Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 71452, 8 pages
doi:10.1155/2007/71452

Research Article

A Note on the q-Genocchi Numbers and Polynomials

Taekyun Kim

Received 15 March 2007; Revised 7 May 2007; Accepted 24 May 2007

Recommended by Paolo Emilio Ricci

We discuss new concept of the q-extension of Genocchi numbers and give some relations between q-Genocchi polynomials and q-Euler numbers.

1. Introduction

The Genocchi numbers G_n, $n = 0,1,2,\ldots$, which can be defined by the generating function

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi, \tag{1.1}$$

have numerous important applications in number theory, combinatorics, and numerical analysis, among other areas, [1–13]. It is easy to find the values $G_1 = 1$, $G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given by $G_{2m} = 2(1-2^{2n})B_{2n} = 2nE_{2n-1}(0)$, where B_n is a Bernoulli number and $E_n(x)$ is an Euler polynomial. The first few Genocchi numbers for $n = 2,4,\ldots$ are $-1,-3,17,-155,2073,\ldots$. The Euler polynomials are well known as

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \text{ (see [1, 3, 7–9])}. \tag{1.2}$$

By (1.1) and (1.2) we easily see that

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{G_{k+1}}{k+1} x^{n-k}, \quad \text{where } \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \text{ (cf. [4–6])}. \tag{1.3}$$

For $m,n \geq 1$ and, m odd, we have

$$(n^m - n)G_m = \sum_{k=1}^{m-1} \binom{m}{k} n^k G_k Z_{m-k}(n-1), \tag{1.4}$$

where $Z_m(n) = 1^m - 2^m + 3^m - \cdots + (-1)^{n-1}n^m$, see [3, 13]. From (1.15) we derive

$$2t = \sum_{n=0}^{\infty} \left((G+1)^n + G_n\right)\frac{t^n}{n!}, \tag{1.5}$$

where we use the technique method notation by replacing G^m by $G_m (m \geq 0)$, symbolically. By comparing the coefficients on both sides in (1.5), we see that

$$G_0 = 0, \qquad (G+1)^n + G_n = \begin{cases} 2 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{1.6}$$

Let p be a fixed odd prime, and let $\mathbb{C}_p$ denote the p-adic completion of the algebraic closure of $\mathbb{Q}_p$(= p-adic number field). For d is a fixed positive integer with $(p,d) = 1$, let

$$\begin{gathered} X = X_d = \varprojlim_{N} \frac{\mathbb{Z}}{dp^N\mathbb{Z}}, \\ X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{gathered} \tag{1.7}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

Ordinary q-calculus is now very well understood from many different points of view. Let us consider a complex number $q \in \mathbb{C}$ with $|q| < 1$ (or $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$) as an indeterminate. The q-basic numbers are defined by

$$[x]_q = \frac{q^x - 1}{q-1}, \quad [x]_{-q} = \frac{-(-q)^x + 1}{q+1}, \quad \text{for } x \in \mathbb{R}. \tag{1.8}$$

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in \mathrm{UD}(\mathbb{Z}_p)$, if the difference quotients

$$F_f(x,y) = \frac{f(x) - f(y)}{x - y} \tag{1.9}$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$.

For $f \in \mathrm{UD}(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p) \tag{1.10}$$

representing a q-analogue of Riemann sums for f, (cf. [5]). The integral of f on $\mathbb{Z}_p$ will be defined as limit ($n \to \infty$) of those sums, when it exists. The p-adic q-integral of the function $f \in \mathrm{UD}(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{0\le x<p^N} f(x)q^x, \text{ (see [5, 10–12])}. \tag{1.11}$$

In the previous paper [4, 9], the author constructed the q-extension of Euler polynomials by using p-adic q-fermionic integral on $\mathbb{Z}_p$ as follows:

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [t+x]_q^n d\mu_{-q}(t), \quad \text{where } \mu_{-q}(x+p^N\mathbb{Z}_p) = \frac{(-q)^x}{[p^N]_{-q}}. \tag{1.12}$$

From (1.12), we note that

$$E_{n,q}(x) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^{l+1}} q^{lx}, \text{ see [4]}. \tag{1.13}$$

