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Research Article

A New One-Step Iterative Process for Common Fixed Points in Banach Spaces

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We introduce a new one-step iterative process and use it to approximate the common fixed points of two asymptotically nonexpansive mappings through some weak and strong convergence theorems. Our process is computationally simpler than the processes currently being used in literature for the purpose.

1. Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers. Let E be a real Banach space, C a nonempty convex subset of E. A mapping $T : C \to C$ is called asymptotically nonexpansive if there is a sequence $\{k_n\} \subset [1,\infty)$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \quad \forall x, y \in C,\ \forall n \in \mathbb{N}, \tag{1.1}$$

where $\sum_{k=1}^{\infty}(k_n - 1) < \infty$. A point $x \in C$ is a fixed point of T, provided that $Tx = x$.

To approximate the common fixed points of two mappings, the following Ishikawa-type two-step iterative process is widely used (see, e.g., [1–9], and references cited therein):

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1 - a_n)x_n + a_n S^n y_n, \\ y_n &= (1 - b_n)x_n + b_n T^n x_n, \quad n \in \mathbb{N}, \end{aligned} \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\}$ are in $[0,1]$ satisfying certain conditions. Note that approximating fixed points of two mappings has a direct link with the minimization problem (see, e.g., [10]).

In this paper, we introduce a new one-step iterative process to compute the common fixed points of two asymptotically nonexpansive mappings. Let $S, T : C \to C$ be two asymptotically nonexpansive mappings. Then, our process reads as follows:

$$x_1 = x \in C,$$
$$x_{n+1} = a_n S^n x_n + (1 - a_n) T^n x_n, \quad n \in \mathbb{N}, \tag{1.3}$$

where $\{a_n\}$ is a sequence in $[0, 1]$.

This process is computationally simpler than (1.2) to approximate common fixed points of two mappings. It is worth noting that our process is of independent interest. Neither (1.2) implies (1.3) nor conversely. However, both (1.2) and (1.3) reduce to Mann-type iterative process when $T = I$, that is, the identity mapping is as follows:

$$x_1 = x \in C,$$
$$x_{n+1} = a_n S^n x_n + (1 - a_n) x_n, \quad n \in \mathbb{N}. \tag{1.4}$$

Remark 1.1. The question may arise that one needs two different sequences $\{s_n\}$ and $\{t_n\}$ for the mappings S and T used in (1.3), but it is readily answered when one takes $k_n = \sup\{s_n, t_n\}$. Henceforth, we will take only one sequence $\{k_n\}$ which works equally good for both mappings S and T.

Let us recall the following definitions.

A Banach space E is said to satisfy Opial's condition [11], if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \quad \forall y \in E \text{ with } y \neq x. \tag{1.5}$$

Examples of Banach spaces satisfying this condition are Hilbert spaces and all spaces l^p $(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$fails to satisfy Opial's condition.

A mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

A Banach space E is said to satisfy the Kadec Klee property if for every sequence $\{x_n\}$ in E converging weakly to x together with $\|x_n\|$ converging strongly to $\|x\|$, $\{x_n\}$converges strongly to x. Uniformly convex Banach spaces, Banach spaces of finite dimension, and reflexive locally uniform convex Banach spaces are some of the examples which satisfy the Kadec Klee property.

Next, we state the following useful lemmas.

Lemma 1.2 (see [12]). *Let $\{\delta_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that $\beta_n \geq 1$ and*

$$\delta_{n+1} \leq \beta_n \delta_n + \gamma_n \quad \forall n \in \mathbb{N}. \tag{1.6}$$

If $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} (\beta_n - 1) < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Lemma 1.3 (see [13]). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty}\|x_n\| \leq r$, $\limsup_{n\to\infty}\|y_n\| \leq r$, and $\lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then, $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 1.4 (see [14, 15]). *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let T be an asymptotically nonexpansive mapping of C into itself. Then, $(I-T)$ is demiclosed with respect to zero.*

Lemma 1.5 (see [16]). *Let C be a convex subset of a uniformly convex Banach space E. Then, there is a strictly increasing and continuous convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for every Lipschitzian map $U : C \to C$ with Lipschitz constant $L \geq 1$, the following inequality holds:*

$$\begin{aligned}&\|U(tx+(1-t)y) - (tUx + (1-t)Uy\| \\ &\quad \leq Lg^{-1}(\|x-y\| - L^{-1}\|Ux - Uy\|) \quad \forall x, y \in C,\ t \in [0,1].\end{aligned} \tag{1.7}$$

Let $\omega_w(\{x_n\})$ denote the set of all weak subsequential limits of a bounded sequence $\{x_n\}$ in E. Then, the following is actually Lemma 3.2 of Falset et al. [16].

