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Research Article

Boundary Blow-Up Solutions to $p(x)$-Laplacian Equations with Exponential Nonlinearities

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This paper investigates the $p(x)$-Laplacian equations with exponential nonlinearities $-\Delta_{p(x)}u + e^{f(x,u)} = 0$ in Ω, $u(x) \to +\infty$ as $d(x,\partial\Omega) \to 0$, where $-\Delta_{p(x)}u = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. The singularity of boundary blow-up solutions is discussed, and the existence of boundary blow-up solutions is given.

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1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. We refer to [1, 2], the background of these problems. Many results have been obtained on this kind of problems, for example, [1–15]. In this paper, we consider the $p(x)$-Laplacian equations with exponential nonlinearities

$$\begin{gathered} -\Delta_{p(x)}u + e^{f(x,u)} = 0 \quad \text{in } \Omega, \\ u(x) \longrightarrow +\infty \quad \text{as } d(x,\partial\Omega) \longrightarrow 0, \end{gathered} \tag{P}$$

where $-\Delta_{p(x)}u = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$, $\Omega = B(0,R) \subset \mathbb{R}^N$ is a bounded radial domain ($B(0,R) = \{x \in \mathbb{R}^N \mid |x| < R\}$). Our aim is to give the existence and asymptotic behavior of solutions for problem (P).

Throughout the paper, we assume that $p(x)$ and $f(x,u)$ satisfy that

(H_1) $p(x) \in C^1(\overline{\Omega})$ is radial and satisfies

$$1 < p^- \le p^+ < +\infty, \quad \text{where } p^- = \inf_{\Omega} p(x),\ p^+ = \sup_{\Omega} p(x); \tag{1.1}$$

(H_2) $f(x,u)$ is radial with respect to x, $f(x,\cdot)$ is increasing and $f(x,0)=0$ for any $x\in\Omega$;

(H_3) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function and satisfies

$$\left|f(x,t)\right|\le C_1+C_2|t|^{\gamma(x)},\quad \forall(x,t)\in\Omega\times\mathbb{R},\tag{1.2}$$

where C_1,C_2 are positive constants, $0\le\gamma\in C(\overline{\Omega})$.

The operator $-\Delta_{p(x)}u=-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. Especially, if $p(x)\equiv p$ (a constant), (P) is the well-known p-Laplacian problem (see [16–18]).

Because of the nonhomogeneity of $p(x)$-Laplacian, $p(x)$-Laplacian problems are more complicated than those of p-Laplacian ones (see [6]); and another difficulty of this paper is that $f(x,u)$ cannot be represented as $h(x)f(u)$.

2. Preliminary

In order to deal with $p(x)$-Laplacian problems, we need some theories on spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, and properties of $p(x)$-Laplacian, which we will use later (see [3, 7]). Let

$$L^{p(x)}(\Omega)=\left\{u\mid u\text{ is a measurable real-valued function, }\int_\Omega\left|u(x)\right|^{p(x)}dx<\infty\right\}.\tag{2.1}$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)}=\inf\left\{\lambda>0\mid\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\le 1\right\}.\tag{2.2}$$

The space $(L^{p(x)}(\Omega),|\cdot|_{p(x)})$ becomes a Banach space. We call it generalized Lebesgue space. The space $(L^{p(x)}(\Omega),|\cdot|_{p(x)})$ is a separable, reflexive, and uniform convex Banach space (see [3, Theorems 1.10, 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega)\mid\left|\nabla u\right|\in L^{p(x)}(\Omega)\right\},\tag{2.3}$$

and it can be equipped with the norm

$$\|u\|=|u|_{p(x)}+\left|\nabla u\right|_{p(x)},\quad\forall u\in W^{1,p(x)}(\Omega).\tag{2.4}$$

$W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive, and uniform convex Banach spaces (see [3, Theorem 2.1]).