The q-extension of Genocchi numbers is defined as

$$g_q^*(t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,q}^* \frac{t^n}{n!}, \text{ see [4]}. \tag{1.14}$$

The following formula is well known in [4, 7]:

$$E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \frac{G_{k+1,q}^*}{k+1}. \tag{1.15}$$

The modified q-Euler numbers are defined as

$$\xi_{0,q} = \frac{[2]_q}{2}, \qquad (q\xi+1)^k + \xi_{k,q} = \begin{cases} [2]_q & \text{if } k=0, \\ 0 & \text{if } k \neq 0, \end{cases} \tag{1.16}$$

with the usual convention of replacing ξ^i by $\xi_{i,q}$, see [10]. Thus, we derive the generating function of $\xi_{n,q}$ as follows:

$$F_q(t) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k]_q t} = \sum_{n=0}^{\infty} \xi_{n,q} \frac{t^n}{n!}. \tag{1.17}$$

Now we also consider the q-Euler polynomials $\xi_{n,q}(x)$ as

$$F_q(t,x) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t} = \sum_{n=0}^{\infty} \xi_{n,q}(x) \frac{t^n}{n!}. \tag{1.18}$$

From (1.18) we note that

$$\xi_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} \xi_{l,q} q^{lx} [x]_q^{n-l}, \text{ see [10]}. \tag{1.19}$$

In the recent, several authors studied the q-extension of Genocchi numbers and polynomials (see [1, 2, 5–7, 12]). In this paper we discuss the new concept of the q-extension of Genocchi numbers and give the same relations between q-Genocchi numbers and q-Euler numbers.

2. q-extension of Genocchi numbers

In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. Now we consider the q-extension of Genocchi numbers as follows:

$$g_q(t) = [2]_q t \sum_{k=0}^{\infty} (-1)^k e^{[k]_q t} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}. \tag{2.1}$$

In (2.1), it is easy to show that $\lim_{q\to 1} g_q(t) = 2t/(e^t+1) = \sum_{n=0}^{\infty} G_n (t^n/n!)$. From (2.1) we derive

$$\begin{aligned} g_q(t) &= [2]_q t \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{\infty} [k]_q^m \frac{t^m}{m!} = [2]_q \sum_{k=0}^{\infty} (-1)^k \sum_{m=1}^{\infty} m[k]_q^{m-1} \frac{t^m}{m!} \\ &= [2]_q \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{\infty} m[k]_q^{m-1} \frac{t^m}{m!}. \end{aligned} \tag{2.2}$$

By (2.2), we easily see that

$$g_q(t) = [2]_q \sum_{m=0}^{\infty} \left(m \left(\frac{1}{1-q} \right)^{m-1} \sum_{l=0}^{m-1} \binom{m-1}{l} (-1)^l \frac{1}{1+q^l} \right) \frac{t^m}{m!}. \tag{2.3}$$

From (2.1) and (2.3) we note that

$$\sum_{m=0}^{\infty} G_{m,q} \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(m[2]_q \left(\frac{1}{1-q} \right)^{m-1} \sum_{l=0}^{m-1} \binom{m-1}{l} \frac{(-1)^l}{1+q^l} \right) \frac{t^m}{m!}. \tag{2.4}$$

By comparing the coefficients on both sides in (2.4), we have the following theorem.

Theorem 2.1. *For $m \geq 0$,*

$$G_{m,q} = m[2]_q \left(\frac{1}{1-q} \right)^{m-1} \sum_{l=0}^{m-1} \binom{m-1}{l} \frac{(-1)^l}{1+q^l}. \tag{2.5}$$

From Theorem 2.1, we easily derive the following corollary.

Corollary 2.2. *For $k \in \mathbb{N}$,*

$$G_{0,q} = 0, \qquad (qG+1)^k + G_{k,q} = \begin{cases} \dfrac{[2]_q^2}{2} & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases} \tag{2.6}$$

with the usual convention of replacing G^i by $G_{i,q}$.

Remark 2.3. We note that Corollary 2.2 is the q-extension of (1.6). By (1.15)–(1.19) and Corollary 2.2, we obtain the following theorem.