Lemma 1.6. *Let E be a uniformly convex Banach space with its dual E^* satisfying the Kadec Klee property. Assume that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$ and for all $p_1, p_2 \in \omega_w(\{x_n\})$. Then, $\omega_w(\{x_n\})$ is a singleton.*

2. Some preparatory lemmas

In this section, we will prove the following important lemmas. In the sequel, we will write $F = F(S) \cap F(T)$ for the set of all common fixed points of the mappings S and T.

Lemma 2.1. *Let C be a nonempty closed convex subset of a normed space E. Let $S, T : C \to C$ be asymptotically nonexpansive mappings. Let $\{x_n\}$ be the process as defined in (1.3), where $\{a_n\}$ is a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $F \neq \phi$, then $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in F$.*

Proof. Let $x^* \in F$, then

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|a_n S^n x_n + (1-a_n)T^n x_n - x^*\| \\ &= \|a_n(S^n x_n - x^*) + (1-a_n)(T^n x_n - x^*)\| \\ &\leq a_n\|S^n x_n - x^*\| + (1-a_n)\|(T^n x_n - x^*)\| \\ &\leq a_n k_n\|x_n - x^*\| + (1-a_n)k_n\|x_n - x^*\| \\ &= k_n\|x_n - x^*\|.\end{aligned} \tag{2.1}$$

Thus, by Lemma 1.2, $\lim_{n\to\infty}\|x_n - x^*\|$ exists for each $x^* \in F$. □

Lemma 2.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to C$ be asymptotically nonexpansive mappings, and let $\{x_n\}$ be the process as defined in*

(1.3) *satisfying*

$$\|x_n - S^n x_n\| \leq \|S^n x_n - T^n x_n\|, \quad n \in \mathbb{N}. \tag{2.2}$$

If $F \neq \phi$, *then* $\lim_{n\to\infty}\|Sx_n - x_n\| = 0 = \lim_{n\to\infty}\|Tx_n - x_n\|$.

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Suppose that

$$\lim_{n\to\infty}\|x_n - x^*\| = c \tag{2.3}$$

for some $c \geq 0$. Then, $\|S^n x_n - x^*\| \leq k_n\|x_n - x^*\|$ implies that

$$\limsup_{n\to\infty}\|S^n x_n - x^*\| \leq c. \tag{2.4}$$

Similarly, we have

$$\limsup_{n\to\infty}\|T^n x_n - x^*\| \leq c. \tag{2.5}$$

Further, $\lim_{n\to\infty}\|x_{n+1} - x^*\| = c$gives that

$$\lim_{n\to\infty}\|a_n(S^n x_n - x^*) + (1 - a_n)(T^n x_n - x^*)\| = c. \tag{2.6}$$

Applying Lemma 1.3, we obtain that

$$\lim_{n\to\infty}\|S^n x_n - T^n x_n\| = 0. \tag{2.7}$$

But then by the condition $\|x_n - S^n x_n\| \leq \|S^n x_n - T^n x_n\|$,

$$\limsup_{n\to\infty}\|x_n - S^n x_n\| \leq 0. \tag{2.8}$$

That is,

$$\lim_{n\to\infty}\|x_n - S^n x_n\| = 0. \tag{2.9}$$

Also, then $\|x_n - T^n x_n\| \leq \|x_n - S^n x_n\| + \|S^n x_n - T^n x_n\|$ implies that

$$\lim_{n\to\infty}\|x_n - T^n x_n\| = 0. \tag{2.10}$$

Now, by definition of $\{x_n\}$, $\|x_{n+1} - T^n x_n\| \leq a_n\|S^n x_n - T^n x_n\|$ so that

$$\lim_{n\to\infty}\|x_{n+1} - T^n x_n\| = 0. \tag{2.11}$$

Then, $\|x_{n+1} - S^n x_n\| \leq \|x_{n+1} - T^n x_n\| + \|S^n x_n - T^n x_n\|$ implies

$$\lim_{n\to\infty} \|x_{n+1} - S^n x_n\| = 0. \tag{2.12}$$

Similarly, by $\|x_{n+1} - x_n\| \leq \|x_{n+1} - T^n x_n\| + \|x_n - T^n x_n\|$, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.13}$$

