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Research Article

On a Multiple Hilbert-Type Integral Inequality with the Symmetric Kernel

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We build a multiple Hilbert-type integral inequality with the symmetric kernel $K(x,y)$ and involving an integral operator T. For this objective, we introduce a norm $\| x\|_{\alpha}^{n}$ ($x \in \mathbb{R}_{+}^{n}$), two pairs of conjugate exponents (p,q) and (r,s), and two parameters. As applications, the equivalent form, the reverse forms, and some particular inequalities are given. We also prove that the constant factors in the new inequalities are all the best possible.

1. Introduction, notations, and lemmas

If $p > 1$, $1/p + 1/q = 1$, $f(x), g(x) \geq 0$, $f \in L^{p}(0,\infty)$, $g \in L^{q}(0,\infty)$, $0 < (\int_0^\infty f^{p}(x)dx)^{1/p} < \infty$, and $0 < (\int_0^\infty g^{q}(y)dy)^{1/p} < \infty$, then

$$\iint_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^{p}(x)dx\right)^{1/p}\left(\int_0^\infty g^{q}(y)dy\right)^{1/q}, \tag{1.1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Equation (1.1) is the famous Hardy-Hilbert's inequality proved by Hardy-Riesz [1] in 1925.

By introducing the norms $\|f\|_p$, $\|g\|_q$, and an integral operator $T : L^{p}(0,\infty) \to L^{p}(0,\infty)$, Yang [2] rewrite (1.1) as

$$(Tf,g) < \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q, \tag{1.2}$$

where (Tf,g) is the formal inner product of Tf and g. For $f \in L^p(0,\infty)$ (or $g \in L^q(0,\infty)$), the integral operator T is defined by $(Tf)(y) := \int_0^\infty (f(x)/(x+y))dx$ (or $(Tg)(x) := \int_0^\infty (g(y)/(x+y))dy$) and $\|f\|_p := \{\int_0^\infty |f(x)|^p dx\}^{1/p}$, $\|g\|_q := \{\int_0^\infty |g(y)|^q dy\}^{1/q}$, then

$$(Tf,g) := \int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)g(y)dy = \iint_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy. \tag{1.3}$$

Inequality (1.2) posts the relationship of Hilbert inequality and the integral operator T. Recently, inequality (1.1) has been extended by [3–6] by using the way of weight function and introducing some parameters. A reverse Hilbert-Pachpatte's inequality was first proved by Zhao in [7]. Yang and Zhong [8–10] gave some reverse inequalities concerning some extensions of Hardy-Hilbert's inequality (1.1).

Because of the requirement of higher-dimensional harmonic analysis and higher-dimensional operator theory, multiple Hardy-Hilbert integral inequalities have been studied by some mathematicians (see [11–15]).

Our major objective of this paper is to build a multiple Hilbert-type integral inequality with the symmetric kernel $K(x,y)$ and involving an integral operator T. In order to fulfil the aim, we introduce the norm $\|x\|_\alpha^n$ ($x \in \mathbb{R}_+^n$), two pairs of conjugate exponents (p,q), (r,s), and two parameters α, λ. As applications, the equivalent form, the reverse forms, and some particular inequalities are given. We also prove that the constant factors in the new inequalities are all the best possible.

For these purposes, we introduce the following notations.

If $p>1$, $1/p+1/q=1$, $r>1$, $1/r+1/s=1$, $\alpha>0$, $\lambda>0$, and $n \in \mathbb{Z}_+$, we set

$$\begin{gathered}\mathbb{R}_+^n := \{x=(x_1,\dots,x_n) : x_1,\dots,x_n>0\},\\ \|x\|_\alpha := \left(x_1^\alpha+\cdots+x_n^\alpha\right)^{1/\alpha}.\end{gathered} \tag{1.4}$$

If $f(x)$ and $\omega(x)>0$ are measurable in $\mathbb{R}_+^n$, define the norm of f with the weight function $\omega(x)$ as

$$\|f\|_{p,\omega} := \left\{\int_{\mathbb{R}_+^n} \omega(x)\,|f(x)|^p dx\right\}^{1/p}. \tag{1.5}$$

If $0<\|f\|_{p,\omega}<\infty$, it is marked by $f \in L_\omega^p(\mathbb{R}_+^n)$ (for $0<p<1$ or $q<0$, we still use (1.5) with this formal mark in the following).

Suppose that $K(x,y)$ is a measurable and symmetric function satisfying $K(x,y) = K(y,x)>0$ (for all $(x,y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$). For $f,g \geq 0$, define an integral operator T as

$$(Tf)(y) := \int_{\mathbb{R}_+^n} K(x,y)f(x)dx \quad (y \in \mathbb{R}_+^n), \tag{1.6}$$

or

$$(Tg)(x) := \int_{\mathbb{R}_+^n} K(x,y)g(y)dy \quad (x \in \mathbb{R}_+^n). \tag{1.7}$$

Then we have the formal inner product as

$$(Tf,g)=(Tg,f)=\iint_{\mathbb{R}_+^n} K(x,y)f(x)g(y)\,dx\,dy. \tag{1.8}$$

We also define the following weight functions:

$$C_{\alpha,\lambda,n}(s,x):=\int_{\mathbb{R}_+^n} K(x,y)\frac{\|x\|_\alpha^{\lambda/r}}{\|y\|_\alpha^{n-\lambda/s}}dy, \tag{1.9}$$

$$\overline{C}_{\alpha,\lambda,n}(q,s,\varepsilon,x):=\int_{\mathbb{R}_+^n} K(x,y)\frac{\|x\|_\alpha^{\lambda/r+\varepsilon/q}}{\|y\|_\alpha^{n-\lambda/s+\varepsilon/q}}dy, \tag{1.10}$$

and the notation as

$$\widetilde{C}:=\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n-\varepsilon}\int_{0<\|y\|_\alpha\le 1}K(x,y)\frac{\|x\|_\alpha^{\lambda/r+\varepsilon/q}}{\|y\|_\alpha^{n-\lambda/s+\varepsilon/q}}dx\,dy, \tag{1.11}$$

where $\varepsilon>0$ in (1.10) and (1.11) are small enough.

Lemma 1.1 (cf. [16]). *Assume that $p>0$, $1/p+1/q=1$, $F,G\ge 0$, and $F\in L^p(E)$, $G\in L^q(E)$. We have the following Hölder's inequalities:*

(1) *if $p>1$, then*

$$\int_E F(t)G(t)dt\le\left(\int_E F^p(t)dt\right)^{1/p}\left(\int_E G^q(t)dt\right)^{1/q}; \tag{1.12}$$

(2) *if $0<p<1$, then*

$$\int_E F(t)G(t)dt\ge\left(\int_E F^p(t)dt\right)^{1/p}\left(\int_E G^q(t)dt\right)^{1/q}, \tag{1.13}$$

where equality holds if and only if there exists nonnegative real numbers A and B ($A^2+B^2\neq 0$) such that $AF^p(t)=BG^q(t)$ a.e. in E.

Lemma 1.2 (cf. [17]). *If $p_i>0$ ($i=1,2,\ldots,n$), $\alpha>0$, and $\Psi(u)$ is a measurable function, then*

$$\begin{aligned}&\int\cdots\int_{\{(x_1,\ldots,x_n)\in\mathbb{R}_+^n;(x_1^\alpha+\cdots+x_n^\alpha)\le 1\}}\Psi(x_1^\alpha+\cdots+x_n^\alpha)x_1^{p_1-1}\cdots x_n^{p_n-1}dx_1\cdots dx_n\\&\quad=\frac{\Gamma(p_1/\alpha)\cdots\Gamma(p_n/\alpha)}{\alpha^n\Gamma((p_1+\cdots+p_n)/\alpha)}\int_0^1\Psi(u)u^{((p_1+\cdots+p_n)/\alpha)-1}du,\end{aligned} \tag{1.14}$$

where $\Gamma(\cdot)$ is the Gamma function.

By (1.14), it is easy to obtain following result.

