) \\ &\quad + 2\widehat{M}\left(\frac{x}{2}\right) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, x\right) + \widehat{\varphi}'\left(\frac{x}{2}, -\frac{x}{2}, x\right),\end{aligned} \tag{2.61}$$

$$\|f_5(x) - A(x) + A_1(x)\| \le \widehat{M}(x) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}\right) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right),$$

$$\|f_6(x) - A(x) + A_2(x)\| \le \widehat{M}'(x) + \widehat{\varphi}'\left(\frac{x}{2}, -\frac{x}{2}, \frac{3x}{2}\right) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in V \setminus \{0\}$, *where*

$$\widehat{M}(x) := \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} M(2^l x), \qquad \widehat{M'}(x) := \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} M'(2^l x),$$
$$\widehat{\varphi}'(x,y,z) := \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} \varphi'(2^l x, 2^l y, 2^l z). \tag{2.62}$$

Moreover, the functions A, A_1, A_2 *are given by*

$$A(x) = \lim_{n\to\infty} \frac{f_1(2^n x) + f_4(2^n x)}{2^{n+1}},$$
$$A_1(x) = \lim_{n\to\infty} \frac{f_2(2^n x) - f_5(2^n x)}{2^{n+1}}, \tag{2.63}$$
$$A_2(x) = \lim_{n\to\infty} \frac{f_3(2^n x) - f_6(2^n x)}{2^{n+1}}$$

for all $x \in V$.

Proof. Replace x by $-x$ in (2.59) to obtain

$$\| - f_1(x-y-z) - f_2(x+y) - f_3(x+z) + f_4(x+y+z) + f_5(x-y) + f_6(x-z)\| \le \varphi(-x,y,z) \tag{2.64}$$

for all $x, y, z \in V \setminus \{0\}$. Let the functions $F, G, H : V \to X$ be defined by

$$F(x) = \frac{1}{2}[f_1(x) + f_4(x)], \qquad G(x) = \frac{1}{2}[f_2(x) + f_5(x)], \qquad H(x) = \frac{1}{2}[f_3(x) + f_6(x)] \tag{2.65}$$

for all $x, y, z \in V$. From (2.59) and (2.64), we get

$$\|F(x+y+z) + G(x-y) + H(x-z) - F(x-y-z) - G(x+y) - H(x+z)\| \le \varphi'(x,y,z) \tag{2.66}$$

for all $x, y, z \in V \setminus \{0\}$. From (2.66), we have

$$\|H(2x) - G(2x)\| \le \varphi'(x,x,-x), \tag{2.67}$$

for all $x \in V \setminus \{0\}$. It follows from (2.66) and (2.67) that

$$\begin{aligned} \|G(4x) - 2G(2x)\| &= \| - F(3x) - F(x) + G(2x) + G(4x) - H(2x)\| \\ &\quad + \|2H(2x) - 2G(2x)\| + \|F(3x) + F(x) - G(2x) - H(2x)\| \\ &\le \varphi'(x,x,x) + 2\varphi'(x,x,-x) + \varphi'(x,3x,-x) \end{aligned} \tag{2.68}$$

for all $x \in V \setminus \{0\}$. Replacing x by $x/2$ and dividing it by 2 in the above inequality, we obtain

$$\left\|G(x) - \frac{G(2x)}{2}\right\| \le \frac{M(x)}{2} \tag{2.69}$$

for all $x \in V \setminus \{0\}$. Applying Lemma 2.1, we obtain

$$\left\| G(x) - \lim_{n\to\infty} \frac{G(2^n x)}{2^n} \right\| \le \widehat{M}(x) \tag{2.70}$$

for all $x \in V \setminus \{0\}$. Similarly we have

$$\left\| H(x) - \lim_{n\to\infty} \frac{H(2^n x)}{2^n} \right\| \le \widehat{M}'(x) \tag{2.71}$$

for all $x \in V \setminus \{0\}$. From (2.67), we get

$$\lim_{n\to\infty} \frac{G(2^n x)}{2^n} = \lim_{n\to\infty} \frac{H(2^n x)}{2^n} \tag{2.72}$$

for all $x \in V$ and we can define a function $A : V \to X$ by

$$A(x) = \lim_{n\to\infty} \frac{G(2^n x)}{2^n} = \lim_{n\to\infty} \frac{H(2^n x)}{2^n} \tag{2.73}$$

for all $x \in V \setminus \{0\}$. It follows from (2.66) and (2.70) that

$$\begin{aligned} \|F(x) - A(x)\| &= \left\| F(x) - H\left(\frac{x}{4}\right) + F\left(\frac{x}{2}\right) - G\left(\frac{x}{2}\right) - H\left(\frac{3x}{4}\right) \right\| \\ &\quad + \left\| H\left(\frac{3x}{4}\right) - F\left(\frac{x}{2}\right) - G\left(\frac{x}{2}\right) + H\left(\frac{x}{4}\right) \right\| + \left\| 2G\left(\frac{x}{2}\right) - 2A\left(\frac{x}{2}\right) \right\| \\ &\le \varphi'\left(\frac{x}{4}, \frac{x}{4}, -\frac{x}{2}\right) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{2}\right) + 2\widehat{M}\left(\frac{x}{2}\right) \end{aligned} \tag{2.74}$$