If $u\in W_{\mathrm{loc}}^{1,p(x)}(\Omega)\cap C(\Omega)$, u is called a solution of (P) if it satisfies

$$\int_Q|\nabla u|^{p(x)-2}\nabla u\nabla q\,dx+\int_Q f(x,u)q\,dx=0,\quad\forall q\in W_0^{1,p(x)}(Q),\tag{2.5}$$

for any domain $Q\Subset\Omega$, and $\max(k-u,0)\in W_0^{1,p(x)}(\Omega)$ for any $k\in\mathbb{N}^+$.

Let $W_{0,\mathrm{loc}}^{1,p(x)}(\Omega)=\{u\mid$ there exists an open domain $Q\Subset\Omega$ s.t. $u\in W_0^{1,p(x)}(Q)\}$. For any $u\in W_{\mathrm{loc}}^{1,p(x)}(\Omega)\cap C(\Omega)$ and $\varphi\in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega)$, define $A:W_{\mathrm{loc}}^{1,p(x)}(\Omega)\cap C(\Omega)\to(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$ as $\langle Au,\varphi\rangle=\int_\Omega(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi+e^{f(x,u)}\varphi)dx$.

Lemma 2.1 (see [5, Theorem 3.1]). *Let $h \in W^{1,p(x)}(\Omega) \cap C(\Omega)$, $X = h + W_{0,\mathrm{loc}}^{1,p(x)}(\Omega) \cap C(\Omega)$. Then, $A : X \to (W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$ is strictly monotone.*

Let $g \in (W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, if $\langle g, \varphi \rangle \geq 0$, for all $\varphi \in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega)$, $\varphi \geq 0$ a.e. in Ω, then denote $g \geq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$; correspondingly, if $-g \geq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, then denote $g \leq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$.

Definition 2.2. Let $u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C(\Omega)$. If $Au \geq 0$ $(Au \leq 0)$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, then u is called a weak supersolution (weak subsolution) of (P).

Copying the proof of [9], we have the following lemma.

Lemma 2.3 (comparison principle). *Let $u, v \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C(\Omega)$ satisfy $Au - Av \geq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$. Let $\varphi(x) = \min\{u(x) - v(x), 0\}$. If $\varphi(x) \in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega)$ (i.e., $u \geq v$ on $\partial\Omega$), then $u \geq v$ a.e. in Ω.*

Lemma 2.4 (see [4, Theorem 1.1]). *Under the conditions (H_1) and (H_3), if $u \in W^{1,p(x)}(\Omega)$ is a bounded weak solution of $-\Delta_{p(x)} u + e^{f(x,u)} = 0$ in Ω, then $u \in C_{\mathrm{loc}}^{1,\vartheta}(\Omega)$, where $\vartheta \in (0,1)$ is a constant.*

3. Main results and proofs

If u is a radial solution of (P), then (P) can be transformed into

$$\begin{gathered} \left(r^{N-1}|u'|^{p(r)-2}u'\right)' = r^{N-1}e^{f(r,u)}, \quad r \in (0,R), \\ u(0) = u_0, \quad u'(0) = 0, \quad u'(r) \geq 0 \quad \text{for } 0 < r < R. \end{gathered} \tag{3.1}$$

It means that $u(r)$ is increasing.

Theorem 3.1. *If there exists a constant $\sigma \in [R/2, R)$ such that*

$$f(r,u) \geq \alpha u^s \quad (\text{as } u \longrightarrow +\infty) \text{ for } r \in [\sigma, R) \text{ uniformly}, \tag{3.2}$$

where α and s are positive constants, then there exists a continuous function $\Phi_1(x)$ which satisfies $\Phi_1(x) \to +\infty$ (as $d(x,\partial\Omega) \to 0$), and such that, if u is a weak solution of problem (P), then $u(x) \leq \Phi_1(x)$.

