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*Research Article*

# New LMI-Based Conditions for Quadratic Stabilization of LPV Systems

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This paper is concerned with quadratic stabilization problem of linear parameter varying (LPV) systems, where arbitrary time-varying dependent parameters are belonging to a polytope. It provides improved linear matrix inequality- (LMI-) based conditions to compute a gain-scheduling state-feedback gain that makes closed-loop system quadratically stable. The proposed conditions, based on the philosophy of Pólya's theorem, are written as a sequence of progressively less and less conservative LMI. More importantly, by adding an additional decision variable, at each step, these new conditions provide less conservative or at least the same results than previous methods in the literature.



## 1. Introduction

Linear parameter varying (LPV) systems are formalized as a certain type of nonlinear systems, and a control strategy has been developed for these systems based on classical gain-scheduled adaptive methodology [1, 2]. There are many examples of dependent physical parameters including inertia, stiffness, or viscosity coefficients in mechanical systems, aero dynamical coefficients in flight control, resistor and capacitor values in electrical circuits and so forth. Therefore, it is often desirable to obtain guarantees of stability and performance against dependent parameter when analyzing these control systems. In the past decade, main papers and special publications concerning LPV controller design problem have appeared in [3–10].

Quadratic stability has been widely used to assess closed-loop stability and performance. This approach allows us to describe several problems of stability analysis and synthesis as LMI optimization problems, which can be solved in polynomial time by interior point algorithms [3, 11–15]. It is known that the results based on quadratic stability are frequently conservative in the context of analysis and synthesis for LPV systems when compared to the results from conditions based on parameter-dependent Lyapunov functions (see, e.g., [3, 6–9, 16–18]). But quadratic stability remains attractive due to its low numerical complexity, being largely employed as a first step in the investigation of stability performance and control design of LPV systems.

As to quadratic stabilization studies of LPV systems, even though stability analysis problem is NP-hard in general, a number of more or less conservative analysis methods are presented to assess quadratic stability [3–8], where a fixed quadratic Lyapunov function is found to prove stability of LPV systems. More recently, a kind of necessary and sufficient LMI-based condition has been proposed to compute a quadratically stabilizing state feedback controller for continuous-time linear systems with arbitrary time-varying parameters belonging to a polytope [19]. These conditions are based on an extension of Pólya's theorem [20] and are written as a sequence of progressively less and less conservative LMI. However, at each step, the LMI-based conditions are still sufficient, and have some conservatism. It leads to higher computational times.

The main contribution of this paper is to provide new necessary and sufficient LMI-based conditions to compute a quadratically stabilizing gain-scheduling state feedback for LPV systems. The proposed conditions are based on the systematic construction of homogeneous polynomial solution for parameter-dependent LMI too. At each step, a set of LMIs provides sufficient conditions for the existence of such a gain-scheduling state feedback. Necessity is asymptotically attained through a relaxation based on the philosophy of Pólya's theorem. More importantly, by adding an additional decision variable, at each step, these new conditions can provide less conservative or at least the same results than the most recent existing methods in the literature. Consequently, the feasible solutions can be obtained in much lower steps.

## 2. Preliminary

Consider an LPV system $P(\partial(t))$ described by state space equations as

$$\dot{x}(t) = A\big(\partial(t)\big)x(t) + B\big(\partial(t)\big)u(t). \tag{2.1}$$

Here, state-space matrices have compatible dimensions of time-varying parameters $\partial(t) = \begin{bmatrix}\partial_1(t) & \partial_2(t) & \cdots & \partial_n(t)\end{bmatrix}^T \in \Re^n$.

Moreover, we have the following assumptions.

(1) The state-space matrices $A(\partial(t))$ and $B(\partial(t))$ are continuous and bounded functions and depend affinely on $\partial(t)$.

