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Research Article

Nonlinear Integral Inequalities in Two Independent Variables and Their Applications

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Recommended by Wing-Sum Cheung

This paper generalizes results of Cheung and Ma (2005) to more general inequalities with more than one distinct nonlinear term. From our results, some results of Cheung and Ma (2005) can be deduced as some special cases. Our results are also applied to show the boundedness of the solutions of a partial differential equation.

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1. Introduction

The integral inequalities play a fundamental role in the study of existence, uniqueness, boundedness, stability, invariant manifolds, and other qualitative properties of solutions of the theory of differential and integral equations. There are a lot of papers investigating them such as [1–8]. In particular, Pachpatte [2] discovered some new integral inequalities involving functions of two variables. These inequalities are applied to study the boundedness and uniqueness of the solutions of the following terminal value problem for the hyperbolic partial differential equation (1.1) with conditions (1.2):

$$D_1 D_2 u(x,y) = h(x,y,u(x,y)) + r(x,y), \tag{1.1}$$

$$u(x,\infty) = \sigma_\infty(x), \qquad u(\infty,y) = \tau_\infty(y), \qquad u(\infty,\infty) = k. \tag{1.2}$$

Cheung [9], and Dragomir and Kim [10, 11] established additional Gronwall-Ou-Iang type integral inequalities involving functions of two independent variables. Meng and Li [12] generalized the results of Pachpatte [2] to certain new integrals. Recently, Cheung and Ma[13] discussed the following inequalities

$$u(x,y) \le a(x,y) + c(x,y)\int_0^x \int_y^\infty d(s,t)w(u(s,t))\,dt\,ds,$$
$$u(x,y) \le a(x,y) + c(x,y)\int_x^\infty \int_y^\infty d(s,t)w(u(s,t))\,dt\,ds, \tag{1.3}$$

where $a(x,y)$ and $c(x,y)$ have certain monotonicity.

Our main aim here, motivated by the work of Cheung and Ma [13], is to discuss more general integral inequalities with n nonlinear terms:

$$u(x,y) \le a(x,y) + \sum_{i=1}^n \int_0^x \int_y^\infty d_i(x,y,s,t)w_i(u(s,t))\,dt\,ds, \tag{1.4}$$

$$u(x,y) \le a(x,y) + \sum_{i=1}^n \int_x^\infty \int_y^\infty d_i(x,y,s,t)w_i(u(s,t))\,dt\,ds, \tag{1.5}$$

where we do not require the monotonicity of $a(x,y)$ and $d_i(x,y,s,t)$. Furthermore, we also show that some results of Cheung and Ma [13] can be deduced from our results as some special cases. Our results are also applied to show the boundedness of the solutions of a partial differential equation.

2. Main results

Let $\mathbb{R} = (-\infty,\infty)$ and $\mathbb{R}_+ = [0,\infty)$. $D_1z(x,y)$ and $D_2z(x,y)$ denote the first-order partial derivatives of $z(x,y)$ with respect to x and y, respectively.

As in [1, 5, 6], we define $w_1 \propto w_2$ for $w_1, w_2 : A \subset \mathbb{R} \to \mathbb{R}\backslash\{0\}$ if w_2/w_1 is nondecreasing on A. This concept helps us compare monotonicity of different functions. Suppose that

(C_1) $w_i(u)$ $(i = 1,\dots,n)$ is a nonnegative, nondecreasing, and continuous function for $u \in \mathbb{R}_+$ with $w_i(u) > 0$ for $u > 0$ such that $w_1 \propto w_2 \propto \cdots \propto w_n$;

(C_2) $a(x,y)$ is a nonnegative and continuous function for $x,y \in \mathbb{R}_+$;

(C_3) $d_i(x,y,s,t)$ $(i = 1,\dots,n)$ is a continuous and nonnegative function for $x,y,s,t \in \mathbb{R}_+$.

Take the notation $W_i(u) := \int_{u_i}^u (dz/w_i(z))$, for $u \ge u_i$, where $u_i > 0$ is a given constant. Clearly, W_i is strictly increasing, so its inverse W_i^{-1} is well defined, continuous, and increasing in its corresponding domain.