Next,

$$\begin{aligned}\|x_{n+1} - Sx_{n+1}\| &\leq \|x_{n+1} - S^{n+1}x_{n+1}\| + \|S^{n+1}x_{n+1} - S^{n+1}x_n\| + \|S^{n+1}x_n - Sx_{n+1}\| \\ &\leq \|x_{n+1} - S^{n+1}x_{n+1}\| + k_{n+1}\|x_{n+1} - x_n\| + k_1\|S^n x_n - x_{n+1}\|\end{aligned} \tag{2.14}$$

yields

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{2.15}$$

Moreover,

$$\begin{aligned}\|Sx_{n+1} - Tx_{n+1}\| &\leq \|Sx_{n+1} - S^{n+1}x_{n+1}\| + \|S^{n+1}x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_{n+1}\| \\ &\leq k_1\|x_{n+1} - S^n x_{n+1}\| + \|S^{n+1}x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + k_{n+1}\|x_{n+1} - x_n\| + k_1\|T^n x_n - x_{n+1}\| \\ &\leq k_1(\|x_{n+1} - S^n x_n\| + \|S^n x_n - S^n x_{n+1}\|) \\ &\quad + \|S^{n+1}x_{n+1} - T^{n+1}x_{n+1}\| + k_{n+1}\|x_{n+1} - x_n\| \\ &\quad + k_1\|T^n x_n - x_{n+1}\| \\ &\leq k_1(\|x_{n+1} - S^n x_n\| + k_n\|x_n - x_{n+1}\|) \\ &\quad + \|S^{n+1}x_{n+1} - T^{n+1}x_{n+1}\| + k_{n+1}\|x_{n+1} - x_n\| \\ &\quad + k_1\|T^n x_n - x_{n+1}\|\end{aligned} \tag{2.16}$$

gives by (2.7), (2.11), (2.12), and (2.13) that

$$\lim_{n\to\infty} \|Sx_n - Tx_n\| = 0. \tag{2.17}$$

In turn, by (2.15) and (2.17), we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.18}$$

This completes the proof. □

Lemma 2.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S, T : C \to C$ be asymptotically nonexpansive mappings and $\{x_n\}$ as defined in* (1.3). *Then, for any $p_1, p_2 \in F$, $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$.*

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$ and so $\{x_n\}$ is bounded. Thus, there exists a real number $r > 0$ such that $\{x_n\} \subseteq D \equiv \overline{B_r(0)} \cap C$, so that D is a closed convex bounded nonempty subset of C. Put

$$u_n(t) = \|tx_n + (1-t)p_1 - p_2\|. \tag{2.19}$$

Notice that $\lim_{n\to\infty}u_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty}u_n(1) = \|x_n - p_2\|$ exist as in the proof of Lemma 2.1.

Define $W_n : D \to D$ by

$$W_n x = a_n S^n x + (1-a_n)T^n x. \tag{2.20}$$

It is easy to verify that $W_n x_n = x_{n+1}$, $W_n p = p$ for all $p \in F$ and

$$\|W_n x - W_n y\| \le k_n\|x - y\| \quad \forall x, y \in C,\ n \in \mathbb{N}. \tag{2.21}$$

Set

$$\begin{gathered} R_{n,m} = W_{n+m-1}W_{n+m-2}\cdots W_n, \qquad m \in \mathbb{N}, \\ v_{n,m} = \|R_{n,m}(tx_n + (1-t)p_1) - (tR_{n,m}x_n + (1-t)p_1)\|. \end{gathered} \tag{2.22}$$

Then, $\|R_{n,m}x - R_{n,m}y\| \le \prod_{j=n}^{n+m-1}k_j\|x - y\|$, $R_{n,m}x_n = x_{n+m}$, and $R_{n,m}p = p$ for all $p \in F$.