(2) The real parameters $\partial(t)$, that can be known in advance or online measurement values, exist in LPV plant and vary in a polytope $\Theta$ as

$$\partial(t) \in \Theta := \mathrm{Co}\{\omega_1, \omega_2, \ldots, \omega_N\} = \left\{\sum_{i=1}^{r} \alpha_i(t)\omega_i : \alpha_i(t) \geq 0,\ \sum_{i=1}^{r} \alpha_i(t) = 1,\ r = 2^n\right\}. \tag{2.2}$$

With above assumptions, the LPV plant is called polytopic, when it ranges in a matrix polytope, LPV system $P(\partial(t))$ can be expressed as

$$A(\partial) = \sum_{i=1}^{r} \alpha_i(t) A(\omega_i), \quad B(\partial) = \sum_{i=1}^{r} \alpha_i(t) B(\omega_i) \quad \text{with } \alpha_i \geq 0, \ \sum_{i=1}^{r} \alpha_i = 1. \tag{2.3}$$

The aim of this paper is to establish new LMI-based conditions of a gain-scheduling state feedback that quadratically stabilizes the class of system (2.1). The control law is given with a state feedback as

$$u(t) = -K(\partial)x(t), \quad K(\partial) = \sum_{j=1}^{r} \alpha_i(t) K(\omega_j). \tag{2.4}$$

Substituting (2.4) into (2.1), the closed-loop system can be written as

$$\dot{x}(t) = A_{\text{cl}}(\partial)x(t), \quad \partial \in \Theta, \tag{2.5}$$

where $A_{\text{cl}}(\partial) = A(\partial) - B(\partial)K(\partial)$.

According to quadratic stability theory [3], the closed-loop system (2.5) is said to be quadratically stable if and only if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$PA_{\text{cl}}(\partial) + A_{\text{cl}}^{T}(\partial)P < 0, \quad \partial \in \Theta. \tag{2.6}$$

The concept of quadratic stability has been widely used for stability evaluation, control, and filter design for continuous and discrete, time-varying and time-invariant systems. The next lemma presents convex LMI conditions of infinite dimension that are necessary and sufficient to assure the existence of such a state-feedback gain.

**Lemma 2.1.** *LPV system* (2.1) *is quadratically stabilizable if and only if there exist a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and a parameter-dependent matrix $N(\partial) \in \mathbb{R}^{m\times n}$ such that*

$$A(\partial)Q + QA^{T}(\partial) - B(\partial)N(\partial) - N^{T}(\partial)B^{T}(\partial) < 0_n, \quad \partial(t) \in \Theta. \tag{2.7}$$

*In this case, the state-feedback gain is given by*

$$K(\partial) = N(\partial)Q^{-1}. \tag{2.8}$$

*Proof.* Using the change of variables $N(\partial) = K(\partial)Q$, (2.7) can be rewritten as

$$(A(\partial) - B(\partial)K(\partial))Q + Q(A(\partial) - B(\partial)K(\partial))^T < 0, \tag{2.9}$$

which, pre- and post-multiplied by $Q^{-1}$, yields (2.6) with $P = Q^{-1}$. Conversely, pre- and post-multiplying (2.6) by $P^{-1}$ and making $Q = P^{-1}$ give the equivalence condition

$$(A(\partial) - B(\partial)K(\partial))Q + Q(A(\partial) - B(\partial)K(\partial))^T < 0, \tag{2.10}$$

which yields (2.7) by making $K(\partial) = N(\partial)Q^{-1}$.

To simplify notation, we define

$$\begin{aligned} G_{ij} &= QA_i^T + A_iQ - N_j^TB_i^T - B_iN_j, \quad i,j = 1,2,\dots,r, \\ K_i &= N_iQ^{-1}, \quad i = 1,2,\dots,r \quad \text{with } A_i = A(\omega_i), \quad N_j = N(\omega_j). \end{aligned} \tag{2.11}$$

□

## 3. Main result

In this section, by adding an additional decision variable, a useful lemma is introduced below.