Theorem 2.1. *In addition to the assumptions (C_1), (C_2), and (C_3), suppose that $a(x,y)$ and $d_i(x,y,s,t)$ are bounded in $y \in \mathbb{R}_+$ for each fixed $x,s,t \in \mathbb{R}_+$. If $u(x,y)$ is a continuous and nonnegative function satisfying (1.4) for $x,y \in \mathbb{R}_+$, then*

$$u(x,y) \le W_n^{-1}\left[W_n(b_n(x,y)) + \int_0^x \int_y^\infty \widetilde{d}_n(x,y,s,t)\,dt\,ds \right] \tag{2.1}$$

for all $0 \leq x \leq x_1$, $y_1 \leq y < \infty$, *where* $b_n(x,y)$ *is determined recursively by*

$$\begin{aligned} b_1(x,y) &= \widetilde{a}(x,y), \\ b_{i+1}(x,y) &= W_i^{-1}\left[W_i(b_i(x,y)) + \int_0^x \int_y^\infty \widetilde{d}_i(x,y,s,t)\,dt\,ds\right], \\ \widetilde{a}(x,y) = \sup_{0\leq\tau\leq x}\sup_{y\leq\mu<\infty} a(\tau,\mu), &\qquad \widetilde{d}_i(x,y,s,t) = \sup_{0\leq\tau\leq x}\sup_{y\leq\mu<\infty} d_i(\tau,\mu,s,t), \end{aligned} \tag{2.2}$$

$W_1(0) := 0$, *and* $x_1, y_1 \in \mathbb{R}_+$ *are chosen such that*

$$W_i(b_i(x_1,y_1)) + \int_0^{x_1}\int_{y_1}^\infty \widetilde{d}_i(x,y,s,t)\,dt\,ds \leq \int_{u_i}^\infty \frac{dz}{w_i(z)} \tag{2.3}$$

for $i = 1,\ldots,n$.

Remark 2.2. x_1 and y_1 are confined by (2.3). In particular, (2.1) is true for all $x, y \in \mathbb{R}_+$ when all w_i $(i = 1,\ldots,n)$ satisfy $\int_{u_i}^\infty (dz/w_i(z)) = \infty$.

Remark 2.3. As in [6, 5, 1], different choices of u_i in W_i do not affect our results.

Proof of Theorem 2.1. From the assumptions, we know that $\widetilde{a}(x,y)$ and $\widetilde{d}_i(x,y,s,t)$ are well defined. Moreover, $\widetilde{a}(x,y)$ and $\widetilde{d}_i(x,y,s,t)$ are nonnegative, nondecreasing in x, nonincreasing in y; and satisfy $\widetilde{a}(x,y) \geq a(x,y)$ and $\widetilde{d}_i(x,y,s,t) \geq d_i(x,y,s,t)$ for each $i = 1,\ldots,n$.

We first discuss the case that $a(x,y) > 0$ for all $x, y \in \mathbb{R}_+$. Thus, $b_1(x,y)$ is positive, nondecreasing in x, nonincreasing in y; and satisfies $b_1(x,y) \geq a(x,y)$ for all $x, y \in \mathbb{R}_+$. From (1.4), we have

$$u(x,y) \leq b_1(x,y) + \sum_{i=1}^n \int_0^x \int_y^\infty \widetilde{d}_i(x,y,s,t)\,w_i(u(s,t))\,dt\,ds. \tag{2.4}$$

Choose arbitrary $\widetilde{x}_1, \widetilde{y}_1$ such that $0 \leq \widetilde{x}_1 \leq x_1$, $y_1 \leq \widetilde{y}_1 < \infty$. From (2.4), we obtain

$$u(x,y) \leq b_1(\widetilde{x}_1,\widetilde{y}_1) + \sum_{i=1}^n \int_0^x \int_y^\infty \widetilde{d}_i(\widetilde{x}_1,\widetilde{y}_1,s,t)\,w_i(u(s,t))\,dt\,ds \tag{2.5}$$

for all $0 \leq x \leq \widetilde{x}_1 \leq x_1$, $y_1 \leq \widetilde{y}_1 \leq y < \infty$.

Having (2.5), we claim

$$u(x,y) \leq W_n^{-1}\left[W_n(\widetilde{b}_n(\widetilde{x}_1,\widetilde{y}_1,x,y)) + \int_0^x \int_y^\infty \widetilde{d}_n(\widetilde{x}_1,\widetilde{y}_1,s,t)\,dt\,ds\right] \tag{2.6}$$

for all $0 \leq x \leq \min\{\widetilde{x}_1, x_2\}$, $\max\{\widetilde{y}_1, y_2\} \leq y < \infty$, where

$$\begin{aligned} \widetilde{b}_1(\widetilde{x}_1,\widetilde{y}_1,x,y) &= b_1(\widetilde{x}_1,\widetilde{y}_1), \\ \widetilde{b}_{i+1}(\widetilde{x}_1,\widetilde{y}_1,x,y) &= W_i^{-1}\left[W_i(\widetilde{b}_i(\widetilde{x}_1,\widetilde{y}_1,x,y)) + \int_0^x \int_y^\infty \widetilde{d}_i(\widetilde{x}_1,\widetilde{y}_1,s,t)\,dt\,ds\right] \end{aligned} \tag{2.7}$$

for $i = 1,\dots,n-1$ and $x_2, y_2 \in \mathbb{R}_+$ are chosen such that

$$W_i(\widetilde{b}_i(\widetilde{x}_1,\widetilde{y}_1,x_2,y_2)) + \int_0^{x_2}\int_{y_2}^{\infty} \widetilde{d}_i(\widetilde{x}_1,\widetilde{y}_1,s,t)\,dt\,ds \le \int_{u_i}^{\infty} \frac{dz}{w_i(z)} \tag{2.8}$$

for $i = 1,\dots,n$.

