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Research Article

Weighted Composition Operators between Mixed Norm Spaces and H_α^∞ Spaces in the Unit Ball

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Let φ be an analytic self-map and let u be a fixed analytic function on the open unit ball B in $\mathbb{C}^n$. The boundedness and compactness of the weighted composition operator $uC_\varphi f = u\cdot(f\circ\varphi)$ between mixed norm spaces and H_α^∞ are studied.

1. Introduction

Let B be the open unit ball in $\mathbb{C}^n$, $\partial B = S$ its boundary, $\mathbb{D}$ the unit disk in $\mathbb{C}$, dV the normalized Lebesgue volume measure on B, $d\sigma$ the normalized surface measure on S, and $H(B)$ the class of all functions analytic on B.

An analytic self-map $\varphi : B\to B$ induces the composition operator C_φ on $H(B)$, defined by $C_\varphi(f)(z) = f(\varphi(z))$ for $f\in H(B)$. It is interesting to provide a functional theoretic characterization of when φ induces a bounded or compact composition operator on various spaces. The book [1] contains a plenty of information on this topic. Let u be a fixed analytic function on the open unit ball. Define a linear operator uC_φ, called a weighted composition operator, by $uC_\varphi f = u\cdot(f\circ\varphi)$, where f is an analytic function on B. We can regard this operator as a generalization of the multiplication operator $M_u(f) = uf$ and a composition operator.

A positive continuous function ϕ on $[0,1)$ is called normal if there exist numbers s and t, $0 < s < t$, such that $\phi(r)/(1-r)^s$ decreasingly converges to zero and $\phi(r)/(1-r)^t$ increasingly tends to ∞, as $r\to 1^-$ (see, e.g., [2]).

For $0 < p < \infty$, $0 < q < \infty$, and a normal function ϕ, let $H(p,q,\phi)$ denote the space of all $f\in H(B)$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f,r) \frac{\phi^p(r)}{1-r} dr \right)^{1/p} < \infty, \tag{1.1}$$

where $M_q(f,r) = (\int_S |f(r\zeta)|^q d\sigma(\zeta))^{1/q}$, $0 \le r < 1$.

For $1 \le p < \infty$, $H(p,q,\phi)$, equipped with the norm $\|\cdot\|_{H(p,q,\phi)}$, is a Banach space. When $0 < p < 1$, $\|f\|_{H(p,q,\phi)}$ is a quasinorm on $H(p,q,\phi)$, and $H(p,q,\phi)$ is a Frechet space but not a Banach space. Note that if $0 < p = q < \infty$, then $H(p,p,\phi)$ becomes a Bergman-type space, and if $\phi(r) = (1-r)^{(\gamma+1)/p}$, $\gamma > -1$, then $H(p,p,\phi)$ is equivalent to the classical weighted Bergman space $A_\gamma^p(B)$.

For $\alpha \ge 0$, we define the weighted space $H_\alpha^\infty(B) = H_\alpha^\infty$ as the subspace of $H(B)$ consisting of all f such that $\|f\|_{H_\alpha^\infty} = \sup_{z\in B}(1-|z|^2)^\alpha |f(z)| < \infty$. Note that for $\alpha = 0$, H_α^∞ becomes the space of all bounded analytic functions $H^\infty(B)$. We also define a little version of H_α^∞, denoted by $H_{\alpha,0}^\infty(B)$, as the subset of H_α^∞ consisting of all $f \in H(B)$ such that $\lim_{|z|\to 1-0}(1-|z|^2)^\alpha |f(z)| = 0$. It is easy to see that $H_{\alpha,0}^\infty$ is a subspace of H_α^∞. Note also that for $\alpha = 0$, in view of the maximum modulus theorem, we obtain $H_{0,0}^\infty = \{0\}$.