**Lemma 3.1.** *LPV system* (2.1) *is quadratically stabilizable if and only if there exist a symmetric positive definite matrix* $W \in \mathbb{R}^{n\times n}$ *and parameter-dependent matrices* $N(\partial) \in \mathbb{R}^{m\times n}$, $Y(\partial) = Y^T(\partial) \in \mathbb{R}^{n\times n}$ *such that one of the following equivalent conditions holds*

$$\text{(i)} \quad \varphi(\partial) = A(\partial)Q + QA^T(\partial) - B(\partial)N(\partial) - N^T(\partial)B^T(\partial) < Y(\partial) \le 0_n, \tag{3.1}$$

$$\begin{aligned} \text{(ii)} \quad \varphi_d(\partial) &= (\alpha_1 + \alpha_2 + \cdots + \alpha_r)^d (A(\partial)Q + QA^T(\partial) - B(\partial)N(\partial) - N^T(\partial)B^T(\partial)) \\ &< (\alpha_1 + \alpha_2 + \cdots + \alpha_r)^d Y(\partial) \le 0_n, \quad \forall d \in \mathbb{Z}_+. \end{aligned} \tag{3.2}$$

*Proof.* Condition (i) is obtained directly through the use of a quadratic Lyapunov function associated to the closed-loop system (2.5). For any fixed $\partial(t) \in \Theta$ and for all $d \in \mathbb{Z}_+$, the equivalence between (i) and (ii) is immediate since $\partial(t) \in \Theta$ implies $(\sum_{i=1}^r \alpha_i)^d = 1$ for all $d \in \mathbb{Z}_+$. □

*Remark 3.2.* When the parameter-dependent matrix $Y(\partial)$ is assumed to be zero, the condition (i) is reduced to the most recent existing conditions [19]. The existing conditions are written as a sequence of progressively less and less conservative LMI. With the increase of this positive integer $d$, necessity is asymptotically attained. However, at each step, the LMI-based conditions are still sufficient and have some conservatism. It leads to higher computational times. Here, an additional decision variable $Y(\partial)$ is introduced to decrease the conservatism at each step. It provides more design freedom to get a feasible solution. In the following, we only give the LMI-based conditions with $d = 1,\ d = 2$.

**Theorem 3.3** ($d = 1$). *LPV system* (2.1) *is quadratically stabilizable via the gain-scheduling controller* (2.4) *if there exist matrices $Q > 0$, $N_i$, $i = 1, 2, \ldots, r$ and $Y_{ijl} = Y_{lji}^T$, $i, j, l = 1, 2, \ldots, r$, satisfying*

$$\begin{gathered} G_{ii} < Y_{iii}, \quad i = 1, 2, \ldots, r, \\ G_{ii} + G_{ij} + G_{ji} < Y_{iij} + Y_{iji} + Y_{iij}^T, \quad i = 1, 2, \ldots, r,\ j \neq i,\ j = 1, 2, \ldots, r, \\ G_{ij} + G_{il} + G_{ji} + G_{jl} + G_{li} + G_{lj} < Y_{ijl} + Y_{ilj} + Y_{jil} + Y_{ijl}^T + Y_{ilj}^T + Y_{jil}^T, \\ i = 1, 2, \ldots, r-2, \quad j = i+1, \ldots, r-1, \quad j = j+1, \ldots, r, \\ \begin{bmatrix} Y_{1i1} & Y_{1i2} & \cdots & Y_{1ir} \\ Y_{2i1} & Y_{2i2} & \cdots & Y_{2ir} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{ri1} & Y_{ri2} & \cdots & Y_{rir} \end{bmatrix} \leq 0, \quad i = 1, 2, \ldots, r. \end{gathered} \tag{3.3}$$

*Moreover, in this case, local state-feedback gains are $K_j = N_j Q^{-1}$, $j = 1, 2, \ldots, r$.*

*Remark 3.4.* The details concerning these LMI-based results above can be referred to [21, Theorem 5] for Takagi-Sugeno fuzzy systems. New proposed LMI-based conditions are presented according to condition (ii) of Lemma 3.1 in the case of $d = 2$. Meanwhile, a simple proof is also given.