Note that we may take $x_2 = x_1$ and $y_2 = y_1$. In fact, $\widetilde{b}_i(\widetilde{x}_1,\widetilde{y}_1,x,y)$ and $\widetilde{d}_i(\widetilde{x}_1,\widetilde{y}_1,x,y)$ are nondecreasing in $\widetilde{x}_1$, nonincreasing in $\widetilde{y}_1$ for fixed x, y. Furthermore, it is easy to check that $\widetilde{b}_i(\widetilde{x}_1,\widetilde{y}_1,\widetilde{x}_1,\widetilde{y}_1) = b_i(\widetilde{x}_1,\widetilde{y}_1)$ for $i = 1,\dots,n$. If x_2, y_2 are replaced by x_1, y_1 on the left side of (2.8), we have from (2.3)

$$\begin{aligned}
&W_i(\widetilde{b}_i(\widetilde{x}_1,\widetilde{y}_1,x_1,y_1)) + \int_0^{x_1}\int_{y_1}^{\infty} \widetilde{d}_i(\widetilde{x}_1,\widetilde{y}_1,s,t)\,dt\,ds \\
&\quad\le W_i(\widetilde{b}_i(x_1,y_1,x_1,y_1)) + \int_0^{x_1}\int_{y_1}^{\infty} \widetilde{d}_i(x_1,y_1,s,t)\,dt\,ds \\
&\quad= W_i(b_i(x_1,y_1)) + \int_0^{x_1}\int_{y_1}^{\infty} \widetilde{d}_i(x_1,y_1,s,t)\,dt\,ds \le \int_{u_i}^{\infty}\frac{dz}{w_i(z)}.
\end{aligned} \tag{2.9}$$

Thus, it means that we can take $x_2 = x_1$, $y_2 = y_1$.

In the following, we will use mathematical induction to prove (2.6).

For $n = 1$, let

$$z(x,y) = \int_0^{x}\int_{y}^{\infty} \widetilde{d}_1(\widetilde{x}_1,\widetilde{y}_1,s,t)\,w_1(u(s,t))\,dt\,ds. \tag{2.10}$$

Then $z(x,y)$ is differentiable, nonnegative, nondecreasing for $x \in [0,\widetilde{x}_1]$, and nonincreasing for $y \in [\widetilde{y}_1,\infty)$ and $z(0,y) = z(x,\infty) = 0$. From (2.5), we have the following:

$$\begin{aligned}
u(x,y) &\le b_1(\widetilde{x}_1,\widetilde{y}_1) + z(x,y), \\
D_1 z(x,y) &= \int_y^{\infty} \widetilde{d}_1(\widetilde{x}_1,\widetilde{y}_1,x,t)\,w_1(u(x,t))\,dt \\
&\le \int_y^{\infty} \widetilde{d}_1(\widetilde{x}_1,\widetilde{y}_1,x,t)\,w_1(b_1(\widetilde{x}_1,\widetilde{y}_1) + z(x,t))\,dt \\
&\le w_1(b_1(\widetilde{x}_1,\widetilde{y}_1) + z(x,y)) \int_y^{\infty} \widetilde{d}_1(\widetilde{x}_1,\widetilde{y}_1,x,t)\,dt.
\end{aligned} \tag{2.11}$$

Since w_1 is nondecreasing and $b_1(\widetilde{x}_1,\widetilde{y}_1) + z(x,y) > 0$, we get

$$\begin{aligned}
\frac{D_1(b_1(\widetilde{x}_1,\widetilde{y}_1) + z(x,y))}{w_1(b_1(\widetilde{x}_1,\widetilde{y}_1) + z(x,y))} &= \frac{D_1 z(x,y)}{w_1(b_1(\widetilde{x}_1,\widetilde{y}_1) + z(x,y))} \\
&\le \frac{w_1(b_1(\widetilde{x}_1,\widetilde{y}_1) + z(x,y)) \int_y^{\infty} \widetilde{d}_1(\widetilde{x}_1,\widetilde{y}_1,x,t)\,dt}{w_1(b_1(\widetilde{x}_1,\widetilde{y}_1) + z(x,y))} \\
&= \int_y^{\infty} \widetilde{d}_1(\widetilde{x}_1,\widetilde{y}_1,x,t)\,dt.
\end{aligned} \tag{2.12}$$

