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Research Article

Uniform Boundedness for Approximations of the Identity with Nondoubling Measures

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Let μ be a nonnegative Radon measure on $\mathbb{R}^d$ which satisfies the growth condition that there exist constants $C_0 > 0$ and $n \in (0,d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$, $\mu(B(x,r)) \leq C_0 r^n$, where $B(x,r)$ is the open ball centered at x and having radius r. In this paper, the authors establish the uniform boundedness for approximations of the identity introduced by Tolsa in the Hardy space $H^1(\mu)$ and the BLO-type space RBLO (μ). Moreover, the authors also introduce maximal operators $\dot{\mathcal{M}}_s$ (homogeneous) and $\mathcal{M}_s$ (inhomogeneous) associated with a given approximation of the identity S, and prove that $\dot{\mathcal{M}}_s$ is bounded from $H^1(\mu)$ to $L^1(\mu)$ and $\mathcal{M}_s$ is bounded from the local atomic Hardy space $h^{1,\infty}_{\mathrm{atb}}(\mu)$ to $L^1(\mu)$. These results are proved to play key roles in establishing relations between $H^1(\mu)$ and $h^{1,\infty}_{\mathrm{atb}}(\mu)$, BMO-type spaces RBMO (μ) and rbmo (μ) as well as RBLO (μ) and rblo (μ), and also in characterizing rbmo (μ) and rblo (μ).

1. Introduction

Recall that a nondoubling measure μ on $\mathbb{R}^d$ means that μ is a nonnegative Radon measure which only satisfies the following growth condition, namely, there exist constants $C_0 > 0$ and $n \in (0,d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu(B(x,r)) \leq C_0 r^n, \tag{1.1}$$

where $B(x,r)$ is the open ball centered at x and having radius r. Such a measure μ is not necessary to be doubling, which is a key assumption in the classical theory of harmonic analysis. In recent years, it was shown that many results on the Calderón-Zygmund theory remain valid for nondoubling measures; see, for example, [1–9]. One of the main motivations for extending the classical theory to the nondoubling context was the solution of several questions related to analytic capacity, like Vitushkin's conjecture or Painlevé's problem; see [10–12] or survey papers [13–16] for more details.

In particular, Tolsa [8] constructed a class of approximations of the identity and used it to develop a Littlewood-Paley theory with nondoubling measures in $L^p(\mu)$ with $p \in (1,\infty)$ and establish some $T(1)$ theorems. The main purpose of this paper is to investigate behaviors of approximations of the identity and some kind of maximal operators associated with it at the extremal cases, namely, when $p = 1$ or $p = \infty$. To be precise, in this paper, we first establish the uniform boundedness for approximations of the identity in the Hardy space $H^1(\mu)$ of Tolsa [7, 9] and the BLO-type space $\mathrm{RBLO}(\mu)$ of Jiang [1], respectively. We then introduce the homogeneous maximal operator $\dot{\mathcal{M}}_S$ and inhomogeneous maximal operator $\mathcal{M}_S$ and prove that $\dot{\mathcal{M}}_S$ is bounded from $H^1(\mu)$ to $L^1(\mu)$ and $\mathcal{M}_S$ is bounded from the local atomic Hardy space $h^{1,\infty}_{\mathrm{atb}}(\mu)$ to $L^1(\mu)$. These results are proved in [17] to play key roles in establishing relations between $H^1(\mu)$ and $h^{1,\infty}_{\mathrm{atb}}(\mu)$, BMO-type spaces $\mathrm{RBMO}(\mu)$ and $\mathrm{rbmo}(\mu)$ as well as BLO-type spaces $\mathrm{RBLO}(\mu)$ and $\mathrm{rblo}(\mu)$, and also in characterizing $\mathrm{rbmo}(\mu)$ and $\mathrm{rblo}(\mu)$. An interesting open problem is if $H^1(\mu)$ and $h^{1,\infty}_{\mathrm{atb}}(\mu)$ can be characterized by $\dot{\mathcal{M}}_S$ and $\mathcal{M}_S$, respectively.