Proof. Let $R_0 \in (\sigma, R)$. Denote

$$\Theta(r,a,\lambda) = \int_r^{R_0} \left[\frac{a\left(a\ln(R-R_0-\lambda)^{-1}\right)^{1/s-1}}{s(R-R_0-\lambda)}\right]^{(p(R_o)-1)/(p(t)-1)} \left[\frac{(R_o)^{N-1}}{t^{N-1}}\sin\varepsilon(t-\sigma)\right]^{1/(p(t)-1)} dt. \tag{3.3}$$

Define the function $g(r,a)$ on $[0,R)$ as

$$g(r,a) = \begin{cases} \left(a\ln(R-r)^{-1}\right)^{1/s} + k, & R_0 \leq r < R, \\ k - \Theta(r,a,0) + \left(a\ln(R-R_0)^{-1}\right)^{1/s}, & \sigma < r < R_0, \\ k - \Theta(\sigma,a,0) + \left(a\ln(R-R_0)^{-1}\right)^{1/s}, & r \leq \sigma, \end{cases} \tag{3.4}$$

where $a > (1/\alpha)\sup_{|x|\geq R_0} p(x)$ is a constant, $R_0 \in (\sigma, R)$, and $R - R_0$ is small enough, $\varepsilon = \pi/2(R_0 - \sigma)$ and $k = ((2p^+/\alpha)\ln(R-R_0)^{-1})^{1/s} + \Theta(\sigma, 2a, 0)$.

Obviously, for any positive constant a, $g(r,a) \in C^1[0,R)$.

When $R_0 < r < R$, we have

$$\left(r^{N-1}|g'|^{p(r)-2}g'\right)' = r^{N-1}\left(\frac{a^{1/s}}{s}\right)^{p(r)-1}\frac{p(r)-1}{(R-r)^{p(r)}}\left(\ln(R-r)^{-1}\right)^{(1/s-1)(p(r)-1)}(1+\Pi(r)), \tag{3.5}$$

where

$$\begin{aligned}\Pi(r) &= \frac{(1/s-1)}{\ln(R-r)^{-1}} + \frac{\left[r^{N-1}(a^{1/s}/s)^{p(r)-1}\right]'}{r^{N-1}(a^{1/s}/s)^{p(r)-1}(p(r)-1)}(R-r)\\&\quad + \frac{-p'(r)\ln(R-r)}{(p(r)-1)}(R-r) + \frac{(1/s-1)p'(r)\ln\ln(R-r)^{-1}}{(p(r)-1)}(R-r).\end{aligned} \tag{3.6}$$

If $(R-R_0)$ is small enough, it is easy to see $|\Pi(r)| \leq 1/2$; from (3.5), we have

$$\begin{aligned}\left(r^{N-1}|g'|^{p(r)-2}g'\right)' &\leq 2r^{N-1}\left(\frac{a^{1/s}}{s}\right)^{p(r)-1}(p(r)-1)(R-r)^{-p(r)}\left(\ln(R-r)^{-1}\right)^{(1/s-1)(p(r)-1)}\\&\leq r^{N-1}\left(\frac{1}{R-r}\right)^{\alpha a} = r^{N-1}e^{\alpha g^s} \leq r^{N-1}e^{f(r,g)}, \quad \forall r \in (R_0, R).\end{aligned} \tag{3.7}$$

Obviously, if $R - R_0$ is small enough, then $g \geq ((2p^+/\alpha)\ln(R-R_0)^{-1})^{1/s}$ is large enough, so we have

$$\begin{aligned}\left(r^{N-1}|g'|^{p(r)-2}g'\right)' &= \varepsilon(R_o)^{N-1}\left[\frac{a\left(a\ln(R-R_0)^{-1}\right)^{1/s-1}}{s(R-R_0)}\right]^{(p(R_o)-1)}\cos(\varepsilon(r-\sigma))\\&\leq r^{N-1}e^{\alpha g^s} \leq r^{N-1}e^{f(r,g)}, \quad \sigma < r < R_0.\end{aligned} \tag{3.8}$$

Obviously,

$$\left(r^{N-1}|g'|^{p(r)-2}g'\right)' = 0 \leq r^{N-1}e^{f(r,g)}, \quad 0 \leq r < \sigma. \tag{3.9}$$

Since $g(|x|,a)$ is a C^1 function on $B(0,R)$, if $0 < R - R_0$ is small enough (R_0 depends on R, p, s, α), from (3.7), (3.8), and (3.9), we can see that $g(|x|,a)$ is a supersolution of (P).