For the case of the unit disk, in [3], Ohno has characterized the boundedness and compactness of weighted composition operators between H^∞ and the Bloch space $\mathscr{B}$ and the little Bloch space $\mathscr{B}_0$. In [4], Li and Stević extend the main results in [3] in the settings of the unit ball. In [5], A. K. Sharma and S. D. Sharma studied the boundedness and compactness of $uC_\varphi : H_\alpha^\infty(\mathbb{D}) \to A_\gamma^p(\mathbb{D})$ for the case of $p \ge 1$. For related results in the setting of the unit ball, see, for example, [1, 6, 7] and the references therein.

Here, we study the weighted composition operators between the mixed norm spaces $H(p,q,\phi)$ and H_α^∞ (or $H_{\alpha,0}^\infty$). As corollaries, we obtain the complete characterizations of the boundedness and compactness of composition operators between Bergman spaces and H^∞.

In this paper, positive constants are denoted by C; they may differ from one occurrence to the next. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary results

In this section, we give some auxiliary results which will be used in proving the main results of the paper. They are incorporated in the lemmas which follow.

Lemma 2.1. *Assume that $f \in H(p,q,\phi)(B)$. Then there is a positive constant C independent of f such that*

$$|f(z)| \le C \frac{\|f\|_{H(p,q,\phi)}}{(1-|z|)^{n/q}\phi(|z|)}. \tag{2.1}$$

Proof. By the monotonicity of the integral means, the following asymptotic relations:

$$\phi(|z|) \asymp \phi(|w|), \qquad w \in B(z, 3(1-|z|)/4),$$
$$1-r \asymp 1-|z|, \qquad r \in [(1+|z|)/2, (3+|z|)/4], \tag{2.2}$$

and [8, Theorem 7.2.5], we have

$$\begin{aligned} \|f\|^p_{H(p,q,\phi)} &\geq \int_{(1+|z|)/2}^{(3+|z|)/4} M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr \geq M_q^p(f,(1+|z|)/2)\int_{(1+|z|)/2}^{(3+|z|)/4} \frac{\phi^p(r)}{1-r}dr \\ &\geq C(1-|z|)^{pn/q}\phi^p(|z|)\,|f(z)|^p, \end{aligned} \tag{2.3}$$

from which the result follows. □

Corollary 2.2. *Assume that $f \in H(p,q,\phi)(B)$. Then*

$$\lim_{|z|\to 1-0}(1-|z|)^{n/q}\phi(|z|)\,|f(z)| = 0. \tag{2.4}$$

Proof. It can be proved in a standard way (see, e.g., [9, Theorem 2]) that

$$\lim_{r\to 1-0}\|f-f_r\|_{H(p,q,\phi)} = 0, \tag{2.5}$$

where $f_r(z) = f(rz)$, $r \in (0,1)$. Also since $f \in H(p,q,\phi)$, by the monotonicity of the integral means, we have $f_r \in H(p,q,\phi)$, for every $r \in (0,1)$.

From this and by inequality (2.1), we have that for each $r \in (0,1)$,

$$(1-|z|)^{n/q}\phi(|z|)\,|f(z)| \leq |f_r(z)|\,(1-|z|)^{n/q}\phi(|z|) + C\|f-f_r\|_{H(p,q,\phi)}. \tag{2.6}$$

From (2.5), we have that for every $\varepsilon > 0$ there is an $r_0 \in (0,1)$ such that

$$\|f-f_r\|_{H(p,q,\phi)} < \varepsilon, \quad r \in [r_0, 1). \tag{2.7}$$

If we take $r = r_0$ in (2.6) and employ (2.7) and the normality of ϕ, the result follows. □

Lemma 2.3. *For $\beta > -1$ and $m > 1+\beta$, one has*

$$\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^m}dr \leq C(1-\rho)^{1+\beta-m}, \quad 0 < \rho < 1. \tag{2.8}$$

The following criterion for compactness is followed by standard arguments.

Lemma 2.4. *The operator $uC_\varphi : H(p,q,\phi)\to H^\infty_\alpha$ (or $H^\infty_\alpha \to H(p,q,\phi)$) is compact if and only if for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $H(p,q,\phi)$ (corresp. H^∞_α), which converges to zero uniformly on compact subsets of B as $k\to\infty$, one has $\|uC_\varphi f_k\|_{H^\infty_\alpha}\to 0$ as $k\to\infty$ (corresp. $\|uC_\varphi f_k\|_{H(p,q,\phi)}\to 0$ as $k\to\infty$).*

In order to investigate the compactness of the operator $uC_\varphi : H(p,q,\phi)\to H^\infty_{\alpha,0}$, we need the following lemma which can be proved similar to [10, Lemma 1].