The organization of this paper is as follows. In Section 2, we recall some necessary definitions and notation, including the definitions and characterizations of the spaces $H^1(\mu)$, $\mathrm{RBLO}(\mu)$, $h^{1,\infty}_{\mathrm{atb}}(\mu)$, and approximations of the identity. Section 3 is devoted to prove that approximations of the identity are uniformly bounded on $H^1(\mu)$ and $\mathrm{RBLO}(\mu)$. In Section 4, we introduce the homogeneous maximal operator $\dot{\mathcal{M}}_S$ and the inhomogeneous maximal operator $\mathcal{M}_S$ associated with a given approximation of the identity S, and prove that $\dot{\mathcal{M}}_S$ is bounded from $H^1(\mu)$ to $L^1(\mu)$ and $\mathcal{M}_S$ is bounded from $h^{1,\infty}_{\mathrm{atb}}(\mu)$ to $L^1(\mu)$.

Since the approximation of the identity in [8] strongly depends on "dyadic" cubes constructed by Tolsa in [8, 9], it is expectable that properties of these "dyadic" cubes will play a key role in the proofs of all these results in this paper. In [17], we introduce a quantity on these "dyadic" cubes, which further clarifies the geometric properties of "dyadic" cubes of Tolsa in [8, 9]; see Lemma 2.18 below. These properties together with some known properties of "dyadic" cubes (see, e.g., [8, Lemmas 3.4 and 4.2]) indeed play key roles in the whole paper.

We finally make some convention. Throughout the paper, we always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constant with subscript such as C_1 does not change in different occurrences. The notation $Y \lesssim Z$ means that there exists a constant $C > 0$ such that $Y \le CZ$, while $Y \gtrsim Z$ means that there exists a constant $C > 0$ such that $Y \ge CZ$. The symbol $A \sim B$ means that $A \lesssim B \lesssim A$. Moreover, for any $D \subset \mathbb{R}^d$, we denote by χ_D the characteristic function of D. We also set $\mathbb{N} = \{1,2,\dots\}$.

2. Preliminaries

Throughout this paper, by a cube $Q \subset \mathbb{R}^d$, we mean a closed cube whose sides are parallel to the axes and centered at some point of $\mathrm{supp}(\mu)$, and we denote its side length by $l(Q)$

and its center by x_Q. If $\mu(\mathbb{R}^d) < \infty$, we also regard $\mathbb{R}^d$ as a cube. Let α, β be two positive constants, $\alpha \in (1,\infty)$ and $\beta \in (\alpha^n,\infty)$. We say that a cube Q is an (α,β)-doubling cube if it satisfies $\mu(\alpha Q) \le \beta\mu(Q)$, where and in what follows, given $\lambda > 0$ and any cube Q, λQ denotes the cube concentric with Q and having side length $\lambda l(Q)$. It was pointed out by Tolsa (see [7, pages 95-96] or [8, Remark 3.1]) that if $\beta > \alpha^n$, then for any $x \in \operatorname{supp}(\mu)$ and any $R > 0$, there exists some (α,β)-doubling cube Q centered at x with $l(Q) \ge R$, and that if $\beta > \alpha^d$, then for μ-almost everywhere $x \in \mathbb{R}^d$, there exists a sequence of (α,β)-doubling cubes $\{Q_k\}_{k\in\mathbb{N}}$ centered at x with $l(Q_k) \to 0$ as $k \to \infty$. Throughout this paper, by a doubling cube Q, we always mean a $(2,2^{d+1})$-doubling cube. For any cube Q, let $\widetilde{Q}$ be the smallest doubling cube which has the form 2^kQ with $k \in \mathbb{N}\cup\{0\}$.

Given two cubes Q, $R \subset \mathbb{R}^d$, let x_Q be the center of Q, and Q_R be the smallest cube concentric with Q containing Q and R. The following coefficients were first introduced by Tolsa in [7]; see also [8, 9].

Definition 2.1. Given two cubes $Q,R \subset \mathbb{R}^d$, we define

$$\delta(Q,R) = \max\left\{\int_{Q_R\setminus Q} \frac{1}{|x-x_Q|^n}\,d\mu(x),\ \int_{R_Q\setminus R} \frac{1}{|x-x_R|^n}\,d\mu(x)\right\}. \tag{2.1}$$

We may treat points $x \in \mathbb{R}^d$ as if they were cubes (with side length $l(x) = 0$). So, for any $x,y \in \mathbb{R}^d$ and cube $Q \subset \mathbb{R}^d$, the notation $\delta(x,Q)$ and $\delta(x,y)$ make sense.

We now recall the notion of cubes of generations in [8, 9].