**Theorem 3.5** ($d = 2$). *LPV system* (2.1) *is quadratically stabilizable via the gain-scheduling controller* (2.4)*, if there exist matrices $Q > 0$; $N_i$, $i = 1, 2, \ldots, r$; $Y_{ijmn} = Y_{njmi}^T$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, r$, $m = 1, 2, \ldots, r$, $n = 1, 2, \ldots, r$ satisfying*

$$G_{ii} < Y_{iiii}, \quad i = 1, 2, \ldots, r,\ i \neq j, \tag{3.4}$$

$$2G_{ii} + G_{ij} + G_{ji} < Y_{iiij} + Y_{iiij}^T + Y_{iiji} + Y_{ijii}, \quad i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, r,\ i \neq j, \tag{3.5}$$

$$G_{ii} + G_{ij} + G_{ji} < Y_{iijj} + Y_{iijj}^T + Y_{jiij}, \quad i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, r,\ i \neq j, \tag{3.6}$$

$$\begin{aligned} &2G_{ii} + 2G_{ij} + 2G_{im} + 2G_{ji} + 2G_{jm} + G_{mi} + G_{mj} \\ &\quad < Y_{iijm} + Y_{ijim} + Y_{ijmi} + Y_{miji} + Y_{mjii} + Y_{miij} \\ &\quad\quad + Y_{iijm}^T + Y_{ijim}^T + Y_{imji} + Y_{miji}^T + Y_{mjii}^T + Y_{miij}^T, \end{aligned} \tag{3.7}$$

$$i = 1, 2, \ldots, r-3, \quad j = 1, 2, \ldots, r-2, \quad m = 1, 2, \ldots, r-1,$$

$$2\left(G_{ij}+G_{im}+G_{in}+G_{ji}+G_{jm}+G_{jn}+G_{mi}+G_{mj}+G_{mn}+G_{ni}+G_{nj}+G_{nm}\right)$$
$$<\begin{pmatrix} Y_{ijmn}+Y_{ijnm}+Y_{imjn}+Y_{imnj}+Y_{injm}+Y_{inmj} \\ +Y_{jinm}+Y_{jimn}+Y_{jmin}+Y_{jmni}+Y_{jnmi}+Y_{jnim} \\ Y_{ijmn}^T+Y_{ijnm}^T+Y_{imjn}^T+Y_{imnj}^T+Y_{injm}^T+Y_{inmj}^T \\ +Y_{jinm}^T+Y_{jimn}^T+Y_{jmin}^T+Y_{jmni}^T+Y_{jnmi}^T+Y_{jnim}^T \end{pmatrix}, \tag{3.8}$$
$$i=1,2,\ldots,r-3,\quad j=1,2,\ldots,r-2,\quad m=1,2,\ldots,r-1,\quad n=1,2,\ldots,r,$$

$$\begin{bmatrix} Y_{1ij1} & Y_{1ij2} & \cdots & Y_{1ijr} \\ Y_{2ij1} & Y_{2ij2} & \cdots & Y_{2ijr} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{rij1} & Y_{rij2} & \cdots & Y_{rijr} \end{bmatrix} \leq 0,\quad i=1,2,\ldots,r,\ j=1,2,\ldots,r. \tag{3.9}$$

*In this case, if the conditions above are feasible, local state feedback gains are* $K_j = N_j Q^{-1}$, $j = 1,2,\ldots,r$.

*Proof.* Consider a candidate of quadratic function $V(x(t)) = x^T(t)P^{-1}x(t)$. The equilibrium of (2.5) is quadratically stable if

$$\dot{V}(x(t)) = x^T(t)\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\left(QA_i^T + A_iQ - N_j^TB_i^T - B_iN_j\right)\right\}x(t) < 0 \quad \forall x(t)\neq 0. \tag{3.10}$$

From inequality (3.10) above, the equilibrium of (2.5) is quadratically stable if