Applying Lemma 1.5 with $x = x_n$, $y = p_1$, $U = R_{n,m}$, and using the facts that $\sum_{k=1}^{\infty}(k_n - 1) < \infty$ and $\lim_{n\to\infty}\|x_n - p\|$exist for all $p \in F$, we obtain $v_{n,m} \to 0$ as $n \to \infty$ and for all $m \ge 1$.

Finally, from the inequality,

$$\begin{aligned} u_{n+m}(t) &= \|tx_{n+m} + (1-t)p_1 - p_2\| \\ &= \|tR_{n,m}x_n + (1-t)p_1 - p_2\| \\ &\le v_{n,m} + \|R_{n,m}(tx_n + (1-t)p_1) - p_2\| \\ &\le v_{n,m} + \prod_{j=n}^{n+m-1}k_j\|tx_n + (1-t)p_1 - p_2\| \\ &= v_{n,m} + \prod_{j=n}^{n+m-1}k_j u_n(t), \end{aligned} \tag{2.23}$$

it follows that

$$\limsup_{n\to\infty} u_n(t) \le \liminf_{n\to\infty} u_n(t). \tag{2.24}$$

Hence, $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$. □

3. Common fixed point approximations by weak convergence

Here, we will approximate common fixed points of the mappings S and T through the weak convergence of the process $\{x_n\}$ defined in (1.3). Our first result in this direction uses the Opial's condition and the second one the Kadec Klee property.

Theorem 3.1. *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, S, T, and let $\{x_n\}$ be as in Lemma 2.2. If $F \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $x^* \in F$, then as proved in Lemma 2.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Now, we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 2.2, $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and $(I - S)$ are demiclosed with respect to zero from Lemma 1.4. Therefore, we obtain $Sz_1 = z_1$. Similarly, $Tz_1 = z_1$. Again, in the same way, we can prove that $z_2 \in F$. Next, we prove the uniqueness. For this, suppose that $z_1 \neq z_2$, then by the Opial's condition

$$\begin{aligned}\lim_{n\to\infty}\|x_n - z_1\| &= \lim_{n_i\to\infty}\|x_{n_i} - z_1\| < \lim_{n_i\to\infty}\|x_{n_i} - z_2\| = \lim_{n\to\infty}\|x_n - z_2\| \\ &= \lim_{n_j\to\infty}\|x_{n_j} - z_2\| < \lim_{n_j\to\infty}\|x_{n_j} - z_1\| = \lim_{n\to\infty}\|x_n - z_1\|.\end{aligned} \tag{3.1}$$

This is a contradiction. Hence, $\{x_n\}$ converges weakly to a point in F. □

Theorem 3.2. *Let E be a uniformly convex Banach space with its dual E^* satisfying the Kadec Klee property. Let C, S, T, and $\{x_n\}$ be as in Lemma 2.2. If $F \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. By the boundedness of $\{x_n\}$ and reflexivity of E, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some p in C. By Lemma 2.2, we have $\lim_{i\to\infty}\|x_{n_i} - Sx_{n_i}\| = 0 = \lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\|$. This gives $p \in F$. To prove that $\{x_n\}$ converges weakly to p, suppose that $\{x_{n_k}\}$ is another subsequence of $\{x_n\}$ that converges weakly to some q in C. Then, by Lemmas 2.2 and 1.4, $p, q \in W \cap F$, where $W = \omega_w(\{x_n\})$. Since $\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$ by Lemma 2.3, therefore, $p = q$ from Lemma 1.6. Consequently, $\{x_n\}$ converges weakly to $p \in F$ and this completes the proof. □

By putting $T = I$, the identity mapping, in Theorems 3.1 and 3.2, we have the following corollaries. Note that the condition $\|x_n - S^n x_n\| \le \|S^n x_n - T^n x_n\|$, $n \in \mathbb{N}$, becomes trivially true in this case.

Corollary 3.3. *Let E be a uniformly convex Banach space satisfying the Opial's condition and let C, S be as in Lemma 2.1 and $\{x_n\}$ as in (1.4). If $F(S) \neq \phi$, then $\{x_n\}$ converges weakly to a fixed point of S.*

Corollary 3.4. *Let E be a uniformly convex Banach space with dual E^* satisfying the Kadec Klee property. Let C, S be as in Lemma 2.1 and $\{x_n\}$ as in (1.4). If $F(S) \neq \phi$, then $\{x_n\}$ converges weakly to a fixed point of S.*

4. Common fixed point approximations by strong convergence

We first prove a strong convergence theorem in general real Banach spaces as follows.