Integrating both sides of the above inequality from 0 to x, we obtain

$$\begin{aligned} W_1(b_1(\tilde{x}_1,\tilde{y}_1)+z(x,y)) &\le W_1(b_1(\tilde{x}_1,\tilde{y}_1)+z(0,y))+\int_0^x\int_y^\infty \tilde{d}_1(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds \\ &= W_1(b_1(\tilde{x}_1,\tilde{y}_1))+\int_0^x\int_y^\infty \tilde{d}_1(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds. \end{aligned} \tag{2.13}$$

Thus the monotonicity of W_1^{-1} implies

$$u(x,y)\le b_1(\tilde{x}_1,\tilde{y}_1)+z(x,y)\le W_1^{-1}\left[W_1(b_1(\tilde{x}_1,\tilde{y}_1))+\int_0^x\int_y^\infty \tilde{d}_1(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right], \tag{2.14}$$

that is, (2.6) is true for $n=1$.

Assume that (2.6) is true for $n=m$. Consider

$$u(x,y)\le b_1(\tilde{x}_1,\tilde{y}_1)+\sum_{i=1}^{m+1}\int_0^x\int_y^\infty \tilde{d}_i(\tilde{x}_1,\tilde{y}_1,s,t)\,w_i(u(s,t))\,dt\,ds \tag{2.15}$$

for all $0\le x\le\tilde{x}_1$, $\tilde{y}_1\le y<\infty$. Let

$$z(x,y)=\sum_{i=1}^{m+1}\int_0^x\int_y^\infty \tilde{d}_i(\tilde{x}_1,\tilde{y}_1,s,t)\,w_i(u(s,t))\,dt\,ds. \tag{2.16}$$

Then $z(x,y)$ is differentiable, nonnegative, nondecreasing for $x\in[0,\tilde{x}_1]$, and nonincreasing for $y\in[\tilde{y}_1,\infty)$. Obviously, $z(0,y)=z(x,\infty)=0$ and $u(x,y)\le b_1(\tilde{x}_1,\tilde{y}_1)+z(x,y)$. Since w_1 is nondecreasing and $b_1(\tilde{x}_1,\tilde{y}_1)+z(x,y)>0$, we have

$$\begin{aligned} &\frac{D_1(b_1(\tilde{x}_1,\tilde{y}_1)+z(x,y))}{w_1(b_1(\tilde{x}_1,\tilde{y}_1)+z(x,y))} \\ &\quad\le\frac{\sum_{i=1}^{m+1}\int_y^\infty \tilde{d}_i(\tilde{x}_1,\tilde{y}_1,x,t)\,w_i(u(x,t))\,dt}{w_1(b_1(\tilde{x}_1,\tilde{y}_1)+z(x,y))} \\ &\quad\le\frac{\sum_{i=1}^{m+1}\int_y^\infty \tilde{d}_i(\tilde{x}_1,\tilde{y}_1,x,t)\,w_i(b_1(\tilde{x}_1,\tilde{y}_1)+z(x,t))\,dt}{w_1(b_1(\tilde{x}_1,\tilde{y}_1)+z(x,y))} \\ &\quad\le\int_y^\infty \tilde{d}_1(\tilde{x}_1,\tilde{y}_1,x,t)\,dt+\sum_{i=2}^{m+1}\int_y^\infty \tilde{d}_i(\tilde{x}_1,\tilde{y}_1,x,t)\,\phi_i(b_1(\tilde{x}_1,\tilde{y}_1)+z(x,t))\,dt \\ &\quad\le\int_y^\infty \tilde{d}_1(\tilde{x}_1,\tilde{y}_1,x,t)\,dt+\sum_{i=1}^{m}\int_y^\infty \tilde{d}_{i+1}(\tilde{x}_1,\tilde{y}_1,x,t)\,\phi_{i+1}(b_1(\tilde{x}_1,\tilde{y}_1)+z(x,t))\,dt, \end{aligned} \tag{2.17}$$

where $\phi_{i+1}(u) = w_{i+1}(u)/w_1(u)$, $i = 1,\ldots,m$. Integrating the above inequality from 0 to x, we obtain

$$\begin{aligned} W_1(b_1(\tilde{x}_1,\tilde{y}_1)+z(x,y)) \le{}& W_1(b_1(\tilde{x}_1,\tilde{y}_1)) + \int_0^x\int_y^\infty \tilde{d}_1(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds \\ &+ \sum_{i=1}^m \int_0^x\int_y^\infty \tilde{d}_{i+1}(\tilde{x}_1,\tilde{y}_1,s,t)\phi_{i+1}(b_1(\tilde{x}_1,\tilde{y}_1)+z(s,t))\,dt\,ds, \end{aligned} \tag{2.18}$$