for all $x \in V \setminus \{0\}$. Replacing x by $2^n x$, dividing it by 2^n in the above inequality and taking the limit in the resulted inequality as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{F(2^n x)}{2^n} = A(x) \tag{2.75}$$

for all $x \in V \setminus \{0\}$. From (2.73) and (2.75), we have

$$A(x + y + z) + A(x - y) + A(x - z) - A(x - y - z) - A(x + y) - A(x + z) = 0 \tag{2.76}$$

for all $x, y, z \in V \setminus \{0\}$. Replace y and z by $2y$ and x in (2.76) to obtain

$$A(2x + 2y) + A(x - 2y) + A(2y) - A(x + 2y) - A(2x) = 0 \tag{2.77}$$

for all $x, y, z \in V \setminus \{0\}$. Replace y and z by $-2y$ and x in (2.76) to get

$$A(2x - 2y) + A(x + 2y) - A(2y) - A(x - 2y) - A(2x) = 0 \tag{2.78}$$

for all $x, y \in V \setminus \{0\}$. Since $A(0) = 0$ and $A(2x) = 2A(x)$, using the above two equalities, we have

$$A(x-y) + A(x+y) - A(2x) = 0 \tag{2.79}$$

for all $x, y \in V$. Hence, A is an additive function.

Let the functions $F', G', H' : V \to X$ be defined by

$$F'(x) = \frac{1}{2}\left[f_1(x) - f_4(x)\right], \qquad G'(x) = \frac{1}{2}\left[f_2(x) - f_5(x)\right], \qquad H'(x) = \frac{1}{2}\left[f_3(x) - f_6(x)\right] \tag{2.80}$$

for all $x, y, z \in V$. From (2.59) and (2.64), we have

$$\left\|F'(x+y+z) + G'(x-y) + H'(x-z) + F'(x-y-z) + G'(x+y) + H'(x+z)\right\| \le \varphi'(x,y,z) \tag{2.81}$$

for all $x, y, z \in V \setminus \{0\}$. It follows from (2.81) that

$$\begin{aligned} \left\|G'(x) - \frac{G'(2x)}{2}\right\| &\le \frac{1}{2}\left\|F'\left(\frac{3x}{2}\right) - F'\left(\frac{x}{2}\right) - G'(x) + G'(2x) + H'(x)\right\| \\ &\quad + \frac{1}{2}\left\|F'\left(\frac{3x}{2}\right) - F'\left(\frac{x}{2}\right) + G'(x) + H'(x)\right\| \\ &\le \frac{1}{2}\varphi'\left(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}\right) + \frac{1}{2}\varphi'\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \end{aligned} \tag{2.82}$$

for all $x \in V \setminus \{0\}$. Applying Lemma 2.1, we obtain an odd function $A_1 : V \to X$ defined by

$$A_1(x) = \lim_{n\to\infty} \frac{G'(2^n x)}{2^n}; \tag{2.83}$$

and the inequality

$$\left\|G'(x) - A_1(x)\right\| \le \widehat{\varphi}'\left(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}\right) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \tag{2.84}$$

holds for all $x \in V \setminus \{0\}$. Similarly we have an odd function $A_2 : V \to X$ defined by

$$A_2(x) = \lim_{n\to\infty} \frac{H'(2^n x)}{2^n} \tag{2.85}$$

for all $x \in V$ and the inequality

$$\left\|H'(x) - A_2(x)\right\| \le \widehat{\varphi}'\left(\frac{x}{2}, -\frac{x}{2}, \frac{3x}{2}\right) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \tag{2.86}$$

for all $x \in V \setminus \{0\}$. Replace x, y, z by x, x, $-x$ in (2.81) to get

$$\left\|2F'(x) + G'(2x) + H'(2x)\right\| \le \varphi'(x, x, -x) \tag{2.87}$$

for all $x \in V \setminus \{0\}$. Replacing x by $2^{n-1}x$ and dividing it by 2^n in the above inequality, we obtain

$$\left\| \frac{2F'(2^{n-1}x) + G'(2^n x) + H'(2^n x)}{2^n} \right\| \leq \frac{\varphi'(2^n x, 2^n x, -2^n x)}{2^n} \tag{2.88}$$

for all $x \in V \setminus \{0\}$. Taking the limit in the above inequality as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{F'(2^n x)}{2^n} = -A_1(x) - A_2(x) \tag{2.89}$$

for all $x \in V \setminus \{0\}$. It follows from (2.81) that

$$\begin{aligned} \left\| F'(x) - \frac{F'(2x)}{2} \right\| &\leq \frac{1}{2} \left\| F'(2x) + G'(0) - H'\left(\frac{x}{2}\right) - F'(x) + G'(x) + H'\left(\frac{3x}{2}\right) \right\| \\ &\quad + \frac{1}{2} \left\| F'(x) + G'(x) - H'\left(\frac{x}{2}\right) + F'(0) + G'(0) + H'\left(\frac{3x}{2}\right) \right\| \\ &\leq \frac{1}{2} \left[\varphi'\left(\frac{x}{2}, \frac{x}{2}, x\right) + \varphi'\left(\frac{x}{2}, \frac{x}{2}, -x\right) \right] \end{aligned} \tag{2.90}$$