Lemma 1.3. *If $p_i > 0$ $(i = 1,2,\ldots,n)$, $\alpha > 0$, and $\Psi(u)$ is a measurable function, then*

$$\int_{\mathbb{R}_+^n} \Psi(x_1^\alpha + \cdots + x_n^\alpha) x_1^{p_1-1} \cdots x_n^{p_n-1} dx_1 \cdots dx_n = \frac{\Gamma(p_1/\alpha)\cdots\Gamma(p_n/\alpha)}{\alpha^n \Gamma((p_1+\cdots+p_n)/\alpha)} \int_0^\infty \Psi(u) u^{((p_1+\cdots+p_n)/\alpha)-1} du. \tag{1.15}$$

Proof. In view of (1.14), setting $t = \rho^\alpha u$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \Psi(x_1^\alpha + \cdots + x_n^\alpha) x_1^{p_1-1} \cdots x_n^{p_n-1} dx_1 \cdots dx_n \\
&\quad = \lim_{\rho\to\infty} \rho^{p_1+\cdots+p_n} \int \cdots \int_{\{(x_1,\ldots,x_n)\in\mathbb{R}_+^n;((x_1/\rho)^\alpha+\cdots+(x_n/\rho)^\alpha)\le 1\}} \\
&\qquad \times \Psi\left(\rho^\alpha\left(\left(\frac{x_1}{\rho}\right)^\alpha + \cdots + \left(\frac{x_n}{\rho}\right)^\alpha\right)\right)\left(\frac{x_1}{\rho}\right)^{p_1-1} \cdots \left(\frac{x_n}{\rho}\right)^{p_n-1} d\left(\frac{x_1}{\rho}\right)\cdots d\left(\frac{x_n}{\rho}\right) \\
&\quad = \lim_{\rho\to\infty} \rho^{p_1+\cdots+p_n} \frac{\Gamma(p_1/\alpha)\cdots\Gamma(p_n/\alpha)}{\alpha^n \Gamma((p_1+\cdots+p_n)/\alpha)} \int_0^1 \Psi(\rho^\alpha u) u^{((p_1+\cdots+p_n)/\alpha)-1} du \\
&\quad = \frac{\Gamma(p_1/\alpha)\cdots\Gamma(p_n/\alpha)}{\alpha^n \Gamma((p_1+\cdots+p_n)/\alpha)} \int_0^\infty \Psi(t) t^{((p_1+\cdots+p_n)/\alpha)-1} dt,
\end{aligned} \tag{1.16}$$

and (1.15) holds. The lemma is proved. □

By (1.14) and (1.15), we still have the following lemma.

Lemma 1.4. *If $p_i > 0$ $(i = 1,2,\ldots,n)$, $\alpha > 0$, and $\Psi(u)$ is a measurable function, then*

$$\int \cdots \int_{\{(x_1,\ldots,x_n)\in\mathbb{R}_+^n;(x_1^\alpha+\cdots+x_n^\alpha)>1\}} \Psi(x_1^\alpha + \cdots + x_n^\alpha) x_1^{p_1-1} \cdots x_n^{p_n-1} dx_1 \cdots dx_n = \frac{\Gamma(p_1/\alpha)\cdots\Gamma(p_n/\alpha)}{\alpha^n \Gamma((p_1+\cdots+p_n)/\alpha)} \int_1^\infty \Psi(u) u^{(p_1+\cdots+p_n)/\alpha-1} du. \tag{1.17}$$

Lemma 1.5. *For $\varepsilon > 0$ small enough and $n \in \mathbb{Z}_+$, we have*

$$\int_{\|x\|_\alpha>1} \|x\|_\alpha^{-n-\varepsilon} dx = \frac{\Gamma^n(1/\alpha)}{\varepsilon\cdot\alpha^{n-1}\Gamma(n/\alpha)}. \tag{1.18}$$

Proof. By using Lemma 1.4, we have

$$\begin{aligned}
&\int_{\|x\|_\alpha>1} \|x\|_\alpha^{-n-\varepsilon} dx \\
&\quad = \int \cdots \int_{\{(x_1,\ldots,x_n)\in\mathbb{R}_+^n;(x_1^\alpha+\cdots+x_n^\alpha)>1\}} (x_1^\alpha + \cdots + x_n^\alpha)^{-(n+\varepsilon)/\alpha} x_1^{1-1} \cdots x_n^{1-1} dx_1 \cdots dx_n \\
&\quad = \frac{\Gamma^n(1/\alpha)}{\alpha^n \Gamma(n/\alpha)} \int_1^\infty u^{-(n+\varepsilon)/\alpha} u^{n/\alpha-1} du = \frac{\Gamma^n(1/\alpha)}{\alpha^n \Gamma(n/\alpha)} \int_1^\infty u^{-\varepsilon/\alpha-1} du.
\end{aligned} \tag{1.19}$$

Hence (1.18) is valid. The lemma is □

2. Main results

Theorem 2.1. *Suppose that* $p > 1$, $1/p + 1/q = 1$, $r > 1$, $1/r + 1/s = 1$, $\alpha, \lambda > 0$, $n \in \mathbb{Z}_+$, $f, g \geq 0$, $K(x,y)$ *is a measurable and symmetric function,* $\omega(x) = \|x\|_\alpha^{p(n-\lambda/r)-n}$, $\varpi(y) = \|y\|_\alpha^{q(n-\lambda/s)-n}$, $h(y) = \|y\|_\alpha^{p\lambda/s-n}$, *and the integral operator* T *is defined by (1.6) (or (1.7)). If*

$$C_{\alpha,\lambda,n}(s,x) = C_{\alpha,\lambda,n}(s) = C_{\alpha,\lambda,n}(r), \tag{2.1}$$

$$\overline{C}_{\alpha,\lambda,n}(q,s,\varepsilon,x) = C_{\alpha,\lambda,n}(s) + o(1) \quad (\varepsilon \longrightarrow 0^+) \tag{2.2}$$

are all constants independent of x, *and*

$$\widetilde{C} = O(1)(\varepsilon \longrightarrow 0^+), \tag{2.3}$$

where $C_{\alpha,\lambda,n}(s,x)$, $\overline{C}_{\alpha,\lambda,n}(q,s,\varepsilon,x)$ *and* $\widetilde{C}$ *are defined by (1.9), (1.10), and (1.11), respectively. We have the following:*

(1) *if* $f \in L^p_\omega(\mathbb{R}^n_+)$, $g \in L^q_\varpi(\mathbb{R}^n_+)$, *then*

$$(Tf,g) = \iint_{\mathbb{R}^n_+} K(x,y)f(x)g(y)\,dx\,dy < C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}; \tag{2.4}$$

(2) *if* $f \in L^p_\omega(\mathbb{R}^n_+)$, *then* $Tf \in L^p_h(\mathbb{R}^n_+)$ *and*

$$\|Tf\|_{p,h} = \left\{\int_{\mathbb{R}^n_+} \|y\|_\alpha^{p\lambda/s-n}\left(\int_{\mathbb{R}^n_+} K(x,y)f(x)dx\right)^p dy\right\}^{1/p} < C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}, \tag{2.5}$$

where the same constant factor $C_{\alpha,\lambda,n}(s)$ *in (2.4) and (2.5) is the best possible. Inequalities (2.4) and (2.5) are equivalent.*

Proof. (1) Since $p > 1$, we use Hölder's inequality (1.12) in the following:

$$\begin{aligned}(Tf,g) &= \iint_{\mathbb{R}^n_+} \left[K^{1/p}(x,y)f(x)\frac{\|x\|_\alpha^{(1/q)(n-\lambda/r)}}{\|y\|_\alpha^{(1/p)(n-\lambda/s)}}\right]\left[K^{1/q}(x,y)g(y)\frac{\|y\|_\alpha^{(1/p)(n-\lambda/s)}}{\|x\|_\alpha^{(1/q)(n-\lambda/r)}}\right]dx\,dy \\ &\leq \left\{\int_{\mathbb{R}^n_+}\left[\int_{\mathbb{R}^n_+} K(x,y)\frac{\|x\|_\alpha^{\lambda/r}}{\|y\|_\alpha^{n-\lambda/s}}dy\right]\|x\|_\alpha^{p(n-\lambda/r)-n}f^p(x)dx\right\}^{1/p} \\ &\quad\times\left\{\int_{\mathbb{R}^n_+}\left[\int_{\mathbb{R}^n_+} K(x,y)\frac{\|y\|_\alpha^{\lambda/s}}{\|x\|_\alpha^{n-\lambda/r}}dx\right]\|y\|_\alpha^{q(n-\lambda/s)-n}g^q(y)dy\right\}^{1/q}.\end{aligned} \tag{2.6}$$