$$\begin{aligned}
&\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\left(QA_i^T + A_iQ - N_j^TB_i^T - B_iN_j\right)\\
&\quad= \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_jG_{ij} = \left(\sum_{i=1}^{r}\alpha_i\right)^2\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_jG_{ij} = \left(\alpha_1+\alpha_2+\cdots+\alpha_r\right)^2\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_jG_{ij}\\
&\quad= \sum_{i=1}^{r}\alpha_i^4G_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{r}\alpha_i^3\alpha_j\left(2G_{ii}+G_{ij}+G_{ji}\right) + \sum_{\substack{i,j=1\\ i\neq j}}^{r}\alpha_i^2\alpha_j^2\left(G_{ii}+G_{ij}+G_{ji}\right)\\
&\qquad+ \sum_{i=1}^{r-2}\sum_{j=i+1}^{r-1}\sum_{m=j+1}^{r}\alpha_i^2\alpha_j\alpha_m\left(2G_{ii}+2G_{ij}+2G_{im}+2G_{ji}+2G_{jm}+G_{mi}+G_{mj}\right)\\
&\qquad+ \sum_{i=1}^{r-3}\sum_{j=i+1}^{r-2}\sum_{m=j+1}^{r-1}\sum_{n=m+1}^{r}\alpha_i\alpha_j\alpha_m\alpha_n * 2\begin{pmatrix} G_{ij}+G_{im}+G_{in}+G_{ji}+G_{jm}+G_{jn} \\ +G_{mi}+G_{mj}+G_{mn}+G_{ni}+G_{nj}+G_{nm}\end{pmatrix}\\
&\quad\triangleq \Delta
\end{aligned}$$

$$\nabla < \sum_{i=1}^{r} \alpha_i^4 \Upsilon_{iiii} + \sum_{\substack{i,j=1 \\ i\neq j}}^{r} \alpha_i^3 \alpha_j \left(\Upsilon_{iiij} + \Upsilon_{iiji} + \Upsilon_{ijii} + \Upsilon_{jiii}\right) + \sum_{\substack{i,j=1 \\ i\neq j}}^{r} \alpha_i^2 \alpha_j^2 \left(\Upsilon_{iijj} + \Upsilon_{jiji} + \Upsilon_{jiij}\right)$$

$$+ \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} \sum_{m=j+1}^{r} \alpha_i^2 \alpha_j \alpha_m \begin{pmatrix} \Upsilon_{iijm} + \Upsilon_{iimj} + \Upsilon_{imij} + \Upsilon_{ijim} + \Upsilon_{ijmi} + \Upsilon_{imji} \\ +\Upsilon_{miji} + \Upsilon_{jimi} + \Upsilon_{mjii} + \Upsilon_{jmii} + \Upsilon_{miij} + \Upsilon_{jiim} \end{pmatrix}$$

$$+ \sum_{i=1}^{r-3} \sum_{j=i+1}^{r-2} \sum_{m=j+1}^{r-1} \sum_{n=m+1}^{r} \alpha_i \alpha_j \alpha_m \alpha_n \begin{pmatrix} \Upsilon_{ijmn} + \Upsilon_{ijnm} + \Upsilon_{imjn} + \Upsilon_{imnj} + \Upsilon_{injm} + \Upsilon_{inmj} \\ +\Upsilon_{jinm} + \Upsilon_{jimn} + \Upsilon_{jmin} + \Upsilon_{jmni} + \Upsilon_{jnmi} + \Upsilon_{jnim} \\ \Upsilon_{ijmn}^T + \Upsilon_{ijnm}^T + \Upsilon_{imjn}^T + \Upsilon_{imnj}^T + \Upsilon_{injm}^T + \Upsilon_{inmj}^T \\ +\Upsilon_{jinm}^T + \Upsilon_{jimn}^T + \Upsilon_{jmin}^T + \Upsilon_{jmni}^T + \Upsilon_{jnmi}^T + \Upsilon_{jnim}^T \end{pmatrix}$$

$$= \alpha_1 \left( \alpha_1 \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \begin{bmatrix} \Upsilon_{1111} & \Upsilon_{1112} & \cdots & \Upsilon_{111r} \\ \Upsilon_{2111} & \Upsilon_{2112} & \cdots & \Upsilon_{211r} \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{r111} & \Upsilon_{r112} & \cdots & \Upsilon_{ri11r} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix} + \alpha_2 \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \begin{bmatrix} \Upsilon_{1121} & \Upsilon_{1122} & \cdots & \Upsilon_{112r} \\ \Upsilon_{2121} & \Upsilon_{2122} & \cdots & \Upsilon_{212r} \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{r121} & \Upsilon_{r122} & \cdots & \Upsilon_{r12r} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix} \right.$$