for all $x \in V \setminus \{0\}$. Applying Lemma 2.1 and (2.89), we have

$$\| F'(x) + A_1(x) + A_2(x) \| \leq \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, x\right) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, -x\right) \tag{2.91}$$

for all $x \in V \setminus \{0\}$. From (2.81), (2.83), (2.85), and (2.89), we have

$$\begin{aligned} -A_1(x+y+z) - A_2(x+y+z) + A_1(x-y) + A_2(x-z) - A_1(x-y-z) \\ - A_2(x-y-z) + A_1(x+y) + A_2(x+z) = 0 \end{aligned} \tag{2.92}$$

for all $x, y, z \in V \setminus \{0\}$. Replace y and z by $2y$ and x in (2.92) to get

$$-A_1(2x+2y) - A_2(2x+2y) + A_1(x-2y) + A_1(2y) + A_2(2y) + A_2(2x) + A_1(x+2y) = 0 \tag{2.93}$$

for all $x, y \in V \setminus \{0\}$. Replace y and z by x and $2y$ in (2.92) to get

$$-A_1(2x+2y) - A_2(2x+2y) + A_2(x-2y) + A_1(2y) + A_2(2y) + A_1(2x) + A_2(x+2y) = 0 \tag{2.94}$$

for all $x, y \in V \setminus \{0\}$. From the above two equalities, we get

$$(A_1 - A_2)(x-2y) - (A_1 - A_2)(2x) + (A_1 - A_2)(x+2y) = 0 \tag{2.95}$$

for all $x, y \in V \setminus \{0\}$. Since $A(0) = 0$, we have

$$(A_1 - A_2)(x-2y) - (A_1 - A_2)(2x) + (A_1 - A_2)(x+2y) = 0 \tag{2.96}$$

for all $x, y \in V$. Hence $A_1 - A_2$ is additive, that is,

$$(A_1 - A_2)(x+y) = (A_1 - A_2)(x) + (A_1 - A_2)(y) \tag{2.97}$$

for all $x, y \in V$. Replace z by $-y$ in (2.92) to obtain

$$-A_1(x) - A_2(x) + A_1(x-y) + A_2(x+y) - A_1(x) - A_2(x) + A_1(x+y) + A_2(x-y) = 0 \tag{2.98}$$

for all $x, y \in V \setminus \{0\}$. Since $A_1 - A_2$ is additive, we have

$$A_2(2x) - A_2(x+y) - A_2(x-y) = A_1(2x) - A_1(x+y) - A_1(x-y) \tag{2.99}$$

for all $x, y \in V \setminus \{0\}$. From this and (2.98), we get

$$-A_1(4x) + 2A_1(x-y) + 2A_1(x+y) = 0 \tag{2.100}$$

for all $x, y \in V \setminus \{0\}$. From this and $A_1(0) = 0$, we have

$$A_1(x+y) = A_1(x) + A_1(y) \tag{2.101}$$

for all $x, y \in V$. Since A_1 and $A_1 - A_2$ are additive, A_2 is additive.

From (2.74), (2.91), and the definitions of F, F', we have

$$\begin{aligned}\|f_1(x) - A(x) + A_1(x) + A_2(x)\| &\le \|F(x) - A(x)\| + \|F'(x) + A_1(x) + A_2(x)\| \\ &\le \varphi'\left(\frac{x}{4}, \frac{x}{4}, -\frac{x}{2}\right) + \varphi'\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{2}\right) + 2\widehat{M}\left(\frac{x}{2}\right) \\ &\quad + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, x\right) + \widehat{\varphi}'\left(\frac{x}{2}, \frac{x}{2}, -x\right)\end{aligned} \tag{2.102}$$

for all $x \in V \setminus \{0\}$. The rest of inequalities in (2.60) can be shown by the similar method. □

Theorem 2.6. *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ be a function such that*

$$\widehat{\varphi}(x, y, z) := \sum_{l=0}^{\infty} 2^l \varphi\left(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}, \frac{z}{2^{l+1}}\right) < \infty \tag{b$'$}$$

holds for all $x, y, z \in V \setminus \{0\}$. If the odd functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy inequalities (2.59) *for all $x, y, z \in V \setminus \{0\}$, then there exist exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying the inequalities* (2.60) *for all $x \in V \setminus \{0\}$, where*

$$\begin{gathered}\widehat{M}(x) := \sum_{l=0}^{\infty} 2^l M\left(\frac{x}{2^{l+1}}\right), \qquad \widehat{M}'(x) := \sum_{l=0}^{\infty} 2^l M'\left(\frac{x}{2^{l+1}}\right), \\ \widehat{\varphi}'(x, y, z) := \sum_{l=0}^{\infty} 2^l \varphi'\left(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}, \frac{z}{2^{l+1}}\right).\end{gathered} \tag{2.103}$$

Moreover, the functions A, A_1, A_2 are given by

$$\begin{aligned}A(x) &= \lim_{n\to\infty} 2^{n-2}\left(f_1\left(\frac{x}{2^n}\right) + f_4\left(\frac{x}{2^n}\right) - f_1\left(-\frac{x}{2^n}\right) - f_4\left(-\frac{x}{2^n}\right)\right), \\ A_1(x) &= \lim_{n\to\infty} 2^{n-2}\left(f_2\left(\frac{x}{2^n}\right) - f_5\left(\frac{x}{2^n}\right) - f_2\left(-\frac{x}{2^n}\right) + f_5\left(-\frac{x}{2^n}\right)\right), \\ A_2(x) &= \lim_{n\to\infty} 2^{n-2}\left(f_3\left(\frac{x}{2^n}\right) - f_6\left(\frac{x}{2^n}\right) - f_3\left(-\frac{x}{2^n}\right) + f_6\left(-\frac{x}{2^n}\right)\right)\end{aligned} \tag{2.104}$$

for all $x \in V$.