Lemma 2.5. *Assume that $K \subset H^\infty_{\alpha,0}$ is a closed bounded set. Then it is compact if and only if* $\lim_{|z|\to 1-0}\sup_{f\in K}(1-|z|^2)^\alpha|f(z)|=0$.

3. The boundedness and compactness of $uC_\varphi : H(p,q,\phi)\to H^\infty_\alpha$

In this section, we characterize the boundedness and compactness of the weighted composition operator $uC_\varphi : H(p,q,\phi)\to H^\infty_\alpha$.

Theorem 3.1. *Suppose that φ is an analytic self-map of the unit ball, $u\in H(B)$, $0<p,\ q<\infty$, and ϕ is normal on $[0,1)$. Then, $uC_\varphi : H(p,q,\phi)\to H^\infty_\alpha$ is bounded if and only if*

$$\sup_{z\in B}\frac{(1-|z|^2)^\alpha|u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}}<\infty. \tag{3.1}$$

Proof. Suppose that the condition (3.1) holds. Then for arbitrary $z\in B$ and $f\in H(p,q,\phi)$, by Lemma 2.1 we have

$$(1-|z|^2)^\alpha|(uC_\varphi f)(z)|\le C\frac{(1-|z|^2)^\alpha|u(z)|}{(1-|\varphi(z)|^2)^{n/q}\phi(|\varphi(z)|)}\|f\|_{H(p,q,\phi)}. \tag{3.2}$$

Taking the supremum in (3.2) over B and then using condition (3.1), we obtain that the operator $uC_\varphi : H(p,q,\phi)\to H^\infty_\alpha$ is bounded.

Conversely, suppose that $uC_\varphi : H(p,q,\phi)\to H^\infty_\alpha$ is bounded. For fixed $w\in B$, take

$$f_w(z)=\frac{(1-|w|^2)^{t+1}}{\phi(|w|)(1-\langle z,w\rangle)^{n/q+t+1}}. \tag{3.3}$$

By [8, Lemma 1.4.10], since ϕ is normal, and by Lemma 2.3, we obtain

$$\begin{aligned}
&||f_w||^p_{H(p,q,\phi)}\\
&\quad=\int_0^1 M_q^p(f_w,r)\frac{\phi^p(r)}{1-r}dr\le C\int_0^1\frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr\\
&\quad\le C\left(\int_0^{|w|}\frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr+\int_{|w|}^1\frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr\right)\\
&\quad\le C(1-|w|^2)^p\int_0^{|w|}\frac{(1-r)^{pt-1}}{(1-r|w|)^{p(t+1)}}dr+C(1-|w|^2)^p\int_{|w|}^1\frac{(1-r)^{ps-1}}{(1-r|w|)^{p(t+1)}}dr\le C.
\end{aligned} \tag{3.4}$$

Therefore $f_w\in H(p,q,\phi)$, and moreover $\sup_{w\in B}\|f_w\|_{H(p,q,\phi)}\le C$. Hence we have

$$(1-|z|^2)^\alpha|u(z)f_w(\varphi(z))|\le||uC_\varphi f_w||_{H^\infty_\alpha}\le C||f_w||_{H(p,q,\phi)}||uC_\varphi||\le C||uC_\varphi|| \tag{3.5}$$

for every $z\in B$, and $w\in B$. From this with $w=\varphi(z)$, (3.1) follows. □

Theorem 3.2. *Suppose that φ is an analytic self-map of the unit ball, $u \in H(B)$, $0 < p,\ q < \infty$, ϕ is normal on $[0,1)$, and $uC_\varphi : H(p,q,\phi)\to H_\alpha^\infty$ is bounded. Then $uC_\varphi : H(p,q,\phi)\to H_\alpha^\infty$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}} = 0. \tag{3.6}$$

Proof. First assume that condition (3.6) holds. Assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $H(p,q,\phi)$ with $\sup_{k\in\mathbb{N}}\|f_k\|_{H(p,q,\phi)} \le L$ and suppose that $f_k\to 0$ uniformly on compact subsets of B as $k\to\infty$. We prove that $\|uC_\varphi f_k\|_{H_\alpha^\infty}\to 0$ as $k\to\infty$.