THEOREM 2.4. *For* $n \in \mathbb{N}$

$$\xi_{n,q} = \frac{G_{n+1,q}}{n+1}. \tag{2.7}$$

From (1.18) we derive

$$\begin{aligned} F_q(x,t) &= [2]_q \sum_{n=0}^{\infty} (-1)^n e^{[n+x]_q t} = q^x t \frac{[2]_q}{q^x t} e^{[x]_q t} \sum_{n=0}^{\infty} (-1)^n e^{q^x [n]_q t} \\ &= e^{[x]_q t} \sum_{n=0}^{\infty} q^{nx} \frac{G_{n+1,q}}{n+1} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \frac{G_{k+1,q}}{k+1} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.8}$$

By (2.8), we easily see that

$$\xi_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \frac{G_{k+1,q}}{k+1}. \tag{2.9}$$

This formula can be considered as the q-extension of (1.3). Let us consider the q-analogue of Genocchi polynomials as follows:

$$g_q(x,t) = [2]_q t \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \tag{2.10}$$

Thus, we note that $\lim_{q \to 1} g_q(x,t) = (2t/(e^t+1))e^{xt} = \sum_{n=0}^{\infty} G_n(x)(t^n/n!)$. From (2.10), we easily derive

$$G_{n,q}(x) = [2]_q n \left(\frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \frac{(-1)^l}{1+q^l} q^{lx} \binom{n-1}{l}. \tag{2.11}$$

By (2.10) we also see that

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} &= [2]_q t \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t} = [2]_q t \sum_{a=0}^{m-1} (-1)^a \sum_{k=0}^{\infty} (-1)^k e^{[k+(a+x)/m]_{q^m} [m]_q t} \\ &= \frac{[2]_q}{[m]_q [2]_{q^m}} \sum_{a=0}^{m-1} (-1)^a \left([m]_q t [2]_{q^m} \sum_{k=0}^{\infty} (-1)^k e^{[m]_q t [k+(a+x)/m]_{q^m}} \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{[2]_q}{[m]_q [2]_{q^m}} \sum_{a=0}^{m-1} (-1)^a [m]_q^n G_{n,q^m}\left(\frac{x+a}{m} \right) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{[2]_q}{[2]_{q^m}} [m]_q^{n-1} \sum_{a=0}^{m-1} (-1)^a G_{n,q^m}\left(\frac{x+a}{m} \right) \right) \frac{t^n}{n!}, \quad \text{where } m \in \mathbb{N} \text{ odd}. \end{aligned} \tag{2.12}$$

Therefore, we obtain the following theorem.

Theorem 2.5. *Let $m(=\text{odd}) \in \mathbb{N}$. Then the distribution of the q-Genocchi polynomials will be as follows:*

$$G_{n,q}(x) = \frac{[2]_q}{[2]_{q^m}}[m]_q^{n-1}\sum_{a=0}^{m-1}(-1)^a G_{n,q^m}\left(\frac{x+a}{m}\right), \tag{2.13}$$

where n is positive integer.

Theorem 2.5 will be used to construct the p-adic q-Genocchi measures which will be treated in the next section. Let χ be a primitive Dirichlet character with a conductor $d(=\text{odd}) \in \mathbb{N}$. Then the generalized q-Genocchi numbers attached to χ are defined as

$$g_{\chi,q}(t) = [2]_q t\sum_{a=0}^{d-1}\chi(n)(-1)^n e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,\chi,q}\frac{t^n}{n!}. \tag{2.14}$$

From (2.14), we derive

$$G_{n,\chi,q} = \frac{[2]_q}{[2]_{q^d}}[d]_q^{n-1}\sum_{a=0}^{d-1}(-1)^a\chi(a)G_{n,q^d}\left(\frac{a}{d}\right). \tag{2.15}$$

3. p-adic q-Genocchi measures

In this section we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x\log q)$. Let χ be a primitive Dirichlet's character with a conductor $d(=\text{odd}) \in \mathbb{N}$. For any positive integers N, k, and $d(=\text{odd})$, let $\mu_k = \mu_{k,q;G}$ be defined as

$$\mu_k(a+dp^N\mathbb{Z}_p) = (-1)^a[dp^N]_q^{k-1}\frac{[2]_q}{[2]_{q^{dp^N}}}G_{k,q^{dp^N}}\left(\frac{a}{dp^N}\right). \tag{3.1}$$

By using Theorem 2.5 and (3.1), we show that

$$\sum_{i=0}^{p-1}\mu_k(a+idp^N+dp^{N+1}\mathbb{Z}_p) = \mu_k(a+dp^N\mathbb{Z}_p). \tag{3.2}$$

Therefore, we obtain the following theorem.