Definition 2.2. We say that $x \in \mathbb{R}^d$ is a stopping point (or stopping cube) if $\delta(x,Q) < \infty$ for some cube $Q \ni x$ with $0 < l(Q) < \infty$. We say that $\mathbb{R}^d$ is an initial cube if $\delta(Q,\mathbb{R}^d) < \infty$ for some cube Q with $0 < l(Q) < \infty$. The cubes Q such that $0 < l(Q) < \infty$ are called transit cubes.

Remark 2.3. In [8, page 67], it was pointed out that if $\delta(x,Q) < \infty$ for some transit cube Q containing x, then $\delta(x,Q') < \infty$ for any other transit cube Q' containing x. Also, if $\delta(Q,\mathbb{R}^d) < \infty$ for some transit cube Q, then $\delta(Q',\mathbb{R}^d) < \infty$ for any transit cube Q'.

Let A be some big positive constant. In particular, we assume that A is much bigger than the constants ϵ_0, ϵ_1, and γ_0, which appear, respectively, in [8, Lemmas 3.1, 3.2, and 3.3]. Moreover, the constants A, ϵ_0, ϵ_1, and γ_0 depend only on C_0, n, and d. In what follows, for $\epsilon > 0$ and $a,b \in \mathbb{R}$, the notation $a = b \pm \epsilon$ does not mean any precise equality but the estimate $|a-b| \le \epsilon$.

Definition 2.4. Assume that $\mathbb{R}^d$ is not an initial cube. We fix some doubling cube $R_0 \subset \mathbb{R}^d$. This will be our "reference" cube. For each $j \in \mathbb{N}$, let R_{-j} be some doubling cube concentric with R_0, containing R_0, and such that $\delta(R_0,R_{-j}) = jA \pm \epsilon_1$ (which exists because of [8, Lemma 3.3]). If Q is a transit cube, we say that Q is a cube of generation $k \in \mathbb{Z}$ if it is a doubling cube, and for some cube R_{-j} containing Q we have $\delta(Q,R_{-j}) = (j+k)A \pm \epsilon_1$. If $Q \equiv \{x\}$ is a stopping cube, we say that Q is a cube of generation $k \in \mathbb{Z}$ if for some cube R_{-j} containing x we have $\delta(Q,R_{-j}) \le (j+k)A + \epsilon_1$.

We remark that the definition of cubes of generations is proved in [8, page 68] to be independent of the chosen reference $\{R_{-j}\}_{j\in\mathbb{N}\cup\{0\}}$ in the sense modulo some small errors.

Definition 2.5. Assume that $\mathbb{R}^d$ is an initial cube. Then we choose $\mathbb{R}^d$ as our "reference" cube. If Q is a transit cube, we say that Q is a cube of generation $k \geq 1$, if Q is doubling and $\delta(Q, \mathbb{R}^d) = kA \pm \epsilon_1$. If $Q \equiv \{x\}$ is a stopping cube, we say that Q is a cube of generation $k \geq 1$ if $\delta(x, \mathbb{R}^d) \leq kA + \epsilon_1$. Moreover, for all $k \leq 0$, we say that $\mathbb{R}^d$ is a cube of generation k.

In what follows, we also regard that $\mathbb{R}^d$ is a cube centered at all the points $x \in \operatorname{supp}(\mu)$. Using [8, Lemma 3.2], it is easy to verify that for any $x \in \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$, there exists a doubling cube of generation k; see [8, page 68]. Throughout this paper, for any $x \in \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$, we denote by $Q_{x,k}$ a fixed doubling cube centered at x of generation k. By [18, Proposition 2.1] and Definition 2.5, it follows that for any $x \in \operatorname{supp}(\mu)$, $l(Q_{x,k}) \to \infty$ as $k \to -\infty$.

Remark 2.6. We should point out that when $\mathbb{R}^d$ is an initial cube, cubes of generations in [8] were not assumed to be doubling. However, by using [8, Lemma 3.2], it is easy to check that doubling cubes of generations exist even in this case. Moreover, it is not so difficult to verify that $(2, 2^{d+1})$-doubling cubes in [8] can be replaced by (ρ, ρ^{d+1})-doubling cubes for any $\rho \in (1, \infty)$.