First note that since $uC_\varphi(H(p,q,\phi)) \subseteq H_\alpha^\infty$, for $f \equiv 1 \in H(p,q,\phi)$, we obtain $uC_\varphi(1) = u \in H_\alpha^\infty$. From (3.6), we have that for every $\varepsilon > 0$, there is a constant $\delta \in (0,1)$ such that $\delta < |\varphi(z)| < 1$ implies that

$$\frac{(1-|z|^2)^\alpha |u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}} < \varepsilon/L. \tag{3.7}$$

Let $\delta B = \{w \in B : |w| \le \delta\}$. From (3.7), since ϕ is normal, and using the estimate in Lemma 2.1, we have that

$$\begin{aligned} &\|uC_\varphi f_k\|_{H_\alpha^\infty} \\ &\quad\le \sup_{\varphi(z)\in\delta B}(1-|z|^2)^\alpha |u(z)f_k(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)^\alpha |u(z)f_k(\varphi(z))| \\ &\quad\le \|u\|_{H_\alpha^\infty}\sup_{w\in\delta B}|f_k(w)| + \sup_{\delta<|\varphi(z)|<1}\frac{C(1-|z|^2)^\alpha |u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}}\|f_k\|_{H(p,q,\phi)} \\ &\quad\le \|u\|_{H_\alpha^\infty}\sup_{w\in\delta B}|f_k(w)| + C\varepsilon. \end{aligned} \tag{3.8}$$

Since δB is compact and by the assumption, it follows that $\lim_{k\to\infty}\sup_{w\in\delta B}|f_k(w)| = 0$. Using this fact and letting $k\to\infty$ in (3.8), we obtain that $\limsup_{k\to\infty}\|uC_\varphi f_k\|_{H_\alpha^\infty} \le C\varepsilon$. Since ε is an arbitrary positive number, it follows that the last limit is equal to zero. Therefore by Lemma 2.4, the operator $uC_\varphi : H(p,q,\phi)\to H_\alpha^\infty$ is compact.

Conversely, suppose that $uC_\varphi : H(p,q,\phi)\to H_\alpha^\infty$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. If such a sequence does not exist, condition (3.6) is automatically satisfied. Let $f_k(z) = f_{\varphi(z_k)}(z)$, $k \in \mathbb{N}$, where f_w is defined in (3.3). We know that $\sup_{k\in\mathbb{N}}\|f_k\|_{H(p,q,\phi)} \le C$ and f_k converges to 0 uniformly on compacts of B as $k\to\infty$. Since uC_φ is compact, we have $\lim_{k\to\infty}\|uC_\varphi f_k\|_{H_\alpha^\infty} = 0$. From this and since

$$\frac{(1-|z_k|^2)^\alpha |u(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{n/q}} \le \sup_{z\in B}(1-|z|^2)^\alpha |u(z)|\,|f_k(\varphi(z))| = \|uC_\varphi f_k\|_{H_\alpha^\infty}, \tag{3.9}$$

condition (3.6) holds, finishing the proof of the theorem. □

From Theorems 3.1 and 3.2, we easily obtain the following corollaries.