By (1.9), (2.1), and notations (1.5), (1.8), it follows

$$(Tf,g) \leq C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}. \tag{2.7}$$

If (2.6) takes the form of equality, then by Lemma 1.1, there exist real numbers A and B ($A^2+B^2\neq 0$), such that

$$A\frac{\|x\|_\alpha^{(p-1)(n-\lambda/r)}}{\|y\|_\alpha^{n-\lambda/s}}f^p(x)=B\frac{\|y\|_\alpha^{(q-1)(n-\lambda/s)}}{\|x\|_\alpha^{n-\lambda/r}}g^q(y),\quad \text{a.e. in } \mathbb{R}_+^n\times\mathbb{R}_+^n. \tag{2.8}$$

It follows that there exists a constant E, such that

$$A\|x\|_\alpha^{p(n-\lambda/r)}f^p(x)=B\|y\|_\alpha^{q(n-\lambda/s)}g^q(y)=E,\quad \text{a.e. in } \mathbb{R}_+^n\times\mathbb{R}_+^n. \tag{2.9}$$

Without lose of generality, suppose that $A\neq 0$. We have

$$\|x\|_\alpha^{p(n-\lambda/r)-n}f^p(x)=\frac{E}{A\|x\|_\alpha^n},\quad \text{a.e. in } \mathbb{R}_+^n, \tag{2.10}$$

which contradicts the fact that $f\in L_\omega^p(\mathbb{R}_+^n)$. Hence, (2.6) takes the form of strict inequality; so does (2.7). Then we obtain (2.4).

Suppose there exists a number $0<C\leq C_{\alpha,\lambda,n}(s)$, such that (2.4) is still valid if we replace $C_{\alpha,\lambda,n}(s)$ by C. In particular, for $\varepsilon>0$ small enough, setting

$$\begin{aligned} f_\varepsilon(x)&=\begin{cases}\|x\|_\alpha^{-(n-\lambda/r)-\varepsilon/p}, & x\in\{\|x\|_\alpha>1\}\cap R_+^n,\\ 0, & x\in\{0<\|x\|_\alpha\leq 1\}\cap R_+^n;\end{cases}\\ g_\varepsilon(y)&=\begin{cases}\|y\|_\alpha^{-(n-\lambda/s)-\varepsilon/q}, & y\in\{\|y\|_\alpha>1\}\cap R_+^n,\\ 0, & y\in\{0<\|y\|_\alpha\leq 1\}\cap R_+^n,\end{cases}\end{aligned} \tag{2.11}$$

it follows

$$\begin{aligned}(Tf_\varepsilon,g_\varepsilon)&<C\left\{\int_{\mathbb{R}_+^n}\|x\|_\alpha^{p(n-\lambda/r)-n}f_\varepsilon^p(x)dx\right\}^{1/p}\left\{\int_{\mathbb{R}_+^n}\|y\|_\alpha^{q(n-\lambda/s)-n}g_\varepsilon^q(y)dy\right\}^{1/q}\\ &=C\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n-\varepsilon}dx=C\frac{\Gamma^n(1/\alpha)}{\varepsilon\cdot\alpha^{n-1}\Gamma(n/\alpha)}\quad(\text{by }(1.18)).\end{aligned} \tag{2.12}$$

But by (2.2), (1.18), and (2.3), we have

$$\begin{aligned}(Tf_\varepsilon,g_\varepsilon)&=\iint_{\mathbb{R}_+^n}K(x,y)f_\varepsilon(x)g_\varepsilon(y)dx\,dy\\ &=\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n-\varepsilon}\left[\int_{\mathbb{R}_+^n}K(x,y)\|x\|_\alpha^{\lambda/r+\varepsilon/q}\|y\|_\alpha^{-(n-\lambda/s)-\varepsilon/q}dy\right.\\ &\quad\left.-\int_{0<\|y\|_\alpha\leq 1}K(x,y)\|x\|_\alpha^{\lambda/r+\varepsilon/q}\|y\|_\alpha^{-(n-\lambda/s)-\varepsilon/q}dy\right]dx\\ &=\frac{\Gamma^n(1/\alpha)}{\varepsilon\cdot\alpha^{n-1}\Gamma(n/\alpha)}\left[C_{\alpha,\lambda,n}(s)+o(1)\right](1+\tilde{o}(1))\quad(\varepsilon\longrightarrow 0^+).\end{aligned} \tag{2.13}$$

In view of (2.12) and (2.13), we have $[C_{\alpha,\lambda,n}(s)+o(1)](1+\tilde{o}(1))<C$, and then $C_{\alpha,\lambda,n}(s)\leq C$ ($\varepsilon\to 0^+$). Hence the constant factor $C=C_{\alpha,\lambda,n}(s)$ is the best possible.

(2) Setting $g(y)=\|y\|_\alpha^{p\lambda/s-n}(\int_{\mathbb{R}_+^n}K(x,y)f(x)dx)^{p-1}$ $(y\in\mathbb{R}_+^n)$, then we have $g(y)\geq 0$. Using the notation (1.5), by Hölder's inequality (1.12) (as (2.6)) and (2.1), we have

$$\begin{aligned}\|Tf\|_{p,h}^p&=\|g\|_{q,\varpi}^q=\int_{\mathbb{R}_+^n}\|y\|_\alpha^{q(n-\lambda/s)-n}g^q(y)dy\\&=\int_{\mathbb{R}_+^n}\|y\|_\alpha^{p\lambda/s-n}\left(\int_{\mathbb{R}_+^n}K(x,y)f(x)dx\right)^p dy=(Tf,g)\leq C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi},\end{aligned}\tag{2.14}$$

which is equivalent to

$$\|Tf\|_{p,h}^p=\|g\|_{q,\varpi}^q\leq C_{\alpha,\lambda,n}^p(s)\|f\|_{p,\omega}^p.\tag{2.15}$$

In view of $f\in L_\omega^p(\mathbb{R}_+^n)$, it follows that $g\in L_\varpi^q(\mathbb{R}_+^n)$ and $Tf\in L_h^p(\mathbb{R}_+^n)$. Using the result of (2.4), we can find that inequality (2.14) takes the strict form; so does (2.15). Hence we obtain (2.5).

On the other hand, if inequality (2.5) holds, then by using the Hölder's inequality (1.12) again, we find

$$\begin{aligned}(Tf,g)&=\iint_{\mathbb{R}_+^n}K(x,y)f(x)g(y)dx\,dy\\&=\int_{\mathbb{R}_+^n}\left[\|y\|_\alpha^{\lambda/s-n/p}\int_{\mathbb{R}_+^n}K(x,y)f(x)dx\right]\left[\|y\|_\alpha^{n/p-\lambda/s}g(y)\right]dy\\&\leq\left\{\int_{\mathbb{R}_+^n}\|y\|_\alpha^{p\lambda/s-n}\left(\int_{\mathbb{R}_+^n}K(x,y)f(x)dx\right)^p dy\right\}^{1/p}\left\{\int_{\mathbb{R}_+^n}\|y\|_\alpha^{q(n-\lambda/s)-n}g^q(y)dy\right\}^{1/q}.\end{aligned}\tag{2.16}$$

By (2.5), we have (2.4). It follows that (2.5) is equivalent to (2.4). If the constant factor $C_{\alpha,\lambda,n}(s)$ in (2.5) is not the best possible, then by (2.16), we can get a contradiction that the constant factor $C_{\alpha,\lambda,n}(s)$ in (2.4) is not the best possible. The theorem is proved. □

Theorem 2.2. *Let $0<p<1$ $(q<0)$, $1/p+1/q=1$, $r>1$, $1/r+1/s=1$, $\alpha,\lambda>0$, and $n\in\mathbb{Z}_+$. Assume that $f,g\geq 0$, $K(x,y)$, $\omega(x)$, $\varpi(y)$, $h(y)$ are all defined as in Theorem 2.1, setting $\phi(x)=\|x\|_\alpha^{q\lambda/r-n}$, the integral operator T is defined by (1.6) (or (1.7)), and the weight functions $C_{\alpha,\lambda,n}(s,x)$ and $\overline{C}_{\alpha,\lambda,n}(q,s,\varepsilon,x)$ satisfy (2.1) and (2.2). Then we have the following:*

(1) *if $f\in L_\omega^p(\mathbb{R}_+^n)$ and $g\in L_\varpi^q(\mathbb{R}_+^n)$, then*

$$(Tf,g)=\iint_{\mathbb{R}_+^n}K(x,y)f(x)g(y)dx\,dy>C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi};\tag{2.17}$$