$$\left. + \cdots + \alpha_r \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \begin{bmatrix} \Upsilon_{11r1} & \Upsilon_{11r2} & \cdots & \Upsilon_{11rr} \\ \Upsilon_{21r1} & \Upsilon_{21r2} & \cdots & \Upsilon_{21rr} \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{r1r1} & \Upsilon_{r1r2} & \cdots & \Upsilon_{r1rr} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix} \right)$$

$$+\alpha_2 \left( \alpha_1 \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \begin{bmatrix} \Upsilon_{1211} & \Upsilon_{1212} & \cdots & \Upsilon_{121r} \\ \Upsilon_{2211} & \Upsilon_{2212} & \cdots & \Upsilon_{221r} \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{r211} & \Upsilon_{r212} & \cdots & \Upsilon_{r21r} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix} + \partial_2 \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \begin{bmatrix} \Upsilon_{1221} & \Upsilon_{1222} & \cdots & \Upsilon_{122r} \\ \Upsilon_{2221} & \Upsilon_{2222} & \cdots & \Upsilon_{222r} \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{r221} & \Upsilon_{r222} & \cdots & \Upsilon_{r22r} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix} \right.$$

$$\left. + \cdots + \alpha_r \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \begin{bmatrix} \Upsilon_{12r1} & \Upsilon_{12r2} & \cdots & \Upsilon_{12rr} \\ \Upsilon_{22r1} & \Upsilon_{22r2} & \cdots & \Upsilon_{22rr} \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{r2r1} & \Upsilon_{r2r2} & \cdots & \Upsilon_{r2rr} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_r I \end{bmatrix} \right)$$

$$+\cdots+\alpha_r\left(\alpha_1\begin{bmatrix}\alpha_1 I\\ \alpha_2 I\\ \vdots\\ \alpha_r I\end{bmatrix}^T\begin{bmatrix}Y_{1r11} & Y_{1r12} & \cdots & Y_{1r1r}\\ Y_{2r11} & Y_{2r12} & \cdots & Y_{2r1r}\\ \vdots & \vdots & \ddots & \vdots\\ Y_{rr11} & Y_{rr12} & \cdots & Y_{rr1r}\end{bmatrix}\begin{bmatrix}\alpha_1 I\\ \alpha_2 I\\ \vdots\\ \alpha_r I\end{bmatrix}+\alpha_2\begin{bmatrix}\alpha_1 I\\ \alpha_2 I\\ \vdots\\ \alpha_r I\end{bmatrix}^T\begin{bmatrix}Y_{1r21} & Y_{1r22} & \cdots & Y_{1r2r}\\ Y_{2r21} & Y_{2r22} & \cdots & Y_{2r2r}\\ \vdots & \vdots & \ddots & \vdots\\ Y_{rr21} & Y_{rr22} & \cdots & Y_{rr2r}\end{bmatrix}\begin{bmatrix}\alpha_1 I\\ \alpha_2 I\\ \vdots\\ \alpha_r I\end{bmatrix}\right.$$

$$\left.+\cdots+\alpha_r\begin{bmatrix}\alpha_1 I\\ \alpha_2 I\\ \vdots\\ \alpha_r I\end{bmatrix}^T\begin{bmatrix}Y_{1rr1} & Y_{1rr2} & \cdots & Y_{1rrr}\\ Y_{2rr1} & Y_{2rr2} & \cdots & Y_{2rrr}\\ \vdots & \vdots & \ddots & \vdots\\ Y_{rrr1} & Y_{rrr2} & \cdots & Y_{rrrr}\end{bmatrix}\begin{bmatrix}\alpha_1 I\\ \alpha_2 I\\ \vdots\\ \alpha_r I\end{bmatrix}\right)$$

$$=\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\left(\begin{bmatrix}\alpha_1 I\\ \alpha_2 I\\ \vdots\\ \alpha_r I\end{bmatrix}^T\begin{bmatrix}Y_{1ij1} & Y_{1ij2} & \cdots & Y_{1ijr}\\ Y_{2ij1} & Y_{2ij2} & \cdots & Y_{2ijr}\\ \vdots & \vdots & \ddots & \vdots\\ Y_{rij1} & Y_{rij2} & \cdots & Y_{rijr}\end{bmatrix}\begin{bmatrix}\alpha_1 I\\ \alpha_2 I\\ \vdots\\ \alpha F_r I\end{bmatrix}\right). \tag{3.11}$$