Define the function $g_m(r, a-\epsilon)$ on $[0, R-1/m)$ as

$$g_m(r,a-\epsilon) = \begin{cases}\left[(a-\epsilon)\ln\left(R-\dfrac{1}{m}-r\right)^{-1}\right]^{1/s} + k, & R_0 \leq r < R - \dfrac{1}{m},\\ k - \Theta\left(r, a-\epsilon, \dfrac{1}{m}\right) + \left[(a-\epsilon)\ln\left(R-\dfrac{1}{m}-R_0\right)^{-1}\right]^{1/s}, & \sigma < r < R_0,\\ k - \Theta\left(\sigma, a-\epsilon, \dfrac{1}{m}\right) + \left[(a-\epsilon)\ln\left(R-\dfrac{1}{m}-R_0\right)^{-1}\right]^{1/s}, & r \leq \sigma,\end{cases} \tag{3.10}$$

where m is a big-enough integer such that $0 < 1/m \le (R - R_0)/2$, $\varepsilon = \pi/2(R_0 - \sigma)$, $0 < \epsilon < 1$, is a positive small constant such that $\alpha(a - \epsilon) > \sup_{|x|\ge R_0} p(x)$.

Obviously, $g_m(|x|, a-\epsilon)$ is a supersolution of (P) on $B(0, R-1/m)$. If u is a solution of (P), according to the comparison principle, we get that $g_m(|x|, a-\epsilon) \ge u(x)$ for any $x \in B(0, R-1/m)$. For any $x \in B(0, R-1/m) \setminus B(0, R_0)$, we have $g_m(|x|, a-\epsilon) \ge g_{m+1}(|x|, a-\epsilon)$. Thus,

$$u(x) \le \lim_{m\to+\infty} g_m(|x|, a-\epsilon), \quad \forall x \in B(0, R) \setminus B(0, R_0). \tag{3.11}$$

When $d(x, \partial\Omega) > 0$ is small enough, we have

$$\lim_{m\to+\infty} g_m(|x|, a-\epsilon) < \left(a \ln (R-r)^{-1}\right)^{1/s} + k \le g(|x|, a). \tag{3.12}$$

According to the comparison principle, we obtain that $g(|x|, a) \ge u(x)$, for all $x \in B(0, R)$, then $\Phi_1(x) = g(|x|, a)$ is an upper control function of all of the solutions of (P). The proof is completed. □

Theorem 3.2. *If there exists a $\sigma \in [R/2, R)$ such that*

$$f(r, u) \le \beta u^s \quad (\text{as } u \longrightarrow +\infty) \text{ for } r \in [\sigma, R) \text{ uniformly}, \tag{3.13}$$

where β and s are positive constants, then there exists a continuous function $\Phi_2(x)$ which satisfies $\Phi_2(x) \to +\infty$ (as $d(x, \partial\Omega) \to 0$), and such that, if $u(x)$ is a solution of problem (P), then $u(x) \ge \Phi_2(x)$.