(2) *if $f\in L_\omega^p(\mathbb{R}_+^n)$, then*

$$\|Tf\|_{p,h}=\left\{\int_{\mathbb{R}_+^n}\|y\|_\alpha^{p\lambda/s-n}\left(\int_{\mathbb{R}_+^n}K(x,y)f(x)dx\right)^p dy\right\}^{1/p}>C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega};\tag{2.18}$$

(3) *if* $g \in L^q_{\varpi}(\mathbb{R}^n_+)$, *then* $Tg \in L^q_{\phi}(\mathbb{R}^n_+)$, *and*

$$\|Tg\|^q_{q,\phi} = \int_{\mathbb{R}^n_+} \|x\|_\alpha^{q\lambda/r-n}\left(\int_{\mathbb{R}^n_+} K(x,y)g(y)dy\right)^q dx < C^q_{\alpha,\lambda,n}(s)\|g\|^q_{q,\varpi}, \tag{2.19}$$

where the constant factors $C_{\alpha,\lambda,n}(s)$ *and* $C^q_{\alpha,\lambda,n}(s)$ *are the best possible. Inequalities (2.18) and (2.19) are all equivalent to inequality (2.17).*

Proof. (1) Since $0 < p < 1$ $(q < 0)$, we can use the reverse Hölder's inequality (1.13). Using the combination as (2.6) and notation (1.8), we have

$$\begin{aligned}(Tf,g) &= \iint_{\mathbb{R}^n_+}\left[K^{1/p}(x,y)f(x)\frac{\|x\|_\alpha^{(1/q)(n-\lambda/r)}}{\|y\|_\alpha^{(1/p)(n-\lambda/s)}}\right]\left[K^{(1/q)}(x,y)\frac{\|y\|_\alpha^{(1/p)(n-\lambda/s)}}{\|x\|_\alpha^{(1/q)(n-\lambda/r)}}\right]dx\,dy\\ &\geq \left\{\int_{\mathbb{R}^n_+}\left[\int_{\mathbb{R}^n_+}K(x,y)\frac{\|x\|_\alpha^{\lambda/r}}{\|y\|_\alpha^{n-\lambda/s}}dy\right]\|x\|_\alpha^{p(n-\lambda/r)-n}f^p(x)dx\right\}^{1/p}\\ &\quad\times\left\{\int_{\mathbb{R}^n_+}\left[\int_{\mathbb{R}^n_+}K(x,y)\frac{\|y\|_\alpha^{\lambda/s}}{\|x\|_\alpha^{n-\lambda/r}}dx\right]\|y\|_\alpha^{q(n-\lambda/s)-n}g^q(y)dy\right\}^{1/q}.\end{aligned} \tag{2.20}$$

By (1.9), (2.1), and notation (1.5), we have

$$(Tf,g) \geq C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}. \tag{2.21}$$

If (2.20) takes the form of equality, then by using the conclusions of (2.8)–(2.10), we still can get a result which contradicts the condition of $f \in L^p_\omega(\mathbb{R}^n_+)$ (or $g \in L^q_\varpi(\mathbb{R}^n_+)$). It means that (2.20) takes the form of strict inequality; so does (2.21). The form (2.17) is valid.

If there exists a positive number $C \geq C_{\alpha,\lambda,n}(s)$, such that (2.17) is still valid if we replace $C_{\alpha,\lambda,n}(s)$ by C, then in particular, for $\varepsilon > 0$ small enough, setting $f_\varepsilon(x)$ and $g_\varepsilon(y)$ as (2.11), we have

$$(Tf_\varepsilon,g_\varepsilon) > C\|f_\varepsilon\|_{p,\omega}\|g_\varepsilon\|_{q,\varpi} = C\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n-\varepsilon}dx. \tag{2.22}$$

But by (1.10) and (2.2), we have

$$\begin{aligned}(Tf_\varepsilon,g_\varepsilon) &= \iint_{\mathbb{R}^n_+}K(x,y)f_\varepsilon(x)g_\varepsilon(y)dx\,dy\\ &\leq \int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n-\varepsilon}\left[\int_{\mathbb{R}^n_+}K(x,y)\|x\|_\alpha^{\lambda/r+\varepsilon/q}\|y\|_\alpha^{-(n-\lambda/s)-\varepsilon/q}dy\right]dx\\ &= \left[C_{\alpha,\lambda,n}(s)+o(1)\right]\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n-\varepsilon}dx.\end{aligned} \tag{2.23}$$

In view of (2.22) and (2.23), we find $C < C_{\alpha,\lambda,n}(s) + o(1)$, and then $C \le C_{\alpha,\lambda,n}(s)$ $(\varepsilon \to 0^+)$. Hence the constant $C = C_{\alpha,\lambda,n}(s)$ is the best possible.

(2) Setting $g(y) = \|y\|_\alpha^{p\lambda/s-n}(\int_{\mathbb{R}_+^n} K(x,y)f(x)dx)^{p-1}$ $(y \in \mathbb{R}_+^n)$, it follows $g(y) \ge 0$. By Notation (1.5) and in view of (2.21), we have

$$\begin{aligned}\|Tf\|_{p,h}^p = \|g\|_{q,\varpi}^q &= \int_{\mathbb{R}_+^n} \|y\|_\alpha^{q(n-\lambda/s)-n} g^q(y)dy \\ &= \int_{\mathbb{R}_+^n} \|y\|_\alpha^{p\lambda/s-n}\left(\int_{\mathbb{R}_+^n} K(x,y)f(x)dx\right)^p dy = (Tf,g) \ge C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi},\end{aligned} \tag{2.24}$$

$$\|Tf\|_{p,h}^p = \|g\|_{q,\varpi}^q \ge C_{\alpha,\lambda,n}^p(s)\|f\|_{p,\omega}^p. \tag{2.25}$$

If $\|Tf\|_{p,h}^p = \|g\|_{q,\varpi}^q = \infty$, by $f \in L_\omega^p(\mathbb{R}_+^n)$, (2.25) takes the form of strict inequality. (2.18) holds. If $Tf \in L_h^p(\mathbb{R}_+^n)$ $(g \in L_\varpi^q(\mathbb{R}_+^n))$, this tells us that the condition of (2.17) is satisfied, then by using (2.17), it follows that both (2.24) and (2.25) keep the strict forms and (2.18) holds.

On the other hand, if (2.18) is valid, using the reverse Hölder's inequality (1.13) again, we have

$$\begin{aligned}(Tf,g) &= \int_{\mathbb{R}_+^n} \|y\|_\alpha^{\lambda/s-n/p}\left[\int_{\mathbb{R}_+^n} K(x,y)f(x)dx\right]\left[\|y\|_\alpha^{n/p-\lambda/s}g(y)\right]dy \\ &\ge \left\{\int_{\mathbb{R}_+^n} \|y\|_\alpha^{p\lambda/s-n}\left[\int_{\mathbb{R}_+^n} K(x,y)f(x)dx\right]^p dy\right\}^{1/p}\left\{\int_{\mathbb{R}_+^n} \|y\|_\alpha^{q(n-\lambda/s)-n} g^q(y)dy\right\}^{1/q}.\end{aligned} \tag{2.26}$$

By (2.18), we have (2.17). This means that (2.18) is equivalent to (2.17).