or

$$\xi(x,y) \le c_1(x,y) + \sum_{i=1}^m \int_0^x\int_y^\infty \tilde{d}_{i+1}(\tilde{x}_1,\tilde{y}_1,s,t)\phi_{i+1}(W_1^{-1}(\xi(s,t)))\,dt\,ds \tag{2.19}$$

for $0 \le x \le \tilde{x}_1$ and $\tilde{y}_1 \le y < \infty$, the same as (2.6) for $n = m$, where $\xi(x,y) = W_1(b_1(\tilde{x}_1,\tilde{y}_1) + z(x,y))$ and $c_1(x,y) = W_1(b_1(\tilde{x}_1,\tilde{y}_1)) + \int_0^x\int_y^\infty \tilde{d}_1(\tilde{x}_1,\tilde{y}_1,s,t)dt\,ds$.

From the assumption (C_1), each $\phi_{i+1}(W_1^{-1}(u))$, $i = 1,\ldots,m$, is continuous and nondecreasing for u. Moreover, $\phi_2(W_1^{-1}) \propto \phi_3(W_1^{-1}) \propto \cdots \propto \phi_{m+1}(W_1^{-1})$. By the inductive assumption, we have

$$\xi(x,y) \le \Phi_{m+1}^{-1}\left[\Phi_{m+1}(c_m(x,y)) + \int_0^x\int_y^\infty \tilde{d}_{m+1}(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right] \tag{2.20}$$

for all $0 \le x \le \min\{\tilde{x}_1, x_3\}$, $\max\{\tilde{y}_1, y_3\} \le y < \infty$, where $\Phi_{i+1}(u) = \int_{\tilde{u}_{i+1}}^u (dz/\phi_{i+1}(W_1^{-1}(z)))$, $u > 0, \tilde{u}_{i+1} = W_1(u_{i+1})$, Φ_{i+1}^{-1} is the inverse of Φ_{i+1}, $i = 1,\ldots,m$,

$$c_{i+1}(x,y) = \Phi_{i+1}^{-1}\left[\Phi_{i+1}(c_i(x,y)) + \int_0^x\int_y^\infty \tilde{d}_{i+1}(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right], \quad i = 1,\ldots,m, \tag{2.21}$$

and $x_3, y_3 \in \mathbb{R}_+$ are chosen such that

$$\Phi_{i+1}(c_i(x_3,y_3)) + \int_0^{x_3}\int_{y_3}^\infty \tilde{d}_{i+1}(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds \le \int_{\tilde{u}_{i+1}}^{W_1(\infty)} \frac{dz}{\phi_{i+1}(W_1^{-1}(z))} \tag{2.22}$$

for $i = 1,\ldots,m$.

Note that

$$\begin{aligned} \Phi_i(u) &= \int_{\tilde{u}_i}^u \frac{dz}{\phi_i(W_1^{-1}(z))} = \int_{W_1(u_i)}^u \frac{w_1(W_1^{-1}(z))dz}{w_i(W_1^{-1}(z))} \\ &= \int_{u_i}^{W_1^{-1}(u)} \frac{dz}{w_i(z)} = W_i \circ W_1^{-1}(u), \quad i = 2,\ldots,m+1. \end{aligned} \tag{2.23}$$

From (2.20), we have

$$\begin{aligned} u(x,y) &\le b_1(\tilde{x}_1,\tilde{y}_1) + z(x,y) = W_1^{-1}(\xi(x,y)) \\ &\le W_{m+1}^{-1}\left[W_{m+1}(W_1^{-1}(c_m(x,y))) + \int_0^x\int_y^\infty \tilde{d}_{m+1}(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right] \end{aligned} \tag{2.24}$$

for all $0 \le x \le \min\{\tilde{x}_1, x_3\}$, $\max\{\tilde{y}_1, y_3\} \le y < \infty$. Let $\tilde{c}_i(x,y) = W_1^{-1}(c_i(x,y))$. Then,

$$\begin{aligned}\tilde{c}_1(x,y) &= W_1^{-1}(c_1(x,y))\\ &= W_1^{-1}\left[W_1(b_1(\tilde{x}_1,\tilde{y}_1)) + \int_0^x\int_y^\infty \tilde{d}_1(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right]\\ &= \tilde{b}_2(\tilde{x}_1,\tilde{y}_1,x,y).\end{aligned} \tag{2.25}$$