Proof. Let z_1 be a radial solution of

$$-\Delta_{p(x)} z_1(x) = -\mu \quad \text{in } \Omega_1 = B(0, \sigma),\ z_1 = 0 \text{ on } \partial\Omega_1, \tag{3.14}$$

where $\mu > 2$ is a positive constant. We denote $z_1 = z_1(r) = z_1(|x|)$, then z_1 satisfies $z_1(\sigma) = 0$, $z_1'(0) = 0$, and

$$z_1' = \left|\frac{r\mu}{N}\right|^{1/(p(r)-1)}, \qquad z_1 = -\int_r^{\sigma} \left|\frac{r\mu}{N}\right|^{1/(p(r)-1)} dr. \tag{3.15}$$

Denote $h_b(r, \delta)$ on $[\sigma, R_0]$ as

$$\begin{aligned} h_b(r, \delta) = \int_r^{R_0} \Bigg\{ & \frac{(R_o)^{N-1}}{t^{N-1}} \frac{t-\sigma}{R_0-\sigma} \left[\frac{b\left(b \ln (R+\delta-R_0)^{-1}\right)^{1/s-1}}{s(R+\delta-R_0)}\right]^{p(R_o)-1} \\ & + \frac{(\sigma)^{N-1}}{t^{N-1}} \frac{R_0-t}{R_0-\sigma} \left[\left|\frac{t\mu}{N}\right|^{1/(p(t)-1)}\right]^{p(\sigma)-1} \Bigg\}^{1/(p(t)-1)} dt. \end{aligned} \tag{3.16}$$

It is easy to see that

$$-h_b'(\sigma, 0) = z_1'(\sigma) = \left|\frac{\sigma\mu}{N}\right|^{1/(p(\sigma)-1)}, \qquad -h_b'(R_0, 0) = \frac{b\left(b \ln (R-R_0)^{-1}\right)^{1/s-1}}{s(R-R_0)}. \tag{3.17}$$

Define the function $v(r,b)$ on $B(0,R)$ as

$$v(r,b)=\begin{cases}\left(b\ln(R-r)^{-1}\right)^{1/s}-k^*, & R_0\le r<R,\\ \left(b\ln(R-R_0)^{-1}\right)^{1/s}-k^*-h_b(r,0), & \sigma<r<R_0,\\ -\displaystyle\int_r^\sigma\left|\frac{r\mu}{N}\right|^{1/(p(r)-1)}dr+\left(b\ln(R-R_0)^{-1}\right)^{1/s}-k^*-h_b(\sigma,0), & r\le\sigma,\end{cases}\tag{3.18}$$

where $b\in(0,(1/\beta)\inf_{|x|\ge R_0}p(x))$ is a constant, $R_0\in(\sigma,R)$, and $R-R_0$ is small enough, and $k^*=((2p^+/\beta)\ln 2(R-R_0)^{-1})^{1/s}$.

Obviously, for any positive constant b, $v(r,b)\in C^1[0,R)$.

Similar to the proof of Theorem 3.1, when $R-R_0$ is small enough, we have

$$\left(r^{N-1}|v'|^{p(r)-2}v'\right)'\ge r^{N-1}e^{f(r,v)},\quad \forall r\in(R_0,R).\tag{3.19}$$

When $R-R_0$ is small enough, for all $r\in(\sigma,R_0)$, since $f(r,v)\le 0$, then

$$\left(r^{N-1}|v'|^{p(r)-2}v'\right)'\ge\frac{1}{2}\frac{(R_o)^{N-1}}{R_0-\sigma}\left[\frac{b\left(b\ln(R-R_0)^{-1}\right)^{1/s-1}}{s(R-R_0)}\right]^{p(R_0)-1}\ge r^{N-1}e^{f(r,v)}.\tag{3.20}$$

Obviously,

$$\left(r^{N-1}|v'|^{p(r)-2}v'\right)'=r^{N-1}\mu\ge r^{N-1}e^{f(r,v)},\quad \forall r\in(0,\sigma).\tag{3.21}$$

Combining (3.19), (3.20), and (3.21), we can see that $v(r,a)$ is a subsolution of (P).