In [8], Tolsa constructed an approximation of the identity $S \equiv \{S_k\}_{k=-\infty}^{\infty}$ related to doubling cubes $\{Q_{x,k}\}_{x \in \mathbb{R}^d, k \in \mathbb{Z}}$, which consists of integral operators given by kernels $\{S_k(x,y)\}_{k \in \mathbb{Z}}$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying the following properties:

(A-1) $S_k(x,y) = S_k(y,x)$ for all $x, y \in \mathbb{R}^d$;

(A-2) for any $k \in \mathbb{Z}$ and any $x \in \operatorname{supp}(\mu)$, if $Q_{x,k}$ is a transit cube, then

$$\int_{\mathbb{R}^d} S_k(x,y)\,d\mu(y) = 1; \tag{2.2}$$

(A-3) if $Q_{x,k}$ is a transit cube, then $\operatorname{supp}(S_k(x,\cdot)) \subset Q_{x,k-1}$;

(A-4) if $Q_{x,k}$ and $Q_{y,k}$ are transit cubes, then there exists a constant $C > 0$ such that

$$0 \leq S_k(x,y) \leq \frac{C}{\left[l(Q_{x,k}) + l(Q_{y,k}) + |x-y|\right]^n}; \tag{2.3}$$

(A-5) if $Q_{x,k}$, $Q_{x',k}$, and $Q_{y,k}$ are transit cubes, and $x, x' \in Q_{x_0,k}$ for some $x_0 \in \operatorname{supp}(\mu)$, then there exists a constant $C > 0$ such that

$$\left| S_k(x,y) - S_k(x',y) \right| \leq C \frac{|x-x'|}{l(Q_{x_0,k})} \frac{1}{\left[l(Q_{x,k}) + l(Q_{y,k}) + |x-y|\right]^n}. \tag{2.4}$$

Moreover, Tolsa also pointed out that (A-1) through (A-5) also hold if any of $Q_{x,k}$, $Q_{x',k}$, and $Q_{y,k}$ is a stopping cube, and that (A-1), (A-3) through (A-5) also hold if any of $Q_{x,k}$, $Q_{x',k}$, and $Q_{y,k}$ coincides with $\mathbb{R}^d$, except that (A-2) is replaced by (A-2'). If $Q_{x,k} = \mathbb{R}^d$ for some $x \in \operatorname{supp}(\mu)$, then $S_k = 0$. In what follows, without loss of generality, for any $x \in \operatorname{supp}(\mu)$, we always assume that $Q_{x,k}$ is not a stopping cube, since the proofs for stopping cubes are similar.

We next recall the notions of the spaces $H^1(\mu)$ and $\mathrm{RBMO}(\mu)$ in [9] and the space $\mathrm{RBLO}(\mu)$ in [1].

Definition 2.7. Given $f \in L^1_{\mathrm{loc}}(\mu)$, we set

$$\mathcal{M}_\Phi(f)(x) = \sup_{\varphi\sim x} \left| \int_{\mathbb{R}^d} f\varphi\, d\mu \right|, \tag{2.5}$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)} \leq 1$;

(ii) $0 \leq \varphi(y) \leq 1/|y-x|^n$ for all $y \in \mathbb{R}^d$;

(iii) $|\nabla\varphi(y)| \leq 1/|y-x|^{n+1}$ for all $y \in \mathbb{R}^d$, where $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$.

Definition 2.8. The Hardy space $H^1(\mu)$ is the set of all functions $f \in L^1(\mu)$ satisfying that $\int_{\mathbb{R}^d} f\, d\mu = 0$ and $\mathcal{M}_\Phi f \in L^1(\mu)$. Moreover, we define the norm of $f \in H^1(\mu)$ by

$$\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \left\|\mathcal{M}_\Phi(f)\right\|_{L^1(\mu)}. \tag{2.6}$$

On the Hardy space, Tolsa established the following atomic characterization (see [7, 9]).

Definition 2.9. Let $\eta > 1$ and $1 < p \leq \infty$. A function $b \in L^1_{\mathrm{loc}}(\mu)$ is called a p-atomic block if

(i) there exists some cube R such that $\mathrm{supp}(b) \subset R$;

(ii) $\int_{\mathbb{R}^d} b(x)\,d\mu(x) = 0$;

(iii) for $j = 1, 2$, there exist functions a_j supported on cubes $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\left\|a_j\right\|_{L^p(\mu)} \leq \left[\mu(\eta Q_j)\right]^{1/p-1}\left[1+\delta(Q_j,R)\right]^{-1}. \tag{2.7}$$

We then let $|b|_{H^{1,p}_{\mathrm{atb}}(\mu)} = |\lambda_1| + |\lambda_2|$.