Proof. The proof is similar to that of Theorem 2.5. □

Applying Theorems 2.5 and 2.6, we get the following corollary in the sense of Rassias inequality.

Corollary 2.7. *Let $p \neq 1$. If the odd functions $f_i : V \to X$, $i = 1, 2, \ldots, 6$, satisfy*

$$\begin{aligned} &\|f_1(x+y+z) + f_2(x-y) + f_3(x-z) - f_4(x-y-z) - f_5(x+y) - f_6(x+z)\| \\ &\quad \leq \varepsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \end{aligned} \tag{2.105}$$

for all $x, y, z \in V \setminus \{0\}$.

Then there exist exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$\begin{aligned} \|f_1(x) - f_1(0) - A(x) + A_1(x) + A_2(x)\| &\leq \left[\frac{2}{2^p} + \frac{4}{4^p} + \frac{2(3^p + 11) + 4\cdot 2^p + 2\cdot 4^p}{4^p|2^p - 2|}\right]\cdot\varepsilon\cdot\|x\|^p, \\ \|f_2(x) - f_2(0) - A(x) - A_1(x)\| &\leq \frac{2(3^p + 8)}{2^p|2^p - 2|}\cdot\varepsilon\cdot\|x\|^p, \\ \|f_3(x) - f_3(0) - A(x) - A_2(x)\| &\leq \frac{2(3^p + 8)}{2^p|2^p - 2|}\cdot\varepsilon\cdot\|x\|^p, \\ \|f_4(x) - f_4(0) - A(x) - A_1(x) - A_2(x)\| &\leq \left[\frac{2}{2^p} + \frac{4}{4^p} + \frac{2(3^p + 11) + 4\cdot 2^p + 2\cdot 4^p}{4^p|2^p - 2|}\right]\cdot\varepsilon\cdot\|x\|^p, \\ \|f_5(x) - f_5(0) - A(x) + A_1(x)\| &\leq \frac{2(3^p + 8)}{2^p|2^p - 2|}\cdot\varepsilon\cdot\|x\|^p, \\ \|f_6(x) - f_6(0) - A(x) + A_2(x)\| &\leq \frac{2(3^p + 8)}{2^p|2^p - 2|}\cdot\varepsilon\cdot\|x\|^p \end{aligned} \tag{2.106}$$

for all $x \in V \setminus \{0\}$. Moreover, the functions A, A_1, A_2 are given by

$$\begin{aligned} A(x) &= \begin{cases} \lim\limits_{n\to\infty} \dfrac{f_1(2^n x) + f_4(2^n x) - f_1(-2^n x) - f_4(-2^n x)}{2^{n+2}}, & \text{if } p < 1, \\ \lim\limits_{n\to\infty} 2^{n-2}\left(f_1\left(\dfrac{x}{2^n}\right) + f_4\left(\dfrac{x}{2^n}\right) - f_1\left(-\dfrac{x}{2^n}\right) - f_4\left(-\dfrac{x}{2^n}\right)\right), & \text{if } p > 1, \end{cases} \\ A_1(x) &= \begin{cases} \lim\limits_{n\to\infty} \dfrac{f_2(2^n x) - f_5(2^n x) - f_2(-2^n x) + f_5(-2^n x)}{2^{n+2}}, & \text{if } p < 1, \\ \lim\limits_{n\to\infty} 2^{n-2}\left(f_2\left(\dfrac{x}{2^n}\right) - f_5\left(\dfrac{x}{2^n}\right) - f_2\left(-\dfrac{x}{2^n}\right) + f_5\left(-\dfrac{x}{2^n}\right)\right), & \text{if } p > 1, \end{cases} \\ A_2(x) &= \begin{cases} \lim\limits_{n\to\infty} \dfrac{f_3(2^n x) - f_6(2^n x) - f_3(-2^n x) + f_6(-2^n x)}{2^{n+2}}, & \text{if } p < 1, \\ \lim\limits_{n\to\infty} 2^{n-2}\left(f_3\left(\dfrac{x}{2^n}\right) - f_6\left(\dfrac{x}{2^n}\right) - f_3\left(-\dfrac{x}{2^n}\right) + f_6\left(-\dfrac{x}{2^n}\right)\right), & \text{if } p > 1 \end{cases} \end{aligned} \tag{2.107}$$

for all $x \in V$.

Proof. Apply Theorem 2.5 for $p < 1$ and Theorem 2.6 for $p > 1$. □

We establish the following theorem for the general case from Theorems 2.2 and 2.5.