(3) Firstly, setting $f(x) = \|x\|_\alpha^{q\lambda/r-n}(\int_{\mathbb{R}_+^n} K(x,y)g(y)dy)^{q-1}$ $(x \in \mathbb{R}_+^n)$, then it follows $f(x) \ge 0$. Using the notation (1.5) and in view of (1.9), (2.1), and (2.20), we have

$$\begin{aligned}\|Tg\|_{q,\phi}^q = \|f\|_{p,\omega}^p &= \int_{\mathbb{R}_+^n} \|x\|_\alpha^{p(n-\lambda/r)-n} f^p(x)dx \\ &= \int_{\mathbb{R}_+^n} \|x\|_\alpha^{q\lambda/r-n}\left(\int_{\mathbb{R}_+^n} K(x,y)g(y)dy\right)^q dy = (Tg,f) \ge C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}.\end{aligned} \tag{2.27}$$

It follows

$$\|Tg\|_{q,\phi} = \|f\|_{p,\omega}^{p/q} = \left\{\int_{\mathbb{R}_+^n} \|x\|_\alpha^{p(n-\lambda/r)-n} f^p(x)dx\right\}^{1/q} \ge C_{\alpha,\lambda,n}(s)\|g\|_{q,\varpi}, \tag{2.28}$$

and by $q < 0$, we have

$$0 < \|Tg\|_{q,\phi}^q = \|f\|_{p,\omega}^p = \int_{\mathbb{R}_+^n} \|x\|_\alpha^{q\lambda/r-n}\left(\int_{\mathbb{R}_+^n} K(x,y)g(y)dy\right)^q dy \le C_{\alpha,\lambda,n}^q(s)\|g\|_{q,\varpi}^q < \infty. \tag{2.29}$$

This follows that $Tg \in L^q_\phi(\mathbb{R}^n_+)$, $f \in L^p_\omega(\mathbb{R}^n_+)$. And by (2.17), we find that (2.27)–(2.29) are strict inequalities. Thus inequality (2.19) holds.

Secondly, if (2.19) is valid, using the reverse Hölder's inequality (1.13) again, in view of

$$\begin{aligned}(Tf,g) &= \iint_{\mathbb{R}^n_+} K(x,y)f(x)g(y)\,dx\,dy \\ &= \int_{\mathbb{R}^n_+}\left[\|x\|_\alpha^{n/q-\lambda/r}f(x)\right]\left[\|x\|_\alpha^{\lambda/r-n/q}\int_{\mathbb{R}^n_+}K(x,y)g(y)\,dy\right]dx \\ &\geq \left\{\int_{\mathbb{R}^n_+}\|x\|_\alpha^{p(n-\lambda/r)-n}f^p(x)\,dx\right\}^{1/p}\left\{\int_{\mathbb{R}^n_+}\|x\|_\alpha^{q\lambda/r-n}\left[\int_{\mathbb{R}^n_+}K(x,y)g(y)\,dy\right]^q dx\right\}^{1/q},\end{aligned} \tag{2.30}$$

by (2.19) and $q<0$, it follows that (2.17) holds, and (2.19) is equivalent to (2.17).

If the constant factor $C_{\alpha,\lambda,n}(s)$ (or $C^q_{\alpha,\lambda,n}(s)$) in (2.18) (or in (2.19)) is not the best possible, then by (2.26) (or (2.30)), we can get a contradiction that the constant factor $C_{\alpha,\lambda,n}(s)$ in (2.17) is not the best possible. The theorem is proved. □

3. Applications to some particular cases

Corollary 3.1. *Let* $p>0$, $1/p+1/q=1$, $r>1$, $1/r+1/s=1$, $\alpha>0$, $0<\lambda<1$, $n\in\mathbb{Z}_+$, $\omega(x)=\|x\|_\alpha^{p(n-\lambda/r)-n}$, $\varpi(y)=\|y\|_\alpha^{q(n-\lambda/s)-n}$, *and* $f,g\geq 0$. *Then*

(1) *if* $p>1$, $f\in L^p_\omega(\mathbb{R}^n_+)$, *and* $g\in L^q_\varpi(\mathbb{R}^n_+)$, *then*

$$\iint_{\mathbb{R}^n_+}\frac{f(x)g(y)}{\left|\,\|x\|_\alpha-\|y\|_\alpha\right|^\lambda}\,dx\,dy < C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}; \tag{3.1}$$

(2) *if* $p>1$, $f\in L^p_\omega(\mathbb{R}^n_+)$, *then*

$$\int_{\mathbb{R}^n_+}\|y\|_\alpha^{p\lambda/s-n}\left(\int_{\mathbb{R}^n_+}\frac{f(x)}{\left|\,\|x\|_\alpha-\|y\|_\alpha\right|^\lambda}\,dx\right)^p dy < C^p_{\alpha,\lambda,n}(s)\|f\|^p_{p,\omega}; \tag{3.2}$$

(3) *if* $0<p<1$, $f\in L^p_\omega(\mathbb{R}^n_+)$, *and* $g\in L^q_\varpi(\mathbb{R}^n_+)$, *then*

$$\iint_{\mathbb{R}^n_+}\frac{f(x)g(y)}{\left|\,\|x\|_\alpha-\|y\|_\alpha\right|^\lambda}\,dx\,dy > C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}; \tag{3.3}$$

(4) *if* $0<p<1$ *and* $f\in L^p_\omega(\mathbb{R}^n_+)$, *then*

$$\int_{\mathbb{R}^n_+}\|y\|_\alpha^{p\lambda/s-n}\left(\int_{\mathbb{R}^n_+}\frac{f(x)}{\left|\,\|x\|_\alpha-\|y\|_\alpha\right|^\lambda}\,dx\right)^p dy > C^p_{\alpha,\lambda,n}(s)\|f\|^p_{p,\omega}; \tag{3.4}$$

(5) *if* $0<p<1$ *and* $g\in L^q_\varpi(\mathbb{R}^n_+)$, *then*

$$\int_{\mathbb{R}^n_+}\|x\|_\alpha^{q\lambda/r-n}\left(\int_{\mathbb{R}^n_+}\frac{g(y)}{\left|\,\|x\|_\alpha-\|y\|_\alpha\right|^\lambda}\,dy\right)^q dx < C^q_{\alpha,\lambda,n}(s)\|g\|^q_{q,\varpi}, \tag{3.5}$$

where the constant factors $C_{\alpha,\lambda,n}(s) = (\Gamma^n(1/\alpha)/\alpha^{n-1}\Gamma(n/\alpha))[B(\lambda/s,1-\lambda)+B(\lambda/r,1-\lambda)]$ *($B(\cdot,\cdot)$ is the Beta function) and* $C^p_{\alpha,\lambda,n}(s)$, $C^q_{\alpha,\lambda,n}(s)$ *are the best possible. Inequality (3.2) is equivalent to (3.1); inequalities (3.4) and (3.5) are all equivalent to (3.3).*

Proof. Setting $K(x,y) = 1/(|\|x\|_\alpha - \|y\|_\alpha|^\lambda)$, it is a measurable function, satisfying $K(x,y) = K(y,x) > 0$ (for all $(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$). In view of Theorems 2.1 and 2.2, just need to prove that conditions (2.1)–(2.3) are all satisfied.

(a) When $p > 0$, by (1.9) and (1.15), setting $t = u^{1/\alpha}$, we have

$$\begin{aligned}
C_{\alpha,\lambda,n}(s,x) &= \|x\|_\alpha^{\lambda/r}\int_{\mathbb{R}^n_+}\frac{1}{|\|x\|_\alpha - \|y\|_\alpha|^\lambda}\|y\|_\alpha^{-(n-\lambda/s)}dy \\
&= \|x\|_\alpha^{\lambda/r}\frac{\Gamma^n(1/\alpha)}{\alpha^n\Gamma(n/\alpha)}\int_0^\infty\frac{u^{-(n-\lambda/s)/\alpha}}{|\|x\|_\alpha - u^{1/\alpha}|^\lambda}u^{n/\alpha-1}du \\
&= \|x\|_\alpha^{\lambda/r}\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\int_0^{\|x\|_\alpha}\frac{t^{\lambda/s-1}}{(\|x\|_\alpha - t)^\lambda}dt + \int_{\|x\|_\alpha}^\infty\frac{t^{\lambda/s-1}}{(t-\|x\|_\alpha)^\lambda}dt\right].
\end{aligned} \tag{3.6}$$

Setting $v = t/\|x\|_\alpha$, we have

$$\|x\|_\alpha^{\lambda/r}\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\int_0^{\|x\|_\alpha}\frac{t^{\lambda/s-1}}{(\|x\|_\alpha - t)^\lambda}dt = \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\int_0^1\frac{v^{\lambda/s-1}}{(1-v)^\lambda}dv. \tag{3.7}$$

Setting $u = \|x\|_\alpha/t$, it follows that $dt = -\|x\|_\alpha u^{-2}du$ and

$$\|x\|_\alpha^{\lambda/r}\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\int_{\|x\|_\alpha}^\infty\frac{t^{\lambda/s-1}}{(t-\|x\|_\alpha)^\lambda}dt = \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\int_0^1\frac{u^{\lambda/r-1}}{(1-u)^\lambda}du. \tag{3.8}$$

In view of (3.7), (3.8), and $0 < \lambda < 1$, it follows $C_{\alpha,\lambda,n}(q,s,x) = C_{\alpha,\lambda,n}(s) = C_{\alpha,\lambda,n}(r) = (\Gamma^n(1/\alpha)/\alpha^{n-1}\Gamma(n/\alpha))[B(\lambda/s,1-\lambda)+B(\lambda/r,1-\lambda)]$, and condition (2.1) is satisfied.