Corollary 3.3. *Suppose that φ is an analytic self-map of the unit ball, $0 < p,\ q < \infty$, and ϕ is normal on $[0,1)$. Then the following statements hold true.*

(a) *The composition operator $C_\varphi : H(p,q,\phi) \to H_\alpha^\infty$ is bounded if and only if*

$$\sup_{z\in B} \frac{(1-|z|^2)^\alpha}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}} < \infty. \tag{3.10}$$

(b) *If $C_\varphi : H(p,q,\phi) \to H_\alpha^\infty$ is bounded, then $C_\varphi : H(p,q,\phi) \to H_\alpha^\infty$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}} = 0. \tag{3.11}$$

Corollary 3.4. *Suppose that φ is an analytic self-map of the unit ball, $u \in H(B)$, and $0 < p < \infty$. Then the following statements hold true.*

(a) *$uC_\varphi : A^p \to H_\alpha^\infty$ is bounded if and only if*

$$\sup_{z\in B} \frac{(1-|z|^2)^\alpha |u(z)|}{(1-|\varphi(z)|^2)^{(n+1)/p}} < \infty. \tag{3.12}$$

(b) *If $uC_\varphi : A^p \to H_\alpha^\infty$ is bounded, then $uC_\varphi : A^p \to H_\alpha^\infty$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |u(z)|}{(1-|\varphi(z)|^2)^{(n+1)/p}} = 0. \tag{3.13}$$

In particular, $C_\varphi : A^p \to H^\infty$ is bounded if and only if $\sup_{z\in B}|\varphi(z)| < 1$.

Recall that the β-Bloch space $\mathcal{B}^\beta(B) = \mathcal{B}^\beta$ is the space of all $f \in H(B)$ such that $\|f\|_{\mathcal{B}^\beta} = |f(0)| + \sup_{z\in B}(1-|z|^2)^\beta |\mathcal{R}f(z)| < \infty$, where $\mathcal{R}f(z) = \sum_{j=1}^n z_j(\partial f/\partial z_j)(z)$ (see [6]), and the little β-Bloch space $\mathcal{B}_0^\beta(B) = \mathcal{B}_0^\beta$ is the space of all $f \in H(B)$ such that $\lim_{|z|\to 1}(1-|z|^2)^\beta |\mathcal{R}f(z)| = 0$. Using the following well-known asymptotic relationship: $\|f\|_{H_\alpha^\infty} \asymp \|f\|_{\mathcal{B}^{\alpha+1}}$, $\alpha > 0$, we obtain that the next results hold true.

Corollary 3.5. *Suppose that φ is an analytic self-map of the unit ball, $u \in H(B)$, $0 < p,\ q < \infty$, and ϕ is normal on $[0,1)$. Then the following statements hold true.*

(a) *$uC_\varphi : H(p,q,\phi) \to \mathcal{B}^\beta$, $\beta > 1$, is bounded if and only if*

$$\sup_{z\in B} \frac{(1-|z|^2)^{\beta-1} |u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}} < \infty. \tag{3.14}$$

(b) *If $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^\beta$, $\beta > 1$, is bounded, then $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^\beta$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{\beta-1} |u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}} = 0. \tag{3.15}$$

4. The boundedness and compactness of $uC_\varphi : H(p,q,\phi)\to H^\infty_{\alpha,0}$

In this section, we study the boundedness and compactness of the operator $uC_\varphi : H(p,q,\phi)\to H^\infty_{\alpha,0}$.

Theorem 4.1. *Suppose that φ is an analytic self-map of the unit ball, $u \in H(B)$, $0 < p,\ q < \infty$, and ϕ is normal on $[0,1)$. Then $uC_\varphi : H(p,q,\phi)\to H^\infty_{\alpha,0}$ is bounded if and only if condition (3.1) holds and $u \in H^\infty_{\alpha,0}$.*

Proof. First assume that the operator $uC_\varphi : H(p,q,\phi)\to H^\infty_{\alpha,0}$ is bounded. Then from the proof of Theorem 3.1, it follows that (3.1) holds. Clearly $uC_\varphi(1) = u \in H^\infty_{\alpha,0}$.

Now assume that condition (3.1) holds and $u \in H^\infty_{\alpha,0}$. Then in view of Theorem 3.1 , we have that the operator $uC_\varphi : H(p,q,\phi)\to H^\infty_\alpha$ is bounded. Hence it is enough to prove that $uC_\varphi(f) \in H^\infty_{\alpha,0}$ for every $f \in H(p,q,\phi)$.