Thus, if (3.9) holds $\nabla < 0$. In other words, the LPV system (2.1) is quadratically stabilizable via the gain-scheduled controller (2.4). □

*Remark 3.6.* The relationship between Theorems 3.3 and 3.5 is discussed here. One can find that in the case of $j = i$, the conditions suggested herein reduce to the conditions of Theorem 3.3. That is, Theorem 3.3 is a special case of Theorem 3.5 here. Consequently, with the increase of this positive integer $d$, the LMI-based conditions will provide more additional slack matrix variables which bring us more design freedom. Although the numerical complexity is increased much, a sequence of LMI-based conditions which are less and less conservative can be obtained. In Section 4, two simple numerical examples will be illustrated to compare the proposed conditions with the most recent existing conditions, where the additional decision variable is not added.

## 4. Numerical example

To illustrate the effectiveness of the proposals, two simple numerical examples are given here. All of LMIs-based conditions are solved by Matlab LMI toolbox [22].

*Example 4.1.* Consider state-space expressions of two vertexes of an LPV plant as follows:

$$A_1=\begin{bmatrix}1.59 & -7.29\\ 0.01 & 0\end{bmatrix},\qquad B_1=\begin{bmatrix}1\\ 0\end{bmatrix},$$
$$A_2=\begin{bmatrix}0.02 & -4.64\\ 0.35 & 0.21\end{bmatrix},\qquad B_2=\begin{bmatrix}8\\ 0\end{bmatrix}, \tag{4.1}$$
$$A_3=\begin{bmatrix}-9 & -4.33\\ 0 & 0.05\end{bmatrix},\qquad B_3=\begin{bmatrix}3\\ -1\end{bmatrix}.$$

The problem is considered to seek an LPV state feedback controller as (2.4) such that closed-loop system is quadratically stable. The case of $d = 2$ is considered with and without the additional decision variable, respectively.

(i) Without the additional decision variable [19], after 21 iterations, these LMI-based conditions are not feasible. Therefore, an LPV state feedback controller cannot be found. Since the existing methods [19] provide necessary and sufficient conditions for such a state-feedback, there could exist a feasible solution in the case of $d > 2$.

(ii) With the additional decision variable, according to (3.4)–(3.9), after 19 iterations, these LMI-based conditions can be solvable with

$$Q = \begin{bmatrix} 1.037 & -0.122 \\ -0.122 & 0.029 \end{bmatrix},$$
$$N_1 = \begin{bmatrix} -3.466 & 0.351 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} -1.285 & -0.1333 \end{bmatrix}, \qquad N_3 = \begin{bmatrix} 0.8826 & 0.2923 \end{bmatrix}. \tag{4.2}$$

Therefore, the vertex matrices of state feedback are given as

$$\begin{aligned} K_1 &= N_1^{-1}Q = \begin{bmatrix} -3.789 & -3.788 \end{bmatrix}, \\ K_2 &= N_2^{-1}Q = \begin{bmatrix} -3.451 & -18.758 \end{bmatrix}, \\ K_3 &= N_3^{-1}Q = \begin{bmatrix} 3.929 & 26.099 \end{bmatrix}. \end{aligned} \tag{4.3}$$