Theorem 3.1. *Let d be an odd positive integer. For any positive integers N, k, and let $\mu_k = \mu_{k,q;G}$ be defined as*

$$\mu_k(a+dp^N\mathbb{Z}_p) = (-1)^a[dp^N]_q^{k-1}\frac{[2]_q}{[2]_{q^{dp^N}}}G_{k,q^{dp^N}}\left(\frac{a}{dp^N}\right). \tag{3.3}$$

Then μ_k can be extended to a distribution on X.

From the definition of μ_k and (2.15) we note that

$$\int_X \chi(x)d\mu_k(x) = G_{k,\chi,q}. \tag{3.4}$$

By (2.1) and (2.3), it is not difficult to show that

$$G_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} G_{k,q}. \tag{3.5}$$

From (3.1) and (3.5) we derive

$$d\mu_k(a) = \lim_{N\to\infty} \mu_k(a + dp^N \mathbb{Z}_p) = k[a]_q^{k-1} d\mu_{-q}(a). \tag{3.6}$$

Therefore, we obtain the following corollary.

Corollary 3.2. *Let k be a positive integer. Then,*

$$G_{k,\chi,q} = \int_X \chi(x) d\mu_k(x) = k \int_X \chi(x)[x]_q^{k-1} d\mu_{-q}(x). \tag{3.7}$$

Moreover,

$$G_{k,q} = k \int_X [x]_q^{k-1} d\mu_{-q}(x). \tag{3.8}$$

Remark 3.3. In the recent paper (see [1]), Cenkci et al. have studied q-Genocchi numbers and polynomials and p-adic q-Genocchi measures. Starting from T. Kim, L.-C. Jang, and H. K. Pak's construction of q-Genocchi numbers [7], they employed the method developed in a series of papers by Kim [see, e.g., [5, 14–16]] and they considerd another q-analogue of Genocchi numbers $G_k(q)$ as

$$G_k(q) = \frac{q(1+q)}{(1-q)^{k-1}} \sum_{m=0}^{k} \binom{k}{m} \frac{m(-1)^{m+1}}{1+q^m}, \tag{3.9}$$

which is easily derived from the generating function

$$F_q^{(G)}(t) = \sum_{k=0}^{\infty} G_k(q) \frac{t^k}{k!} = q(1+q)t \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]t}. \tag{3.10}$$

However, these q-Genocchi numbers and generating function do not seem to be natural ones; in particular, these numbers cannot be represented as a nice Witt's type formula for the p-adic invariant integral on $\mathbb{Z}_p$ and the generating function does not seems to be simple and useful for deriving many interesting identities related to q-Genocchi numbers. By this reason, we consider q-Genocchi numbers and polynomials which are different. Our q-Genocchi numbers and polynomials to treat in this paper can be represented by p-adic q-fermionic integral on $\mathbb{Z}_p$ [9, 13] and this integral representation also can be considered as Witt's type formula for q-Genocchi numbers. These formulae are useful to study congruences and worthwhile identities for q-Genocchi numbers. By using the generating function of our q-Genocchi numbers, we can derive many properties and identities as same as ordinary Genocchi numbers which were well known.

Acknowledgments

The author wishes to express his sincere gratitude to the referee for his/her valuable suggestions and comments and Professor Paolo E. Ricci for his cooperations and helps. This work was supported by Jangjeon Research Institute for Mathematical Science(JRIMS2005-005-C00001) and Jangjeon Mathematical Society.

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Taekyun Kim: Electrical Engineering Computer Science, Kyungpook National University, Taegu 702-701, South Korea
Email addresses: tkim@knu.ac.kr; tkim64@hanmail.net