Theorem 2.8. *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \times V \setminus \{0\} \to [0,\infty)$ be a function that satisfies conditions* (a) *and* (b). *Suppose that the functions $f_i : V \to X$, $i = 1,2,\dots,6$, satisfy the inequality*

$$\left\| f_1(x+y+z) + f_2(x-y) + f_3(x-z) - f_4(x-y-z) - f_5(x+y) - f_6(x+z) \right\| \le \varphi(x,y,z) \tag{2.108}$$

for all $x,y,z \in V \setminus \{0\}$. Then there exist exactly one quadratic function $Q : V \to X$ and exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$\begin{aligned}
&\left\| f_1(x) - f_1(0) - Q(x) - A(x) + A_1(x) + A_2(x) \right\| \\
&\quad \le \frac{1}{2}\left[\varphi'_e\left(\frac{x}{2}, \frac{3}{2}x, -x\right) + \varphi'_e\left(\frac{x}{2}, \frac{x}{2}, -x\right)\right] + 2\varphi'_e\left(\frac{x}{4}, \frac{x}{4}, -\frac{x}{2}\right) + 2\varphi'_e\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{2}\right) \\
&\qquad + \frac{1}{2}\widetilde{M}_e(2x) + \widetilde{M}_e(x) + 2\widehat{M}_e\left(\frac{x}{2}\right) + \widehat{\varphi}'_e\left(\frac{x}{2}, \frac{x}{2}, x\right) + \widehat{\varphi}'_e\left(\frac{x}{2}, -\frac{x}{2}, x\right), \\
&\left\| f_2(x) - f_2(0) - Q(x) - A(x) - A_1(x) \right\| \\
&\quad \le \widetilde{M}_e(x) + \varphi'_e\left(\frac{x}{4}, \frac{3x}{4}, -\frac{x}{4}\right) + \varphi'_e\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) + \widehat{M}_e(x) + \widehat{\varphi}'_e\left(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}\right) + \widehat{\varphi}'_e\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right), \\
&\left\| f_3(x) - f_3(0) - Q(x) - A(x) - A_2(x) \right\| \\
&\quad \le \widetilde{M}'_e(x) + \varphi'_e\left(\frac{x}{4}, \frac{x}{4}, -\frac{3x}{4}\right) + \varphi'_e\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) + \widehat{M}'_e(x) + \widehat{\varphi}'_e\left(\frac{x}{2}, \frac{x}{2}, -\frac{3x}{2}\right) + \widehat{\varphi}'_e\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right), \\
&\left\| f_4(x) - f_4(0) - Q(x) - A(x) - A_1(x) - A_2(x) \right\| \\
&\quad \le \frac{1}{2}\left[\varphi'_e\left(\frac{x}{2}, \frac{3}{2}x, -x\right) + \varphi'_e\left(\frac{x}{2}, \frac{x}{2}, -x\right)\right] + 2\varphi'_e\left(\frac{x}{4}, \frac{x}{4}, -\frac{x}{2}\right) + 2\varphi'_e\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{2}\right) \\
&\qquad + \frac{1}{2}\widetilde{M}_e(2x) + \widetilde{M}_e(x) + 2\widehat{M}_e\left(\frac{x}{2}\right) + \widehat{\varphi}'_e\left(\frac{x}{2}, \frac{x}{2}, x\right) + \widehat{\varphi}'_e\left(\frac{x}{2}, -\frac{x}{2}, x\right), \\
&\left\| f_5(x) - f_5(0) - Q(x) - A(x) + A_1(x) \right\| \\
&\quad \le \widetilde{M}_e(x) + \varphi'_e\left(\frac{x}{4}, \frac{3x}{4}, -\frac{x}{4}\right) + \varphi'_e\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) + \widehat{M}_e(x) + \widehat{\varphi}'_e\left(\frac{x}{2}, \frac{3x}{2}, -\frac{x}{2}\right) + \widehat{\varphi}'_e\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right), \\
&\left\| f_6(x) - f_6(0) - Q(x) - A(x) + A_2(x) \right\| \\
&\quad \le \widetilde{M}'(x) + \varphi'_e\left(\frac{x}{4}, \frac{x}{4}, -\frac{3x}{4}\right) + \varphi'_e\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) + \widehat{M}'_e(x) + \widehat{\varphi}'_e\left(\frac{x}{2}, \frac{x}{2}, -\frac{3x}{2}\right) + \widehat{\varphi}'_e\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)
\end{aligned} \tag{2.109}$$

for all $x \in V \setminus \{0\}$, where

$$\begin{aligned} \widetilde{M}_e(x) &:= \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} M_e(2^l x), \qquad \widetilde{M}'_e(x) := \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} M'_e(2^l x), \\ \widehat{M}_e &:= \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} M_e(2^l x), \qquad \widehat{M}'_e := \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} M_e(2^l x), \\ \widehat{\varphi}'_e(x,y,z) &:= \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} \varphi'_e(2^l x, 2^l y, 2^l z) \end{aligned} \tag{2.110}$$

for all $x \in V \setminus \{0\}$. Moreover, the function Q is given by

$$Q(x) = \lim_{n\to\infty} \frac{f_k(2^n x) + f_k(-2^n x)}{2\cdot 4^n} \tag{2.111}$$

for $i = 1,2,3,4,5,6$ and the functions A, A_1, A_2 are given by

$$\begin{aligned} A(x) &= \lim_{n\to\infty} \frac{f_1(2^n x) + f_4(2^n x) - f_1(-2^n x) - f_4(-2^n x)}{2^{n+2}}, \\ A_1(x) &= \lim_{n\to\infty} \frac{f_2(2^n x) - f_5(2^n x) - f_2(-2^n x) + f_5(-2^n x)}{2^{n+2}}, \\ A_2(x) &= \lim_{n\to\infty} \frac{f_3(2^n x) - f_6(2^n x) - f_3(-2^n x) + f_6(-2^n x)}{2^{n+2}} \end{aligned} \tag{2.112}$$

for all, $x \in V$.