A function $f \in L^1(\mu)$ is said to belong to the space $H^{1,p}_{\mathrm{atb}}(\mu)$ if there exist p-atomic blocks $\{b_i\}_{i\in\mathbb{N}}$ such that $f = \sum_{i=1}^\infty b_i$ with $\sum_{i=1}^\infty |b_i|_{H^{1,p}_{\mathrm{atb}}(\mu)} < \infty$. The $H^{1,p}_{\mathrm{atb}}(\mu)$ norm of f is defined by $\|f\|_{H^{1,p}_{\mathrm{atb}}(\mu)} = \inf\{\sum_{i=1}^\infty |b_i|_{H^{1,p}_{\mathrm{atb}}(\mu)}\}$, where the infimum is taken over all the possible decompositions of f in p-atomic blocks as above.

Remark 2.10. It was proved in [7, 9] that the definition of $H^{1,p}_{\mathrm{atb}}(\mu)$ in [7] is independent of the chosen constant $\eta > 1$, and for any $1 < p < \infty$, all the atomic Hardy spaces $H^{1,p}_{\mathrm{atb}}(\mu)$ coincide with $H^{1,\infty}_{\mathrm{atb}}(\mu)$ with equivalent norms. Moreover, Tolsa proved that $H^{1,\infty}_{\mathrm{atb}}(\mu)$ coincides with $H^1(\mu)$ with equivalent norms (see [9, Theorem 1.2]). Thus, in the rest of this paper, we identify the atomic Hardy space $H^{1,p}_{\mathrm{atb}}(\mu)$ with $H^1(\mu)$, and when we use the atomic characterization of $H^1(\mu)$, we always assume $\eta = 2$ and $p = \infty$ in Definition 2.9.

Definition 2.11. Let $\eta \in (1,\infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space $\mathrm{RBMO}(\mu)$ if there exists some constant $\widetilde{C} \geq 0$ such that for any cube Q centered at some point of $\mathrm{supp}(\mu)$,

$$\frac{1}{\mu(\eta Q)} \int_Q \left| f(y) - m_{\widetilde{Q}}(f) \right| d\mu(y) \leq \widetilde{C}, \tag{2.8}$$

and for any two doubling cubes $Q \subset R$,

$$|m_Q(f) - m_R(f)| \le \widetilde{C}[1 + \delta(Q,R)], \tag{2.9}$$

where $m_Q(f)$ denotes the mean of f over cube Q, namely, $m_Q(f) = (1/\mu(Q))\int_Q f(y)\,d\mu(y)$. Moreover, we define the RBMO(μ) norm of f by the minimal constant $\widetilde{C}$ as above and denote it by $\|f\|_{\mathrm{RBMO}(\mu)}$.

Remark 2.12. It was proved by Tolsa [7] that the definition of RBMO(μ) is independent of the choices of η. As a result, throughout this paper, we always assume $\eta = 2$ in Definition 2.11.

The following space RBLO(μ) was introduced in [1]. It is obvious that $L^\infty(\mu) \subset$ RBLO$(\mu) \subset$ RBMO(μ).

Definition 2.13. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to belong to the space RBLO(μ) if there exists some constant $\widetilde{C} \ge 0$ such that for any doubling cube Q,

$$\frac{1}{\mu(Q)} \int_Q \Big[f(x) - \operatorname*{ess\,inf}_Q f(y) \Big] d\mu(x) \le \widetilde{C}, \tag{2.10}$$

and for any two doubling cubes $Q \subset R$,

$$m_Q(f) - m_R(f) \le \widetilde{C}[1 + \delta(Q,R)]. \tag{2.11}$$

The minimal constant $\widetilde{C}$ as above is defined to be the norm of f in the space RBLO(μ) and denote it by $\|f\|_{\mathrm{RBLO}(\mu)}$.