(b) When $p > 1$, by (1.10) and (1.15), setting $t = u^{1/\alpha}$, for $0 < \varepsilon < q\lambda/s$, we have

$$\begin{aligned}
\overline{C}_{\alpha,\lambda,n}(q,s,\varepsilon,x) &= \|x\|_\alpha^{\lambda/r+\varepsilon/q}\int_{\mathbb{R}^n_+}\frac{1}{|\|x\|_\alpha - \|y\|_\alpha|^\lambda}\|y\|_\alpha^{-(n-\lambda/s)-\varepsilon/q}dy \\
&= \|x\|_\alpha^{\lambda/r+\varepsilon/q}\frac{\Gamma^n(1/\alpha)}{\alpha^n\Gamma(n/\alpha)}\int_0^\infty\frac{u^{-(n-\lambda/s+\varepsilon/q)/\alpha}}{|\|x\|_\alpha - u^{1/\alpha}|^\lambda}u^{n/\alpha-1}du \\
&= \|x\|_\alpha^{\lambda/r+\varepsilon/q}\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\int_0^{\|x\|_\alpha}\frac{t^{\lambda/s-\varepsilon/q-1}}{(\|x\|_\alpha - t)^\lambda}dt + \int_{\|x\|_\alpha}^\infty\frac{t^{\lambda/s-\varepsilon/q-1}}{(t-\|x\|_\alpha)^\lambda}dt\right].
\end{aligned} \tag{3.9}$$

Setting $v = t/\|x\|_\alpha$ or $u = \|x\|_\alpha/t$, respectively, as (3.7) or (3.8), we find

$$\begin{aligned}\overline{C}_{\alpha,\lambda,n}(q,s,\varepsilon,x) &= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[\int_0^1 \frac{v^{\lambda/s-\varepsilon/q-1}}{(1-v)^\lambda}dv + \int_0^1 \frac{u^{\lambda/r+\varepsilon/q-1}}{(1-u)^\lambda}du\right] \\ &= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\left[B\left(\frac{\lambda}{s}-\frac{\varepsilon}{q},1-\lambda\right)+B\left(\frac{\lambda}{r}+\frac{\varepsilon}{q},1-\lambda\right)\right].\end{aligned} \tag{3.10}$$

It follows that condition (2.2) is satisfied.

Note. When $0 < p < 1$ $(q < 0)$, setting $0 < \varepsilon < -q\lambda/r$, the constant $\overline{C}_{\alpha,\lambda,n}(q,s,\varepsilon,x)$ satisfies (2.2) as well.

(c) If $p > 1$, by (1.11), (1.14), and (1.17), respectively, setting $t = u^{1/\alpha}$ and $v = t/\|x\|_\alpha$, for $0 < \varepsilon < q\lambda/s$ and $0 < \lambda < 1$, we have

$$\begin{aligned}0 < \widetilde{C} &= \int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n-\varepsilon}\int_{0<\|y\|_\alpha\le 1}\frac{\|x\|_\alpha^{\lambda/r+\varepsilon/p}}{\left|\,\|x\|_\alpha-\|y\|_\alpha\right|^\lambda\|y\|_\alpha^{n-\lambda/s+\varepsilon/q}}dx\,dy \\ &= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n-\varepsilon}dx\int_0^{1/\|x\|_\alpha}\frac{v^{\lambda/s-\varepsilon/q-1}}{(1-v)^\lambda}dv \\ &\le \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n}dx\int_0^{1/\|x\|_\alpha}\frac{v^{\lambda/s-\varepsilon/q-1}}{1-v}dv \\ &= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n}dx\int_0^{1/\|x\|_\alpha}\sum_{k=0}^{\infty}v^{k+\lambda/s-\varepsilon/q-1}dv \\ &= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\sum_{k=0}^{\infty}\frac{1}{k+\lambda/s-\varepsilon/q}\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-(n+k+\lambda/s-\varepsilon/q)}dx \\ &= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-2}\Gamma(n/\alpha)}\sum_{k=0}^{\infty}\frac{1}{(k+\lambda/s-\varepsilon/q)^2}.\end{aligned} \tag{3.11}$$

It follows that $\widetilde{C}$ satisfies (2.3).

In view of (3.7)–(3.11), by Theorems 2.1 and 2.2, Corollary 3.1 is proved. □

Corollary 3.2. *Suppose that* $p > 0$, $1/p + 1/q = 1$, $r > 1$, $1/r + 1/s = 1$, $\alpha,\lambda > 0$, $n \in \mathbb{Z}_+$, $\omega(x) = \|x\|_\alpha^{p(n-\lambda/r)-n}$, $\varpi(y) = \|y\|_\alpha^{q(n-\lambda/s)-n}$, *and* $f,g \ge 0$. *Then*

(1) *if* $p > 1$, $f \in L^p_\omega(\mathbb{R}^n_+)$, *and* $g \in L^q_\varpi(\mathbb{R}^n_+)$, *then*

$$\iint_{\mathbb{R}^n_+}\frac{\ln\left(\|y\|_\alpha/\|x\|_\alpha\right)}{\|y\|_\alpha^\lambda-\|x\|_\alpha^\lambda}f(x)g(y)dx\,dy < C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}; \tag{3.12}$$

(2) *if* $p > 1$, $f \in L^p_\omega(\mathbb{R}^n_+)$, *then*

$$\int_{\mathbb{R}^n_+} \|y\|_\alpha^{p\lambda/s-n} \left(\int_{\mathbb{R}^n_+} \frac{\ln(\|y\|_\alpha/\|x\|_\alpha) f(x)}{\|y\|_\alpha^\lambda - \|x\|_\alpha^\lambda} dx \right)^p dy < C^p_{\alpha,\lambda,n}(s) \|f\|^p_{p,\omega}; \tag{3.13}$$

(3) *if* $0 < p < 1$, $f \in L^p_\omega(\mathbb{R}^n_+)$, *and* $g \in L^q_\varpi(\mathbb{R}^n_+)$, *then*

$$\iint_{\mathbb{R}^n_+} \frac{\ln(\|y\|_\alpha/\|x\|_\alpha)}{\|y\|_\alpha^\lambda - \|x\|_\alpha^\lambda} f(x) g(y) dx\, dy > C_{\alpha,\lambda,n}(s) \|f\|_{p,\omega} \|g\|_{q,\varpi}; \tag{3.14}$$

(4) *if* $0 < p < 1$ *and* $f \in L^p_\omega(\mathbb{R}^n_+)$, *then*

$$\int_{\mathbb{R}^n_+} \|y\|_\alpha^{p\lambda/s-n} \left(\int_{\mathbb{R}^n_+} \frac{\ln(\|y\|_\alpha/\|x\|_\alpha) f(x)}{\|y\|_\alpha^\lambda - \|x\|_\alpha^\lambda} dx \right)^p dy > C^p_{\alpha,\lambda,n}(s) \|f\|^p_{p,\omega}; \tag{3.15}$$

(5) *if* $0 < p < 1$ *and* $g \in L^q_\varpi(\mathbb{R}^n_+)$, *then*

$$\int_{\mathbb{R}^n_+} \|x\|_\alpha^{q\lambda/r-n} \left(\int_{\mathbb{R}^n_+} \frac{\ln(\|y\|_\alpha/\|x\|_\alpha) g(y)}{\|y\|_\alpha^\lambda - \|x\|_\alpha^\lambda} dy \right)^q dx < C^q_{\alpha,\lambda,n}(s) \|g\|^q_{q,\varpi}, \tag{3.16}$$

where the constant factor $C_{\alpha,\lambda,n}(s) = (\Gamma^n(1/\alpha)/\lambda^2\alpha^{n-1}\Gamma(n/\alpha))B^2(1/s,1/r)$ *(*$B(\cdot,\cdot)$ *is Beta function) and* $C^p_{\alpha,\lambda,n}(s)$, $C^q_{\alpha,\lambda,n}(s)$ *are all the best possible. Inequality (3.13) is equivalent to (3.12); inequalities (3.15) and (3.16) are all equivalent to (3.14).*

Proof. Setting $K(x,y) = \ln(\|y\|_\alpha/\|x\|_\alpha)/(\|y\|_\varepsilon^\lambda - \|x\|_\alpha^\lambda)$, it is a measurable function, satisfying $K(x,y) = K(y,x) > 0$. As in Corollary 3.1, we just need to prove that conditions (2.1)–(2.3) are all satisfied. Setting $t = u^{1/\alpha}$ and $v = (t/\|x\|_\alpha)^\lambda$, respectively, we can find the results in the following.