Moreover, with the assumption that $\tilde{c}_m(x,y) = \tilde{b}_{m+1}(\tilde{x}_1,\tilde{y}_1,x,y)$, we have

$$\begin{aligned}\tilde{c}_{m+1}(x,y) &= W_1^{-1}\left[\Phi_{m+1}^{-1}\left(\Phi_{m+1}(c_m(x,y)) + \int_0^x\int_y^\infty \tilde{d}_{m+1}(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right)\right]\\ &= W_{m+1}^{-1}\left[W_{m+1}(W_1^{-1}(c_m(x,y))) + \int_0^x\int_y^\infty \tilde{d}_{m+1}(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right]\\ &= W_{m+1}^{-1}\left[W_{m+1}(\tilde{c}_m(x,y)) + \int_0^x\int_y^\infty \tilde{d}_{m+1}(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right]\\ &= W_{m+1}^{-1}\left[W_{m+1}(\tilde{b}_{m+1}(\tilde{x}_1,\tilde{y}_1,x,y)) + \int_0^x\int_y^\infty \tilde{d}_{m+1}(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right]\\ &= \tilde{b}_{m+2}(\tilde{x}_1,\tilde{y}_1,x,y).\end{aligned} \tag{2.26}$$

This proves that

$$\tilde{c}_i(x,y) = \tilde{b}_{i+1}(\tilde{x}_1,\tilde{y}_1,x,y), \quad i = 1,\dots,m. \tag{2.27}$$

Therefore, (2.22) becomes

$$\begin{aligned}&W_{i+1}(\tilde{b}_{i+1}(\tilde{x}_1,\tilde{y}_1,x_3,y_3)) + \int_0^{x_3}\int_{y_3}^\infty \tilde{d}_{i+1}(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\\ &\quad\le \int_{\tilde{u}_{i+1}}^{W_1(\infty)} \frac{dz}{\phi_{i+1}(W_1^{-1}(z))} = \int_{u_{i+1}}^\infty \frac{dz}{w_{i+1}(z)}, \quad i = 1,\dots,m.\end{aligned} \tag{2.28}$$

The above inequalities and (2.8) imply that we may take $x_2 = x_3$, $y_2 = y_3$. From (2.24), we get

$$u(x,y) \le W_{m+1}^{-1}\left[W_{m+1}(\tilde{b}_{m+1}(\tilde{x}_1,\tilde{y}_1,x,y)) + \int_0^x\int_y^\infty \tilde{d}_{m+1}(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right] \tag{2.29}$$

for all $0 \le x \le \tilde{x}_1 \le x_2$, $y_2 \le \tilde{y}_1 \le y < \infty$. This proves (2.6) by mathematical induction.

Taking $x = \tilde{x}_1, y = \tilde{y}_1$, $x_2 = x_1$, and $y_2 = y_1$, we have

$$u(\tilde{x}_1,\tilde{y}_1) \le W_n^{-1}\left[W_n(\tilde{b}_n(\tilde{x}_1,\tilde{y}_1,\tilde{x}_1,\tilde{y}_1)) + \int_0^{\tilde{x}_1}\int_{\tilde{y}_1}^\infty \tilde{d}_n(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right] \tag{2.30}$$

for $0 \le \tilde{x}_1 \le x_1, y_1 \le \tilde{y}_1 < \infty$. It is easy to verify $\tilde{b}_n(\tilde{x}_1,\tilde{y}_1,\tilde{x}_1,\tilde{y}_1) = b_n(\tilde{x}_1,\tilde{y}_1)$. Thus, (2.30) can be written as

$$u(\tilde{x}_1,\tilde{y}_1) \le W_n^{-1}\left[W_n(b_n(\tilde{x}_1,\tilde{y}_1)) + \int_0^{\tilde{x}_1}\int_{\tilde{y}_1}^{\infty} \tilde{d}_n(\tilde{x}_1,\tilde{y}_1,s,t)\,dt\,ds\right]. \tag{2.31}$$

Since $\tilde{x}_1$, $\tilde{y}_1$ are arbitrary, replace $\tilde{x}_1$ and $\tilde{y}_1$ by x and y respectively and we have

$$u(x,y) \le W_n^{-1}\left[W_n(b_n(x,y)) + \int_0^{x}\int_{y}^{\infty} \tilde{d}_n(x,y,s,t)\,dt\,ds\right] \tag{2.32}$$

for all $0 \le x \le x_1$, $y_1 \le y < \infty$.