Remark 2.14. Let $\eta \in (1,\infty)$. It was proved in [17] that we obtain an equivalent norm of RBLO(μ) if (2.10) and (2.11) in Definition 2.13 are, respectively, replaced by that there exists a nonnegative constant $\widetilde{C}$ such that for any cube Q centered at some point of supp(μ),

$$\frac{1}{\mu(\eta Q)} \int_Q \Big[f(x) - \operatorname*{ess\,inf}_{\widetilde{Q}} f(y) \Big] d\mu(x) \le \widetilde{C}, \tag{2.12}$$

and for any two doubling cubes $Q \subset R$,

$$\operatorname*{ess\,inf}_Q f(y) - \operatorname*{ess\,inf}_R f(y) \le \widetilde{C}[1 + \delta(Q,R)]. \tag{2.13}$$

If $\mathbb{R}^d$ is not an initial cube, letting $\{R_{-j}\}_{j=0}^{\infty}$ be as in Definition 2.4, we then define the set $\mathfrak{D} = \{Q \subset \mathbb{R}^d :$ there exists a cube $P \subset Q$ and $j \in \mathbb{N} \cup \{0\}$ such that $P \subset R_{-j}$ with $\delta(P, R_{-j}) \le (j+1)A + \epsilon_1\}$. If $\mathbb{R}^d$ is an initial cube, we define the set $\mathfrak{D} = \{Q \subset \mathbb{R}^d :$ there exists a cube $P \subset Q$ such that $\delta(P, \mathbb{R}^d) \le A + \epsilon_1\}$.

Remark 2.15. In [17], it was pointed out that if $Q \in \mathfrak{D}$, then any R containing Q is also in $\mathfrak{D}$ and the definition of the set $\mathfrak{D}$ is independent of the chosen reference $\{R_{-j}\}_{j \in \mathbb{N} \cup \{0\}}$ in the sense modulo some small error (the error is no more than $2\epsilon_1 + \epsilon_0$); see also [8, page 68]. Moreover, it was also proved in [17] that if μ is the d-dimensional Lebesgue measure on $\mathbb{R}^d$, then for any cube $Q \subset \mathbb{R}^d$, $Q \in \mathfrak{D}$ if and only if $l(Q) \gtrsim 1$.

In [17], we used the set $\mathcal{D}$ to introduce the local Hardy spaces $h^{1,p}_{\mathrm{atb},\eta}(\mu)$, $p \in (1,\infty]$, in the sense of Goldberg [19].

Definition 2.16. For a fixed $\eta \in (1,\infty)$ and $p \in (1,\infty]$, a function $b \in L^1_{\mathrm{loc}}(\mu)$ is called a p-atomic block if it satisfies (i), (ii), and (iii) of Definition 2.9. A function $b \in L^1_{\mathrm{loc}}(\mu)$ is called a p-block if it only satisfies (i) and (iii) of Definition 2.9. In both cases, we let $|b|_{h^{1,p}_{\mathrm{atb},\eta}(\mu)} = \sum_{j=1}^{2} |\lambda_j|$.

Moreover, a function $f \in L^1(\mu)$ is said to belong to the space $h^{1,p}_{\mathrm{atb},\eta}(\mu)$ if there exist p-atomic blocks or p-blocks $\{b_i\}_i$ such that $f = \sum_i b_i$ and $\sum_i |b_i|_{h^{1,p}_{\mathrm{atb},\eta}(\mu)} < \infty$, where b_i is a p-atomic block if $\mathrm{supp}(b_i) \subset R_i$ with $R_i \notin \mathcal{D}$, while b_i is a p-block if $\mathrm{supp}(b_i) \subset R_i$ and $R_i \in \mathcal{D}$. We define the $h^{1,p}_{\mathrm{atb},\eta}(\mu)$ norm of f by letting $\|f\|_{h^{1,p}_{\mathrm{atb},\eta}(\mu)} = \inf\{\sum_i |b_i|_{h^{1,p}_{\mathrm{atb},\eta}(\mu)}\}$, where the infimum is taken over all possible decompositions of f in p-atomic blocks or p-blocks as above.

Remark 2.17. It was proved in [17] that the definition of $h^{1,p}_{\mathrm{atb},\eta}(\mu)$ is independent of the chosen constant $\eta > 1$, and for any $1 < p < \infty$, all the atomic Hardy spaces $h^{1,p}_{\mathrm{atb},\eta}(\mu)$ coincide with $h^{1,\infty}_{\mathrm{atb},\eta}(\mu)$ with equivalent norms. Thus, in the rest of this paper, we always assume $\eta = 2$ and $p = \infty$ in Definition 2.16.

In what follows, for any cube R and $x \in R \cap \mathrm{supp}(\mu)$, let H^x_R be the largest integer k such that $R \subset Q_{x,k}$. The following properties of H^x_R play key roles in the proofs of all theorems in this paper, whose proofs can be found in [17].