(a) When $p > 0$, by (1.9) and (1.15), we have

$$\begin{aligned} C_{\alpha,\lambda,n}(s,x) &= \|x\|_\alpha^{\lambda/r} \int_{\mathbb{R}^n_+} \frac{\ln(\|y\|_\alpha/\|x\|_\alpha)}{\|y\|_\alpha^\lambda - \|x\|_\alpha^\lambda} \|y\|_\alpha^{-(n-\lambda/s)} dy \\ &= \|x\|_\alpha^{\lambda/r} \frac{\Gamma^n(1/\alpha)}{\alpha^n\Gamma(n/\alpha)} \int_0^\infty \frac{[\ln u^{1/\alpha} - \ln\|x\|_\alpha] u^{-(n-\lambda/s)/\alpha}}{u^{\lambda/\alpha} - \|x\|_\alpha^\lambda} u^{n/\alpha-1} du \\ &= \|x\|_\alpha^{\lambda/r} \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \frac{[\ln t - \ln\|x\|_\alpha] t^{\lambda/s-1}}{t^\lambda - \|x\|_\alpha^\lambda} dt \\ &= \frac{\Gamma^n(1/\alpha)}{\lambda^2\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \frac{\ln v}{v-1} v^{1/s-1} dv. \end{aligned} \tag{3.17}$$

It follows $C_{\alpha,\lambda,n}(q,s,x) = C_{\alpha,\lambda,n}(s) = (\Gamma^n(1/\alpha)/\lambda^2\alpha^{n-1}\Gamma(n/\alpha))B^2(1/s,1/r)$ satisfies (2.1).

(b) When $p > 1$, for $0 < \varepsilon < q\lambda/s$, by (1.10) and (1.15), we have

$$\begin{aligned}
\overline{C}_{\alpha,\lambda,n}(q,s,\varepsilon,x) &= \|x\|_\alpha^{\lambda/r+\varepsilon/q}\int_{\mathbb{R}_+^n}\frac{\ln(\|y\|_\alpha/\|x\|_\alpha)}{\|y\|_\alpha^\lambda-\|x\|_\alpha^\lambda}\|y\|_\alpha^{-(n-\lambda/s)-\varepsilon/q}dy \\
&= \|x\|_\alpha^{\lambda/r+\varepsilon/q}\frac{\Gamma^n(1/\alpha)}{\alpha^n\Gamma(n/\alpha)}\int_0^\infty\frac{(\ln u^{1/\alpha}-\ln\|x\|_\alpha)u^{-(n-\lambda/s+\varepsilon/q)/\alpha}}{u^{\lambda/\alpha}-\|x\|_\alpha^\lambda}u^{n/\alpha-1}du \\
&= \frac{\Gamma^n(1/\alpha)}{\lambda^2\alpha^{n-1}\Gamma(n/\alpha)}\int_0^\infty\frac{\ln v}{v-1}v^{1/s-\varepsilon/q\lambda-1}dv \\
&= \frac{\Gamma^n(1/\alpha)}{\lambda^2\alpha^{n-1}\Gamma(n/\alpha)}B^2\left(\frac{1}{s}-\frac{\varepsilon}{q\lambda},\frac{1}{r}+\frac{\varepsilon}{q\lambda}\right).
\end{aligned} \tag{3.18}$$

It follows that (2.2) is valid.

Note. When $0 < p < 1$ $(q < 0)$, setting $0 < \varepsilon < -q\lambda/r$, the constant $\overline{C}_{\alpha,\lambda,n}$ (q,s,ε,x) satisfies (2.2) as well.

(c) If $p > 1$, then for $0 < \varepsilon < q\lambda/s$, by (1.11), (1.14), and (1.17), we have

$$\begin{aligned}
0 < \widetilde{C} &= \int_{\|x\|_\alpha>1}\int_{0<\|y\|_\alpha\le 1}\frac{[\ln(\|y\|_\alpha/\|x\|_\alpha)]\|x\|_\alpha^{-(n-\lambda/r)-\varepsilon/p}}{(\|y\|_\alpha^\lambda-\|x\|_\alpha^\lambda)\|y\|_\alpha^{n-\lambda/s+\varepsilon/q}}dx\,d \\
&= \frac{\Gamma^n(1/\alpha)}{\lambda^2\alpha^{n-1}\Gamma(n/\alpha)}\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n-\varepsilon}dx\int_0^{1/\|x\|_\alpha^\lambda}\frac{\ln v}{v-1}v^{1/s-\varepsilon/q\lambda-1}dv \\
&\le \frac{\Gamma^n(1/\alpha)}{\lambda^2\alpha^{n-1}\Gamma(n/\alpha)}\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n}dx\int_0^{1/\|x\|_\alpha^\lambda}(-\ln v)\sum_{k=0}^\infty v^{k+1/s-\varepsilon/q\lambda-1}dv \\
&= \frac{\Gamma^n(1/\alpha)}{\lambda^2\alpha^{n-1}\Gamma(n/\alpha)}\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-n}dx\sum_{k=0}^\infty\frac{1}{k+1/s-\varepsilon/q\lambda}\int_0^{1/\|x\|_\alpha^\lambda}(-\ln v)dv^{k+1/s-\varepsilon/q\lambda} \\
&= \frac{\Gamma^n(1/\alpha)}{\lambda^2\alpha^{n-1}\Gamma(n/\alpha)}\left\{\sum_{k=0}^\infty\frac{\lambda}{k+1/s-\varepsilon/q\lambda}\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-[n+\lambda(k+1/s-\varepsilon/q\lambda)]}\ln\|x\|_\alpha dx\right. \\
&\qquad\left.+\sum_{k=0}^\infty\frac{1}{(k+1/s-\varepsilon/q\lambda)^2}\int_{\|x\|_\alpha>1}\|x\|_\alpha^{-[n+\lambda(k+1/s-\varepsilon/q\lambda)]}dx\right\} \\
&= \frac{2\Gamma^n(1/\alpha)}{\lambda^3\alpha^{n-2}\Gamma(n/\alpha)}\sum_{k=0}^\infty\frac{1}{(k+1/s-\varepsilon/q\lambda)^3}.
\end{aligned} \tag{3.19}$$

It is obvious that $\widetilde{C}$ is a bounded quantity and satisfies (2.3).

In view of (3.17)–(3.19), by Theorems 2.1 and 2.2, Corollary 3.2 is proved. □

Similarly, by setting $K(x,y) = 1/(\|x\|_\alpha^\lambda+\|y\|_\alpha^\lambda)$ and $K(x,y) = 1/(\mathrm{Max}\{\|x\|_\alpha,\|y\|_\alpha\})^\lambda$, respectively, we have Corollaries 3.3 and 3.4 in the following. In order to compress the length of the paper, the proof for Corollaries 3.3 and 3.4 are here omitted.