In case $a(x,y) = 0$ for some $x,y \in \mathbb{R}_+$. Let $b_{1,\epsilon}(x,y) := b_1(x,y) + \epsilon$ for all $x,y \in \mathbb{R}_+$, where $\epsilon > 0$ is arbitrary, and then $b_{1,\epsilon}(x,y) > 0$. Using the same arguments as above, where $b_1(x,y)$ is replaced with $b_{1,\epsilon}(x,y) > 0$, we get

$$u(x,y) \le W_n^{-1}\left[W_n(b_{n,\epsilon}(x,y)) + \int_0^{x}\int_{y}^{\infty} \tilde{d}_n(x,y,s,t)\,dt\,ds\right]. \tag{2.33}$$

Letting $\epsilon \to 0^+$, we obtain (2.1) by the continuity of $b_{1,\epsilon}$ in ϵ and the continuity of W_i and W_i^{-1} under the notation $W_1(0) := 0$. □

Theorem 2.4. *In addition to the assumptions (C_1), (C_2), and (C_3), suppose that $a(x,y)$ and $d_i(x,y,s,t)$ are bounded in $x,y \in \mathbb{R}_+$ for each fixed $s,t \in \mathbb{R}_+$. If $u(x,y)$ is a continuous and nonnegative function satisfying (1.5) for $x,y \in \mathbb{R}_+$, then*

$$u(x,y) \le W_n^{-1}\left[W_n(b_n(x,y)) + \int_x^{\infty}\int_{y}^{\infty} \hat{d}_n(x,y,s,t)\,dt\,ds\right] \tag{2.34}$$

for all $x_4 \le x < \infty$, $y_4 \le y < \infty$, where $b_n(x,y)$ is determined recursively by

$$\begin{aligned} b_1(x,y) &= \hat{a}(x,y),\\ b_{i+1}(x,y) &= W_i^{-1}\left[W_i(b_i(x,y)) + \int_x^{\infty}\int_{y}^{\infty} \hat{d}_i(x,y,s,t)\,dt\,ds\right], \end{aligned} \tag{2.35}$$

$$\begin{aligned} \hat{a}(x,y) &= \sup_{x\le\tau<\infty}\ \sup_{y\le\mu<\infty} a(\tau,\mu),\\ \hat{d}_i(x,y,s,t) &= \sup_{x\le\tau<\infty}\ \sup_{y\le\mu<\infty} d_i(\tau,\mu,s,t), \end{aligned} \tag{2.36}$$

$W_1(0) := 0$, and $x_4, y_4 \in \mathbb{R}_+$ are chosen such that

$$W_i(b_i(x_4,y_4)) + \int_{x_4}^{\infty}\int_{y_4}^{\infty} \hat{d}_i(x,y,s,t)\,dt\,ds \le \int_{u_i}^{\infty} \frac{dz}{w_i(z)} \tag{2.37}$$

for $i = 1,\dots,n$.

The proof is similar to the argument in the proof of Theorem 2.1 with suitable modification. We omit the details here.

Remark 2.5. Take $d_1(x,y,s,t) = c(x,y)d(s,t)$ and $n = 1$ in (1.4). Suppose that $a(x,y)$ and $c(x,y)$ are continuous, nonnegative, nondecreasing in x and nonincreasing in y; and $d(s,t)$ is nonnegative and continuous. We note that

$$b_1(x,y) = a(x,y), \qquad \widetilde{d}_1(x,y,s,t) = c(x,y)d(s,t). \tag{2.38}$$

From Theorem 2.1, we get

$$u(x,y) \le W_1^{-1}\left[W_1(a(x,y)) + c(x,y)\int_0^x \int_y^\infty d(s,t)\,dt\,ds\right], \tag{2.39}$$

which is exactly (2.6) of Lemma 2.2 in [13].

Remark 2.6. Take $d_1(x,y,s,t) = c(x,y)d(s,t)$ and $n = 1$ in (1.5). Suppose that $a(x,y)$ and $c(x,y)$ are continuous, nonnegative, nonincreasing in x,y; and $d(s,t)$ is nonnegative and continuous. It is easy to check that

$$b_1(x,y) = a(x,y), \qquad \widehat{d}_1(x,y,s,t) = c(x,y)d(s,t). \tag{2.40}$$

From Theorem 2.4, we get

$$u(x,y) \le W_1^{-1}[W_1(a(x,y)) + c(x,y)\int_x^\infty \int_y^\infty d(s,t)\,dt\,ds] \tag{2.41}$$

which is (2.10) of Lemma 2.2 in [13].

3. Applications

Consider the partial differential equation

$$\begin{aligned} D_1D_2v(x,y) = {} & \frac{1}{(x+1)^2(y+1)^2} + \exp(-x)\exp(-y)\sqrt{|v(x,y)|+1} \\ & + x\exp(-x)\exp(-y)\mathfrak{T}v(x,y), \end{aligned} \tag{3.1}$$

$$v(x,\infty) = \sigma(x), v(0,y) = \tau(y), v(0,\infty) = k \tag{3.2}$$

for $x,y \in \mathbb{R}_+$, where $\sigma,\tau \in C(\mathbb{R}_+,\mathbb{R})$, $\sigma(x)$ is nondecreasing in x, $\tau(y)$ is nonincreasing in y, k is a real constant, and $\mathfrak{T}$ is a continuous operator on $C(\mathbb{R}_+ \times \mathbb{R}_+,\mathbb{R})$ such that $|\mathfrak{T}v| \le c_0|v|$ for a constant $c_0 > 0$. Integrating (3.1) with respect to x and y and using the initial conditions (3.2), we get

$$\begin{aligned} v(x,y) = {} & \sigma(x) + \tau(y) - k - \frac{x}{(x+1)(y+1)} \\ & - \int_0^x \int_y^\infty \exp(-s)\exp(-t)\sqrt{|v(s,t)|+1}\,dt\,ds \\ & - \int_0^x \int_y^\infty s\exp(-s)\exp(-t)\mathfrak{T}v(s,t)\,dt\,ds. \end{aligned} \tag{3.3}$$