Define the function $v_m(r,b+\epsilon)$ on $B(0,R)$ as

$$v_m(r,b+\epsilon)=\begin{cases}\left[(b+\epsilon)\ln\left(R+\dfrac{1}{m}-r\right)^{-1}\right]^{1/s}-k^*, & R_0\le r<R,\\ \left[(b+\epsilon)\ln\left(R+\dfrac{1}{m}-R_0\right)^{-1}\right]^{1/s}-k^*-h_{b+\epsilon}\left(r,\dfrac{1}{m}\right), & \sigma<r<R_0,\\ -\displaystyle\int_r^\sigma\left|\frac{\mu r}{N}\right|^{1/(p(r)-1)}dr+\left[(b+\epsilon)\ln\left(R+\frac{1}{m}-R_0\right)^{-1}\right]^{1/s}-k^*-h_{b+\epsilon}\left(\sigma,\frac{1}{m}\right), & r\le\sigma,\end{cases}\tag{3.22}$$

where ϵ is a small-enough positive constant such that $(b+\epsilon)<(1/\beta)\inf_{|x|\ge R_0}p(x)$.

We can see that $v_m(r,b+\epsilon)\in C^1([0,R))$ is a subsolution of (P) on $B(R_0,R)$, according to the comparison principle, we get that $v_m(|x|,b+\epsilon)\le u(x)$ for any $x\in B(0,R)$. For any $x\in B(0,R)\setminus B(0,R_0)$, we have $v_m(|x|,b+\epsilon)\le v_{m+1}(|x|,b+\epsilon)$. Thus,

$$u(x)\ge\lim_{m\to+\infty}v_m(|x|,b+\epsilon),\quad \forall x\in B(0,R)\setminus B(0,R_0).\tag{3.23}$$

When $d(x,\partial\Omega)$ is small enough, we have

$$\lim_{m\to+\infty}v_m(|x|,b+\epsilon)>v(|x|,b).\tag{3.24}$$

From the comparison principle, we obtain $v(|x|,b)\le u(x)$, $\forall x\in B(0,R)$, then $\Phi_2(x)=v(|x|,b)$ is a lower control function of all of the solutions of (P). □

Theorem 3.3. *If* $\inf_{x\in\Omega} p(x) > N$ *and there exists a* $\sigma \in [R/2, R)$ *such that*

$$f(r,u) \geq au^s \quad (\text{as } u \longrightarrow +\infty) \text{ for } r \in [\sigma, R) \text{ uniformly}, \tag{3.25}$$

where a *and* s *are positive constants, then* (P) *possesses a solution.*

Proof. In order to deal with the existence of boundary blow-up solutions of (P), let us consider the problem

$$\begin{gathered} -\Delta_{p(x)} u + e^{f(x,u)} = 0 \quad \text{in } \Omega, \\ u(x) = j \quad \text{for } x \in \partial\Omega, \end{gathered} \tag{3.26}$$

where $j = 1, 2, \ldots$. Since $\inf_{x\in\Omega} p(x) > N$, then $W^{1,p(x)}(\Omega) \hookrightarrow C^{\alpha}(\overline{\Omega})$, where $\alpha \in (0,1)$. The relative functional of (3.26) is

$$\varphi(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx + \int_{\Omega} F(x,u) dx, \tag{3.27}$$

where $F(x,u) = \int_0^u e^{f(x,t)} dt$. Since φ is coercive in $X_j := j + W_0^{1,p(x)}(\Omega)$, then φ possesses a nontrivial minimum point u_j, then problem (3.26) possesses a weak solution u_j. According to the comparison principle, we get $u_j(x) \leq u_{j+1}(x)$ for any $x \in \Omega$ and $j = 1, 2, \ldots$. Since $\Phi_1(x)$ defined in Theorem 3.1 is a supersolution, according to the comparison principle, we have $u_j(x) \leq \Phi_1(x)$ on Ω for all $j = 1, 2, \ldots$. Since $\Phi_1(x)$ is locally bounded, from Lemma 2.4, every weak solution of (P) is a locally $C^{1,\vartheta}_{\text{loc}}$ function. Thus, $\{u_j(x)\}$ possesses a subsequence (we still denote it by $\{u_j(x)\}$), such that $\lim_{j\to\infty} u_j = u$ is a solution of (P). □

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