Proof. From (2.108), we obtain

$$\begin{aligned} &\left\| f_1(-x-y-z) + f_2(-x+y) + f_3(-x+z) - f_4(-x+y+z) - f_5(-x-y) - f_6(-x-z) \right\| \\ &\quad \le \varphi(-x,-y,-z) \end{aligned} \tag{2.113}$$

for all $x, y, z \in V \setminus \{0\}$. From (2.108) and this inequality, one gets

$$\begin{aligned} &\left\| f_{1e}(x+y+z) + f_{2e}(x-y) + f_{3e}(x-z) - f_{4e}(x-y-z) - f_{5e}(x+y) - f_{6e}(x+z) \right\| \le \varphi_e(x,y,z), \\ &\left\| f_{1o}(x+y+z) + f_{2o}(x-y) + f_{3o}(x-z) - f_{4o}(x-y-z) - f_{5o}(x+y) - f_{6o}(x+z) \right\| \le \varphi_e(x,y,z) \end{aligned} \tag{2.114}$$

for all $x, y, z \in V \setminus \{0\}$, where $f_{ke}(x) = (f_k(x) + f_k(-x))/2$, $f_{ko}(x) = (f_k(x) - f_k(-x))/2$ for all $x \in V \setminus \{0\}$, $k = 1,2,3,4,5,6$. Since f_{ke} is an even function, f_{ko} is an odd function, and $f_k = f_{ke} + f_{ko}$, we can apply Theorems 2.2 and 2.5 to get the desired result. □

We establish the following theorem for the general case from Theorems 2.2 and 2.6.

Theorem 2.9. *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \times V \setminus \{0\} \to [0,\infty)$ be a function that satisfies conditions* (a) *and* (b′)*. If the functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy inequalities* (2.108) *for all $x, y, z \in V \setminus \{0\}$, then there exist exactly one quadratic function $Q : V \to X$ and exactly three additive functions*

$A, A_1, A_2 : V \to X$ satisfying the inequalities in Theorem 2.8 for all $x \in V \setminus \{0\}$, where $\widetilde{M}_e$, $\widetilde{M}'_e$ are as in Theorem 2.8 and

$$\widehat{M}_e(x) := \sum_{l=0}^{\infty} 2^l M_e\left(\frac{x}{2^{l+1}}\right), \qquad \widehat{M}'_e(x) := \sum_{l=0}^{\infty} 2^l M'_e\left(\frac{x}{2^{l+1}}\right),$$
$$\widehat{\varphi}'_e(x, y, z) := \sum_{l=0}^{\infty} 2^l \varphi'_e\left(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}, \frac{z}{2^{l+1}}\right) \tag{2.115}$$

for all $x \in V \setminus \{0\}$. Moreover, the function Q is given by (2.111) *and the functions A, A_1, A_2 are given by*

$$A(x) = \lim_{n\to\infty} 2^{n-2}\left(f_1\left(\frac{x}{2^n}\right) + f_4\left(\frac{x}{2^n}\right) - f_1\left(-\frac{x}{2^n}\right) - f_4\left(-\frac{x}{2^n}\right)\right), \tag{2.116}$$

$$A_1(x) = \lim_{n\to\infty} 2^{n-2}\left(f_2\left(\frac{x}{2^n}\right) - f_5\left(\frac{x}{2^n}\right) - f_2\left(-\frac{x}{2^n}\right) + f_5\left(-\frac{x}{2^n}\right)\right),$$
$$A_2(x) = \lim_{n\to\infty} 2^{n-2}\left(f_3\left(\frac{x}{2^n}\right) - f_6\left(\frac{x}{2^n}\right) - f_3\left(-\frac{x}{2^n}\right) + f_6\left(-\frac{x}{2^n}\right)\right) \tag{2.117}$$

for all $x \in V$.

We establish the following theorem for the general case from Theorems 2.3 and 2.6.

Theorem 2.10. *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ be a function that satisfies conditions* (a$'$) *and* (b$'$). *If the functions $f_1, f_2, f_3, f_4, f_5, f_6 : V \to X$ satisfy inequalities* (2.108) *for all $x, y, z \in V \setminus \{0\}$, then there exist exactly one quadratic function $Q : V \to X$ and exactly three additive functions $A, A_1, A_2 : V \to X$ satisfying the inequalities in Theorem 2.8 for all $x \in V \setminus \{0\}$, where $\widehat{M}_e$, $\widehat{M}'_e$, $\widehat{\varphi}'_e$ are as in Theorem 2.9 and*

$$\widetilde{M}_e(x) := \sum_{l=0}^{\infty} 4^l M_e\left(\frac{x}{2^{l+1}}\right), \qquad \widetilde{M}'_e(x) := \sum_{l=0}^{\infty} 4^l M'_e\left(\frac{x}{2^{l+1}}\right) \tag{2.118}$$

for all $x \in V \setminus \{0\}$. Moreover, the function Q is given by

$$Q(x) = \lim_{n\to\infty} 2 \cdot 4^{n-1}\left(f_k(2^{-n}x) + f_k(-2^{-n}x) - 2f_k(0)\right) \tag{2.119}$$

for $i = 1, 2, 3, 4, 5, 6$ and the functions A, A_1, A_2 are given by (2.116) *for all $x \in V$.*