From (2.4), we have that for every $\varepsilon > 0$ there is a $\delta \in (0,1)$ such that for $\delta < |z| < 1$,

$$|f(z)| < \frac{\varepsilon}{(1-|z|^2)^{n/q}\phi(|z|)}. \tag{4.1}$$

On the other hand, since $u \in H^\infty_{\alpha,0}$, for the above chosen ε, there is $r \in (\delta,1)$ such that for $r < |z| < 1$,

$$(1-|z|^2)^\alpha |u(z)| < \varepsilon(1-\delta^2)^{n/q}\phi(\delta). \tag{4.2}$$

From (4.1), we have that

$$(1-|z|^2)^\alpha |u(z)|\,|f(\varphi(z))| \le \varepsilon\frac{(1-|z|^2)^\alpha |u(z)|}{(1-|\varphi(z)|^2)^{n/q}\phi(|\varphi(z)|)}, \tag{4.3}$$

for $r < |z| < 1$ and $\delta < |\varphi(z)| < 1$.

On the other hand, combining (3.2) and (4.2), and using the fact that ϕ is normal, we have

$$(1-|z|^2)^\alpha |(uC_\varphi f)(z)| \le \frac{C(1-\delta^2)^s(1-|z|^2)^\alpha |u(z)|}{(1-|\varphi(z)|^2)^{n/q+s}\phi(\delta)}\|f\|_{H(p,q,\phi)} \le C\varepsilon\|f\|_{H(p,q,\phi)}, \tag{4.4}$$

when $r < |z| < 1$ and $|\varphi(z)| \le \delta$. From (3.1), (4.3), and (4.4), the result follows. □

Theorem 4.2. *Suppose that φ is an analytic self-map of the unit ball, $u \in H(B)$, $0 < p,\ q < \infty$, ϕ is normal on $[0,1)$, and $uC_\varphi : H(p,q,\phi)\to H^\infty_\alpha$ is bounded. Then $uC_\varphi : H(p,q,\phi)\to H^\infty_{\alpha,0}$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha |u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}} = 0. \tag{4.5}$$

Proof. Taking supremum in (3.2) over the unit ball in $H(p,q,\phi)$, using (4.5), and applying Lemma 2.5, we obtain that $uC_\varphi : H(p,q,\phi)\to H^\infty_{\alpha,0}$ is compact.

Assume now that $uC_\varphi : H(p,q,\phi) \to H^\infty_{\alpha,0}$ is compact. Then by Theorem 3.2, we have that condition (3.6) holds, which implies that for every $\varepsilon > 0$ there is an $r \in (0,1)$ such that for $r < |\varphi(z)| < 1$,

$$\frac{(1-|z|^2)^\alpha |u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}} < \varepsilon. \tag{4.6}$$

On the other hand, we know that $u \in H^\infty_{\alpha,0}$. Hence there is a $\sigma \in (0,1)$ such that for $\sigma < |z| < 1$,

$$(1-|z|^2)^\alpha |u(z)| < \varepsilon (1-r^2)^{n/q} \phi(r). \tag{4.7}$$

Hence if $|\varphi(z)| \le r$ and $\sigma < |z| < 1$, then from (4.7) and since ϕ is normal, we get

$$\frac{(1-|z|^2)^\alpha |u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n/q}} < \frac{(1-r^2)^s (1-|z|^2)^\alpha |u(z)|}{\phi(r)(1-|\varphi(z)|^2)^{n/q+s}} < \varepsilon. \tag{4.8}$$

From (4.8), and since for $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, (4.6) holds, we get (4.5). □

5. The boundedness and compactness of $uC_\varphi : H^\infty_\alpha \to H(p,q,\phi)$

In this section, we characterize the boundedness and compactness of the operator $uC_\varphi : H^\infty_\alpha \to H(p,q,\phi)$.