*Example 4.2.* To illustrate the proposed approach, consider the problem of balancing an inverted pendulum on a cart. The equations of motion for the pendulum are as follows [23]:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{g\sin(x_1) - am/x_2^2 \sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)}, \end{aligned} \tag{4.4}$$

where $x_1$ denotes the angle (in radians) of the pendulum from the vertical, and $x_2$ is the angular velocity. $g = 9.8\,\mathrm{m/s^2}$ is the gravity constant, $m$ is the mass of the pendulum, $M$ is the mass of the cart, $2l$ is the length of the pendulum, and $u$ is the force applied to the cart (in Newtons): $a = 1/(m + M)$. We choose $m = 2.0\,\mathrm{kg}$, $M = 8.0\,\mathrm{kg}$, and $2l = 1.0\,\mathrm{m}$. We first represent the nonlinear system above by LPV model. Notice that when $x_1 = \pm\pi/2$, the system is uncontrollable. Hence, we approximate the system with state-space expressions of the vertex as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ \dfrac{g}{4l/3 - aml} & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ -\dfrac{a}{4l/3 - aml} \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0 & 1 \\ \dfrac{2g}{\pi(4l/3 - aml\beta^2)} & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ -\dfrac{a\beta}{4l/3 - aml\beta^2} \end{bmatrix}, \end{aligned} \tag{4.5}$$

where $\beta = \cos(88°)$.

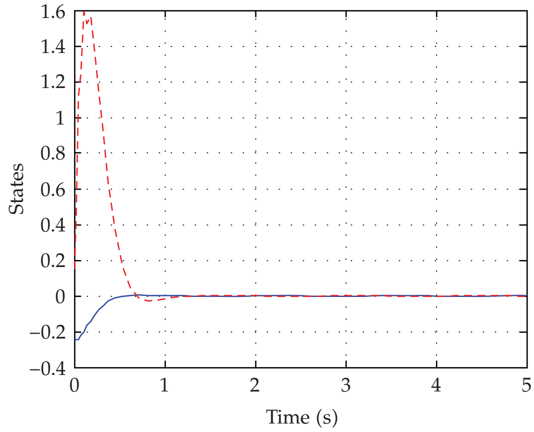


**Figure 1:** Trajectory of the states of this plant with initial values $x(0) = [-0.25 \;\; 0.15]^T$.

According to the approach proposed here, several cases will be considered with the increasing of the scalar $d$.

In the case of $d = 0$, after 39 iterations, due to the conservatism of the conditions, we cannot find a feasible solution to this state feedback.

In the case of $d \geq 1$, we can find feasible solutions to this state feedback.

When $d = 1$, after 5 iterations, we can obtain

$$W = \begin{bmatrix} 0.0302 & -0.185 \\ -0.185 & 1.569 \end{bmatrix},$$
$$Z_1 = \begin{bmatrix} -1.833 & 39.95 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} 17.776 & -80.410 \end{bmatrix}. \tag{4.6}$$

Then, the state feedback is obtained as

$$K(\partial) = \sum_{j=1}^{r} \alpha_j K_j \quad \text{with } K_1 = Z_1 W^{-1} = \begin{bmatrix} 346.9 & 66.42 \end{bmatrix}, \; K_2 = Z_2 W^{-1} = \begin{bmatrix} 996.13 & 66.40 \end{bmatrix}. \tag{4.7}$$

Here, we choose $\alpha_1(t) = \delta$, $\alpha_2(t) = (1 - \delta)$, in which $\delta =: (1.5701 - x_1(t))/3.141$. It is easy to check that the $\alpha_i(t)$ are convex coordinates, since they satisfy $0 \leq \alpha_i(t) \leq 1$, $\sum_{i=1}^{2} \alpha_i(t) = 1$.

The trajectory of the states of this plant can be drawn for the initial values $x(0) = [-0.25 \;\; 0.15]^T$ as shown in Figure 1.

From these numerical examples above, one can see that by adding an additional decision variable, at each step, these new conditions can provide less conservative or at least the same results than the most recent existing methods in the literature. Consequently, the feasible solutions can be obtained in much lower steps.

## 5. Conclusion

A sequence of new LMI-based conditions has been proposed for quadratic stabilization of LPV systems. One can find that with the increase of this positive integer $d$, a sequence of LMI-based conditions which are less and less conservative will be obtained. Here, we only present the conditions in the case of $d = 2$. By adding an additional decision variable, at each step, these new conditions relaxed the conservatism of the previous existing works. As a result, the feasible solutions can be obtained in much lower steps.

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