Corollary 2.11. *Let $p \neq 1, 2$ and $\varepsilon > 0$. Suppose that the functions $f_i : V \to X$, $i = 1, 2, \ldots, 6$, satisfy*

$$\begin{aligned} &\left\| f_1(x+y+z) + f_2(x-y) + f_3(x-z) - f_4(x-y-z) - f_5(x+y) - f_6(x+z) \right\| \\ &\quad \leq \varepsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \end{aligned} \tag{2.120}$$

for all $x, y, z \in V \setminus \{0\}$.

Then there exist exactly one quadratic function $Q : V \to X$ and three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$\begin{aligned}
&\left\| f_1(x) - f_1(0) - Q(x) - A(x) + A_1(x) + A_2(x) \right\| \\
&\quad \leq \left[\frac{(3^p + 11)(2^p + 2)}{2 \cdot 2^p |2^p - 4|} + \frac{11 + 3^p}{2 \cdot 2^p} + 1 + \frac{8}{4^p} + \frac{2(3^p + 11) + 4 \cdot 2^p + 2 \cdot 4^p}{4^p |2^p - 2|} \right] \cdot \varepsilon \cdot \|x\|^p, \\
&\left\| f_2(x) - f_2(0) - Q(x) - A(x) - A_1(x) \right\| \leq \left[\frac{(3^p + 11)}{2^p |2^p - 4|} + \frac{3^p + 5}{4^p} + \frac{2(3^p + 8)}{2^p |2^p - 2|} \right] \cdot \varepsilon \cdot \|x\|^p, \\
&\left\| f_3(x) - f_3(0) - Q(x) - A(x) - A_2(x) \right\| \leq \left[\frac{(3^p + 11)}{2^p |2^p - 4|} + \frac{3^p + 5}{4^p} + \frac{2(3^p + 8)}{2^p |2^p - 2|} \right] \cdot \varepsilon \cdot \|x\|^p, \\
&\left\| f_4(x) - f_4(0) - Q(x) - A(x) - A_1(x) - A_2(x) \right\| \\
&\quad \leq \left[\frac{(3^p + 11)(2^p + 2)}{2 \cdot 2^p |2^p - 4|} + \frac{11 + 3^p}{2 \cdot 2^p} + 1 + \frac{8}{4^p} + \frac{2(3^p + 11) + 4 \cdot 2^p + 2 \cdot 4^p}{4^p |2^p - 2|} \right] \cdot \varepsilon \cdot \|x\|^p \\
&\left\| f_5(x) - f_5(0) - Q(x) - A(x) + A_1(x) \right\| \leq \left[\frac{(3^p + 11)}{2^p |2^p - 4|} + \frac{3^p + 5}{4^p} + \frac{2(3^p + 8)}{2^p |2^p - 2|} \right] \cdot \varepsilon \cdot \|x\|^p, \\
&\left\| f_6(x) - f_6(0) - Q(x) - A(x) + A_2(x) \right\| \leq \left[\frac{(3^p + 11)}{2^p |2^p - 4|} + \frac{3^p + 5}{4^p} + \frac{2(3^p + 8)}{2^p |2^p - 2|} \right] \cdot \varepsilon \cdot \|x\|^p
\end{aligned} \tag{2.121}$$

for all $x \in V \setminus \{0\}$. Moreover, the function Q is given by (2.111) *for $p < 2$ and* (2.119) *for $p > 2$ and the functions A, A_1, A_2 ($k = 1, 2, 3$) are given by* (2.112) *for $p < 1$ and* (2.116) *for $p > 1$.*

Corollary 2.12. *Let $\varepsilon > 0$ be a fixed real number. Suppose that the functions $f_i : V \to X$, $i = 1, 2, \ldots, 6$, satisfy*

$$\left\| f_1(x + y + z) + f_2(x - y) + f_3(x - z) - f_4(x - y - z) - f_5(x + y) - f_6(x + z) \right\| \leq \varepsilon \tag{2.122}$$

for all $x, y, z \in V \setminus \{0\}$.

Then there exist exactly one quadratic function $Q : V \to X$ and three additive functions $A, A_1, A_2 : V \to X$ satisfying

$$\begin{aligned}
&\left\| f_1(x) - f_1(0) - Q(x) - A(x) + A_1(x) + A_2(x) \right\| \leq 17\varepsilon, \\
&\left\| f_2(x) - f_2(0) - Q(x) - A(x) - A_1(x) \right\| \leq \frac{28}{3}\varepsilon, \\
&\left\| f_3(x) - f_3(0) - Q(x) - A(x) - A_2(x) \right\| \leq \frac{28}{3}\varepsilon, \\
&\left\| f_4(x) - f_4(0) - Q(x) - A(x) - A_1(x) - A_2(x) \right\| \leq 17\varepsilon, \\
&\left\| f_5(x) - f_5(0) - Q(x) - A(x) + A_1(x) \right\| \leq \frac{28}{3}\varepsilon, \\
&\left\| f_6(x) - f_6(0) - Q(x) - A(x) + A_2(x) \right\| \frac{28}{3}\varepsilon
\end{aligned} \tag{2.123}$$

for all $x \in V \setminus \{0\}$. *Moreover, the function* Q *is given by* (2.111) *for* $i = 1,2,3,4,5,6$ *and the functions* $A,\ A_1,\ A_2$ *are given by* (2.112) *for all* $x \in V$.