Theorem 5.1. *Suppose that φ is an analytic self-map of the unit ball, $u \in H(B)$, $0 < p, q < \infty$, and ϕ is normal on $[0,1)$. Then $uC_\varphi : H^\infty \to H(p,q,\phi)$ is bounded if and only if $uC_\varphi : H^\infty \to H(p,q,\phi)$ is compact if and only if $u \in H(p,q,\phi)$.*

Proof. First note that every compact operator is bounded. Second, since $f(z) \equiv 1 \in H^\infty$, from the boundedness of $uC_\varphi : H^\infty \to H(p,q,\phi)$, we have $uC_\varphi(1) = u \in H(p,q,\phi)$. Hence we should only prove that $u \in H(p,q,\phi)$ implies the compactness of the operator $uC_\varphi : H^\infty \to H(p,q,\phi)$. To this end, note that $\|uC_\varphi(f)\|_{H(p,q,\phi)} \le \|f\|_\infty \|u\|_{H(p,q,\phi)}$, for every $f \in H^\infty$, which implies the boundedness of the operator $uC_\varphi : H^\infty \to H(p,q,\phi)$.

Now assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in H^∞ such that $\sup_{k\in\mathbb{N}} \|f_k\|_\infty \le L < \infty$ and $f_k \to 0$ uniformly on compacts of B. We show that $\lim_{k\to\infty} \|uC_\varphi(f_k)\|_{H(p,q,\phi)} = 0$. Let

$$I_k(r) = \left(\int_S |u(r\zeta) f_k(\varphi(r\zeta))|^q d\sigma(\zeta) \right)^{p/q}, \quad k \in \mathbb{N}. \tag{5.1}$$

Then since $\varphi \in H(B)$, we have that the set $\varphi(rS)$ is compact for every $r \in [0,1)$. Hence $u(r\zeta) f_k(\varphi(r\zeta)) \to 0$ uniformly on S, and consequently $\lim_{k\to\infty} I_k(r) = 0$, for every $r \in [0,1)$. On the other hand, it is clear that $I_k(r) \le L^p M_q^p(u,r) = g(r)$, $r \in [0,1)$, and since $u \in H(p,q,\phi)$, it follows that $g \in \mathcal{L}^1([0,1),(\phi^p(r)/(1-r))dr)$. Hence by employing the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty} \|uC_\varphi(f_k)\|^p_{H(p,q,\phi)} = \lim_{k\to\infty} \int_0^1 I_k(r) \frac{\phi^p(r)}{1-r} dr = \int_0^1 \lim_{k\to\infty} I_k(r) \frac{\phi^p(r)}{1-r} dr = 0. \tag{5.2}$$

By Lemma 2.4, the compactness of $uC_\varphi : H^\infty \to H(p,q,\phi)$ follows. □

The case $\alpha > 0$ is somewhat complicated and we do not have an equivalent condition for the boundedness of $uC_\varphi : H_\alpha^\infty \to H(p,q,\phi)$ at the moment. Using the argument in the proof of Theorem 5.1 and the family of test functions $f_w(z) = (1 - \langle z, w\rangle)^{-\alpha}$, $w \in B$, we get the following result. We omit the details of the proof.

Theorem 5.2. *Suppose that φ is an analytic self-map of the unit ball, $u \in H(B)$, $0 < \alpha$, p, $q < \infty$, and ϕ is normal on $[0,1)$. Then the following statements hold true.*

(a) *If $uC_\varphi : H_\alpha^\infty \to H(p,q,\phi)$ is bounded, then*

$$\sup_{w\in B}\int_0^1 \left(\int_S \frac{|u(r\zeta)|^q}{|1 - \langle \varphi(r\zeta), w\rangle|^{q\alpha}} d\sigma(\zeta)\right)^{p/q} \frac{\phi^p(r)}{1-r} dr < \infty. \tag{5.3}$$

(b) *The operator $uC_\varphi : H_\alpha^\infty \to H(p,q,\phi)$ is compact if*

$$\int_0^1 \left(\int_S \frac{|u(r\zeta)|^q}{(1 - |\varphi(r\zeta)|^2)^{q\alpha}} d\sigma(\zeta)\right)^{p/q} \frac{\phi^p(r)}{1-r} dr < \infty. \tag{5.4}$$

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