Thus,

$$\begin{aligned}|v(x,y)| \le{}& |\sigma(x)+\tau(y)-k| + \frac{x}{(x+1)(y+1)} \\ &+ \int_0^x \int_y^\infty \exp(-s)\exp(-t)\sqrt{|v(s,t)|+1}\,dt\,ds \\ &+ \int_0^x \int_y^\infty s\exp(-s)\exp(-t)c_0|v(s,t)|\,dt\,ds.\end{aligned} \tag{3.4}$$

Letting $u(x,y) = |v(x,y)|$, we have

$$u(x,y) \le a(x,y) + \int_0^x \int_y^\infty d_1(x,y,s,t)w_1(u)\,dt\,ds + \int_0^x \int_y^\infty d_2(x,y,s,t)w_2(u)\,dt\,ds, \tag{3.5}$$

where $a(x,y) = |\sigma(x)+\tau(y)-k| + x/(x+1)(y+1)$, $w_1(u) = \sqrt{u+1}$, $w_2(u) = c_0u$, $d_1(x,y,s,t) = \exp(-s)\exp(-t)$, $d_2(x,y,s,t) = s\exp(-s)\exp(-t)$. Clearly, $w_2(u)/w_1(u) = c_0(u/\sqrt{u+1})$ is nondecreasing for $u > 0$, that is, $w_1 \propto w_2$. Then for $u_1, u_2 > 0$,

$$\begin{aligned}b_1(x,y) &= a(x,y), \qquad \tilde{d}_1(x,y,s,t) = d_1(x,y,s,t), \qquad \tilde{d}_2(x,y,s,t) = d_2(x,y,s,t), \\ W_1(u) &= \int_{u_1}^u \frac{dz}{\sqrt{z+1}} = 2\left(\sqrt{u+1}-\sqrt{u_1+1}\right), \qquad W_1^{-1}(u) = \left(\frac{u}{2}+\sqrt{u_1+1}\right)^2 - 1, \\ W_2(u) &= \int_{u_2}^u \frac{dz}{c_0 z} = \frac{1}{c_0}\ln\frac{u}{u_2}, \qquad W_2^{-1}(u) = u_2\exp(c_0u), \\ b_2(x,y) &= W_1^{-1}\left[W_1(b_1(x,y)) + \int_0^x \int_y^\infty \tilde{d}_1(x,y,s,t)\,dt\,ds\right] \\ &= W_1^{-1}\left[2\left(\sqrt{b_1(x,y)+1}-\sqrt{u_1+1}\right) + (1-\exp(-x))\exp(-y)\right] \\ &= \left[\sqrt{b_1(x,y)+1} + \frac{1-\exp(-x)}{2}\exp(-y)\right]^2 - 1.\end{aligned} \tag{3.6}$$

By Theorem 2.1, we have

$$\begin{aligned}|v(x,y)| &\le W_2^{-1}\left[W_2(b_2(x,y)) + \int_0^x \int_y^\infty \tilde{d}_2(x,y,s,t)\,dt\,ds\right] \\ &= W_2^{-1}\left[\frac{1}{c_0}\ln\frac{b_2(x,y)}{u_2} + (1-(x+1)\exp(-x))\exp(-y)\right] \\ &= u_2\exp\left[c_0\left(\frac{1}{c_0}\ln\frac{b_2(x,y)}{u_2} + (1-(x+1)\exp(-x))\exp(-y)\right)\right] \\ &= b_2(x,y)\exp\left[c_0(1-(x+1)\exp(-x))\exp(-y)\right] \\ &= \left[\left(\sqrt{|\sigma(x)+\tau(y)-k| + \frac{x}{(x+1)(y+1)}+1} + \frac{1-\exp(-x)}{2}\exp(-y)\right)^2 - 1\right] \\ &\quad\times \exp\left[c_0(1-(x+1)\exp(-x))\exp(-y)\right].\end{aligned} \tag{3.7}$$

This implies that the solution of (3.1) is bounded for $x, y \in \mathbb{R}_+$ provided that $\sigma(x) + \tau(y) - k$ is bounded for all $x, y \in \mathbb{R}_+$.

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