Theorem 4.1. *Let E be a real Banach space and C, $\{x_n\}$, and let S, T be as in Lemma 2.1. If $F \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T if and only if*

$$\liminf_{n\to\infty} D(x_n, F) = 0, \tag{4.1}$$

where $D(x, F) = \inf\{\|x - p\| : p \in F\}$.

Proof. Necessity is obvious. Conversely, suppose that

$$\liminf_{n\to\infty} D(x_n, F) = 0. \tag{4.2}$$

As in the proof of Lemma 2.1, we have

$$\|x_{n+1} - p\| \leq k_n \|x_n - p\|. \tag{4.3}$$

This gives

$$D(x_{n+1}, F) \leq k_n D(x_n, F), \tag{4.4}$$

so that $\lim_{n\to\infty} D(x_n, F)$ exists; but by hypothesis

$$\liminf_{n\to\infty} D(x_n, F) = 0, \tag{4.5}$$

we have $\lim_{n\to\infty} D(x_n, F) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} D(x_n, F) = 0$, there exists a constant n_0 such that for all $n \geq n_0$, we have

$$D(x_n, F) < \frac{\epsilon}{4}. \tag{4.6}$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in F\} < \epsilon/4$. Hence, there exists $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{2}. \tag{4.7}$$

Now, for $m, n \geq n_0$, we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \leq 2\|x_{n_0} - p^*\| < 2\left(\frac{\epsilon}{2}\right) = \epsilon. \tag{4.8}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a Banach space E, therefore, it must converge in C. Let $\lim_{n\to\infty} x_n = q$. Now, $\lim_{n\to\infty} D(x_n, F) = 0$ gives that $D(q, F) = 0$; but as being well known, F is closed, therefore, $q \in F$. □

Fukhar-ud-din and Khan gave the following so-called condition (A') in [17].

Two mappings $S, T : C \to C$, where C is a subset of E, are said to satisfy condition (A') if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Tx\| \geq f(D(x,F))$ or $\|x - Sx\| \geq f(D(x,F))$ for all $x \in C$ where $D(x,F) = \inf\{\|x - x^*\| : x^* \in F\}$.

Our next theorem is an application of Theorem 4.1 and makes use of condition (A').

Theorem 4.2. *Let E be a uniformly convex Banach space, and let $C, \{x_n\}$ be as in Lemma 2.2. Let $S, T : C \to C$ be two asymptotically nonexpansive mappings satisfying condition (A'). If $F \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$. Let it be c for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. Now, $\|x_{n+1} - x^*\| \leq k_n \|x_n - x^*\|$ gives that $D(x_{n+1}, F) \leq k_n D(x_n, F)$ and so $\lim_{n\to\infty} D(x_n, F)$ exists by Lemma 1.2. By using condition (A'), either

$$\lim_{n\to\infty} f(D(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0 \tag{4.9}$$

or

$$\lim_{n\to\infty} f(D(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{4.10}$$

In both the cases,

$$\lim_{n\to\infty} f(D(x_n, F)) = 0. \tag{4.11}$$

Since f is a nondecreasing function and $f(0) = 0$, $\lim_{n\to\infty} D(x_n, F) = 0$. Now, applying Theorem 4.2, we get the result. □

Remark 4.3. When $T = I$, both of the above theorems remain valid for the Mann iterative process (1.4).

Remark 4.4. Above theorems can also be proved using our process with error terms:

$$\begin{gathered} x_1 = x \in C, \\ x_{n+1} = a_n S^n x_n + b_n T^n x_n + c_n u_n, \quad n \in \mathbb{N}, \end{gathered} \tag{4.12}$$

where $a_n + b_n + c_n = 1$, $\sum_{n=1}^{\infty} c_n < \infty$ and $\{u_n\}$ is a bounded sequence in C.

Remark 4.5. Non-self-asymptotically nonexpansive mappings case can also be dealt with similarly using above iterative process even with error terms.

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