Corollary 3.3. *Suppose that* $p > 0$, $1/p + 1/q = 1$, $r > 1$, $1/r + 1/s = 1$, $\alpha, \lambda > 0$, $n \in \mathbb{Z}_+$, $\omega(x) = \|x\|_\alpha^{p(n-\lambda/r)-n}$, $\varpi(y) = \|y\|_\alpha^{q(n-\lambda/s)-n}$, *and* $f, g \geq 0$. *Then*

(1) *if* $p > 1$, $f \in L_\omega^p(\mathbb{R}_+^n)$, *and* $g \in L_\varpi^q(\mathbb{R}_+^n)$, *then*

$$\iint_{\mathbb{R}_+^n} \frac{f(x)g(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx\, dy < C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}; \tag{3.20}$$

(2) *if* $p > 1$, $f \in L_\omega^p(\mathbb{R}_+^n)$, *then*

$$\int_{\mathbb{R}_+^n} \|y\|_\alpha^{p\lambda/s-n} \left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx \right)^p dy < C_{\alpha,\lambda,n}^p(s)\|f\|_{p,\omega}^p; \tag{3.21}$$

(3) *if* $0 < p < 1$, $f \in L_\omega^p(\mathbb{R}_+^n)$, *and* $g \in L_\varpi^q(\mathbb{R}_+^n)$, *then*

$$\iint_{\mathbb{R}_+^n} \frac{f(x)g(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx\, dy > C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}; \tag{3.22}$$

(4) *if* $0 < p < 1$ *and* $f \in L_\omega^p(\mathbb{R}_+^n)$, *then*

$$\int_{\mathbb{R}_+^n} \|y\|_\alpha^{p\lambda/s-n} \left(\int_{\mathbb{R}_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx \right)^p dy > C_{\alpha,\lambda,n}^p(s)\|f\|_{p,\omega}^p; \tag{3.23}$$

(5) *if* $0 < p < 1$ *and* $g \in L_\varpi^q(\mathbb{R}_+^n)$, *then*

$$\int_{\mathbb{R}_+^n} \|x\|_\alpha^{q\lambda/r-n} \left(\int_{\mathbb{R}_+^n} \frac{g(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dy \right)^q dx < C_{\alpha,\lambda,n}^q(s)\|g\|_{q,\varpi}^q, \tag{3.24}$$

where the constant factors $C_{\alpha,\lambda,n}(s) = (\Gamma^n(1/\alpha)/\lambda\alpha^{n-1}\Gamma(n/\alpha))B(1/s, 1/r)$ ($B(\cdot,\cdot)$ *is the Beta function) and* $C_{\alpha,\lambda,n}^p(s)$, $C_{\alpha,\lambda,n}^q(s)$ *are all the best possible. Inequality (3.21) is equivalent to (3.20); inequalities (3.23) and (3.24) are all equivalent to (3.22).*

Corollary 3.4. *Suppose that* $p > 0$, $1/p + 1/q = 1$, $r > 1$, $1/r + 1/s = 1$, $\alpha, \lambda > 0$, $n \in \mathbb{Z}_+$, $\omega(x) = \|x\|_\alpha^{p(n-\lambda/r)-n}$, $\varpi(y) = \|y\|_\alpha^{q(n-\lambda/s)-n}$, *and* $f, g \geq 0$. *Then*

(1) *if* $p > 1$, $f \in L_\omega^p(\mathbb{R}_+^n)$, *and* $g \in L_\varpi^q(\mathbb{R}_+^n)$, *then*

$$\iint_{\mathbb{R}_+^n} \frac{f(x)g(y)}{(\max\{\|x\|_\alpha, \|y\|_\alpha\})^\lambda} dx\, dy < C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}; \tag{3.25}$$

(2) *if* $p > 1$, $f \in L_\omega^p(\mathbb{R}_+^n)$, *then*

$$\int_{\mathbb{R}_+^n} \|y\|_\alpha^{p\lambda/s-n} \left(\int_{\mathbb{R}_+^n} \frac{f(x)}{(\max\{\|x\|_\alpha, \|y\|_\alpha\})^\lambda} dx \right)^p dy < C_{\alpha,\lambda,n}^p(s)\|f\|_{p,\omega}; \tag{3.26}$$

(3) *if* $0 < p < 1$, $f \in L_\omega^p(\mathbb{R}_+^n)$, *and* $g \in L_\varpi^q(\mathbb{R}_+^n)$, *then*

$$\iint_{\mathbb{R}_+^n} \frac{f(x)g(y)}{(\max\{\|x\|_\alpha, \|y\|_\alpha\})^\lambda} dx\, dy > C_{\alpha,\lambda,n}(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}; \tag{3.27}$$

(4) *if* $0 < p < 1$ *and* $f \in L^p_\omega(\mathbb{R}^n_+)$, *then*

$$\int_{\mathbb{R}^n_+} \|y\|_\alpha^{p\lambda/s-n}\left(\int_{\mathbb{R}^n_+} \frac{f(x)}{(\max\{\|x\|_\alpha, \|y\|_\alpha\})^\lambda} dx\right)^p dy > C^p_{\alpha,\lambda,n}(s)\|f\|^p_{p,\omega}; \tag{3.28}$$

(5) *if* $0 < p < 1$ *and* $g \in L^q_\varpi(\mathbb{R}^n_+)$, *then*

$$\int_{\mathbb{R}^n_+} \|x\|_\alpha^{q\lambda/r-n}\left(\int_{\mathbb{R}^n_+} \frac{g(y)}{(\max\{\|x\|_\alpha, \|y\|_\alpha\})^\lambda} dy\right)^q dx < C^q_{\alpha,\lambda,n}(s)\|g\|^q_{q,\varpi}, \tag{3.29}$$

where the constant factors $C_{\alpha,\lambda,n}(s) = sr\Gamma^n(1/\alpha)/\lambda\alpha^{n-1}\Gamma(n/\alpha)$ *and* $C^p_{\alpha,\lambda,n}(s)$, $C^q_{\alpha,\lambda,n}(s)$ *are all the best possible. Inequality (3.26) is equivalent to (3.25); inequalities (3.28) and (3.29) are all equivalent to (3.27).*

Remark 3.5. For $n = 1$, the inequalities in Corollaries 3.1–3.4 reduce to the correspondent inequalities in the 2-dimensional space.

References

[1] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 2nd edition, 1952.

[2] B. Yang, "On the norm of an integral operator and applications," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 1, pp. 182–192, 2006.

[3] I. Brnetić and J. Pečarić, "Generalization of inequalities of Hardy-Hilbert type," *Mathematical Inequalities & Applications*, vol. 7, no. 2, pp. 217–225, 2004.

[4] W. Zhong and B. Yang, "A best extension of Hilbert inequality involving seveial parameters," *Jinan University Journal (Natural Science and Medical Edition)*, vol. 28, no. 1, pp. 20–23, 2007 (Chinese).

[5] B. Yang and L. Debnath, "On the extended Hardy-Hilbert's inequality," *Journal of Mathematical Analysis and Applications*, vol. 272, no. 1, pp. 187–199, 2002.

[6] B. Yang and M. Z. Gao, "An optimal constant in the Hardy-Hilbert inequality," *Advances in Mathematics*, vol. 26, no. 2, pp. 159–164, 1997 (Chinese).

[7] C.-J. Zhao and L. Debnath, "Some new inverse type Hilbert integral inequalities," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 1, pp. 411–418, 2001.

[8] B. Yang, "A reverse of the Hardy-Hilbert's type inequality," *Journal of Southwest China Normal University (Natural Science)*, vol. 30, no. 6, pp. 1012–1015, 2005.

[9] W. Zhong, "A reverse Hilbert's type integral inequality," *International Journal of Pure and Applied Mathematics*, vol. 36, no. 3, pp. 353–360, 2007.

[10] W. Zhong and B. Yang, "On the extended forms of the reverse Hardy-Hilbert's integral inequalities," *Journal of Southwest China Normal University (Natural Science)*, vol. 29, no. 4, pp. 44–48, 2007.

[11] B. Yang, "A multiple Hardy-Hilbert integral inequality," *Chinese Annals of Mathematics*, vol. 24, no. 6, pp. 743–750, 2003.

[12] I. Brnetić and J. Pečarić, "Generalization of Hilbert's integral inequality," *Mathematical Inequalities & Applications*, vol. 7, no. 2, pp. 199–205, 2004.

[13] I. Brnetić, M. Krnić, and J. Pečarić, "Multiple Hilbert and Hardy-Hilbert inequalities with non-conjugate parameters," *Bulletin of the Australian Mathematical Society*, vol. 71, no. 3, pp. 447–457, 2005.

[14] B. Yang and T. M. Rassias, "On the way of weight coefficient and research for the Hilbert-type inequalities," *Mathematical Inequalities & Applications*, vol. 6, no. 4, pp. 625–658, 2003.

[15] Y. Hong, "On multiple Hardy-Hilbert integral inequalities with some parameters," *Journal of Inequalities and Applications*, vol. 2006, Article ID 94960, 11 pages, 2006.
[16] J. Kuang, *Applied Inequalities*, Shangdong Science and Technology Press, Jinan, China, 2004.
[17] G. M. Fichtingoloz, *A Course in Differential and Integral Calculus*, Renmin Education, Beijing, China, 1957.

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