Now we obtain the general solution of (1.6) from Corollary 2.12.

Corollary 2.13. *Suppose that the functions* $f_i : V \to X$, $i = 1,2,\ldots,6$, *satisfy*

$$f_1(x+y+z) + f_2(x-y) + f_3(x-z) - f_4(x-y-z) - f_5(x+y) - f_6(x+z) = 0 \tag{2.124}$$

for all $x,y,z \in V \setminus \{0\}$.

Then there exist exactly one quadratic function $Q : V \to X$ *and three additive functions* $A, A_1, A_2 : V \to X$ *satisfying*

$$\begin{aligned}
f_1(x) &= Q(x) + A(x) - A_1(x) - A_2(x) + f_1(0),\\
f_2(x) &= Q(x) + A(x) + A_1(x) + f_2(0),\\
f_3(x) &= Q(x) + A(x) + A_2(x) + f_3(0),\\
f_4(x) &= Q(x) + A(x) + A_1(x) + A_2(x) + f_4(0),\\
f_5(x) &= Q(x) + A(x) - A_1(x) + f_5(0),\\
f_6(x) &= Q(x) + A(x) - A_2(x) + f_6(0)
\end{aligned} \tag{2.125}$$

for all $x \in V$. *Moreover, the function* Q *is given by*

$$Q(x) = \frac{f_i(x) + f_i(-x)}{2} - f_i(0) \tag{2.126}$$

for $i = 1,2,3,4,5,6$ *and the functions* $A,\ A_1,\ A_2$ $(k = 1,2,3)$ *are given by*

$$\begin{aligned}
A(x) &= \frac{f_1(x) + f_4(x) - f_1(-x) - f_4(-x)}{4},\\
A_1(x) &= \frac{f_2(x) - f_5(x) - f_2(-x) + f_5(-x)}{4},\\
A_2(x) &= \frac{f_3(x) - f_6(x) - f_3(-x) + f_6(-x)}{4}
\end{aligned} \tag{2.127}$$

for all $x \in V$.

References

[1] S. M. Ulam, *Problems in Modern Mathematics*, chapter VI, John Wiley & Sons, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.

[3] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[4] Th. M. Rassias, "Problem 16; 2, Report of the 27th International Symposium on Functional Equations," *Aequationes Mathematicae*, vol. 39, no. 2-3, pp. 292–293, 309, 1990.

[5] Z. Gajda, “On stability of additive mappings,” *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[6] P. Găvruţa, “A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[7] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1998.

[8] Th. M. Rassias, “On the stability of functional equations in Banach spaces,” *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.

[9] Th. M. Rassias, “On the stability of functional equations and a problem of Ulam,” *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.

[10] Th. M. Rassias and P. Šemrl, “On the behavior of mappings which do not satisfy Hyers-Ulam stability,” *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.

[11] K.-W. Jun and Y.-H. Lee, “On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic equation. II,” in *Functional Equations, Inequalities and Applications*, pp. 39–65, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[12] Y.-H. Lee and K.-W. Jun, “A generalization of the Hyers-Ulam-Rassias stability of the Pexider equation,” *Journal of Mathematical Analysis and Applications*, vol. 246, no. 2, pp. 627–638, 2000.

[13] K.-W. Jun, D.-S. Shin, and B.-D. Kim, “On Hyers-Ulam-Rassias stability of the Pexider equation,” *Journal of Mathematical Analysis and Applications*, vol. 239, no. 1, pp. 20–29, 1999.

[14] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1989.

[15] F. Skof, “Local properties and approximation of operators,” *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.

[16] P. W. Cholewa, “Remarks on the stability of functional equations,” *Aequationes Mathematicae*, vol. 27, no. 1, pp. 76–86, 1984.

[17] S. Czerwik, “On the stability of the quadratic mapping in normed spaces,” *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.

[18] J. Wang, “Some further generalizations of the Hyers-Ulam-Rassias stability of functional equations,” *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 406–423, 2001.

[19] K.-W. Jun, J.-H. Bae, and Y.-H. Lee, “On the Hyers-Ulam-Rassias stability of an n-dimensional Pexiderized quadratic equation,” *Mathematical Inequalities* & *Applications*, vol. 7, no. 1, pp. 63–77, 2004.

[20] K.-W. Jun and H.-M. Kim, “On the stability of an n-dimensional quadratic and additive functional equation,” *Mathematical Inequalities* & *Applications*, vol. 9, no. 1, pp. 153–165, 2006.

[21] K.-W. Jun and Y.-H. Lee, “On the Hyers-Ulam-Rassias stability of a generalized quadratic equation,” *Bulletin of the Korean Mathematical Society*, vol. 38, no. 2, pp. 261–272, 2001.

[22] K.-W. Jun and Y.-H. Lee, “On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic inequality,” *Mathematical Inequalities* & *Applications*, vol. 4, no. 1, pp. 93–118, 2001.