Lemma 2.18. *The following properties hold.*

(a) *For any cube R and $x \in R \cap \mathrm{supp}(\mu)$, $Q_{x,H^x_R+1} \subset 3R$ and $5R \subset Q_{x,H^x_R-1}$.*

(b) *For any cube R, $x \in R \cap \mathrm{supp}(\mu)$ and $k \in \mathbb{Z}$ with $k \geq H^x_R + 2$, $Q_{x,k} \subset (7/5)R$.*

(c) *For any cube $R \subset \mathbb{R}^d$ and $x, y \in R \cap \mathrm{supp}(\mu)$, $|H^x_R - H^y_R| \leq 1$.*

(d) *If $\mathbb{R}^d$ is not an initial cube, then for any cube R and $x \in R \cap \mathrm{supp}(\mu)$, $H^x_R \leq 1$ when $R \in \mathcal{D}$ and $H^x_R \geq 0$ when $R \notin \mathcal{D}$. If $\mathbb{R}^d$ is an initial cube, then $0 \leq H^x_R \leq 1$ for any cube $R \in \mathcal{D}$ and $x \in R \cap \mathrm{supp}(\mu)$.*

(e) *For any cube R and $x \in R \cap \mathrm{supp}(\mu)$, there exists a constant $C > 0$ such that $\delta(R, Q_{x,H^x_R}) \leq C$ and $\delta(Q_{x,H^x_R+1}, R) \leq C$.*

3. Uniform boundedness in $H^1(\mu)$ and RBLO(μ)

This section is devoted to establishing the boundedness for approximations of the identity in the spaces $H^1(\mu)$ and RBLO(μ).

Theorem 3.1. *For any $k \in \mathbb{Z}$, let S_k be as in Section 2. Then there exists a constant $C > 0$ independent of k such that for all $f \in H^1(\mu)$,*

$$\|S_k(f)\|_{H^1(\mu)} \leq C\|f\|_{H^1(\mu)}. \tag{3.1}$$

Proof. We use some ideas from [20]. By the Fatou lemma, to show Theorem 3.1, it suffices to prove that for any ∞-atomic block $b = \sum_{j=1}^{2} \lambda_j a_j$ as in Definition 2.9, $\mathcal{M}_\Phi(S_k(b)) \in L^1(\mu)$ and $\|\mathcal{M}_\Phi(S_k(b))\|_{L^1(\mu)} \lesssim \sum_{j=1}^{2} |\lambda_j|$, where $\mathcal{M}_\Phi$ is the maximal operator as in

Definition 2.7. Moreover, if $k \le 0$ and $\mathbb{R}^d$ is an initial cube, then $S_k = 0$, and Theorem 3.1 holds automatically in this case. Therefore, we may assume that $\mathbb{R}^d$ is not an initial cube when $k \le 0$. Using the notation as in Definition 2.9 and choosing any $x_0 \in \operatorname{supp}(\mu) \cap R$, we now consider the following two cases: (1) $k \le H_R^{x_0}$; (2) $k \ge H_R^{x_0} + 1$.

In case (1), write

$$||\mathcal{M}_\Phi(S_k(b))||_{L^1(\mu)} = \int_{8R} \mathcal{M}_\Phi(S_k(b))(x)d\mu(x) + \int_{\mathbb{R}^d \setminus 8R} \cdots \equiv I + II. \tag{3.2}$$

Since $\mathcal{M}_\Phi$ is sublinear, we have that

$$\begin{aligned} I &\le \sum_{j=1}^{2} |\lambda_j| \int_{8R} \mathcal{M}_\Phi(S_k(a_j))(x)d\mu(x) \\ &= \sum_{j=1}^{2} |\lambda_j| \int_{2Q_j} \mathcal{M}_\Phi(S_k(a_j))(x)d\mu(x) + \sum_{j=1}^{2} |\lambda_j| \int_{8R\setminus 2Q_j} \cdots \equiv I_1 + I_2. \end{aligned} \tag{3.3}$$

By (A-2) and (A-4), we see that for any $x \in 2Q_j$, $j = 1, 2$, and $\varphi \sim x$,

$$\left| \int_{\mathbb{R}^d} \varphi(y) S_k(a_j)(y) d\mu(y) \right| \le \iint_{\mathbb{R}^d} \varphi(y) S_k(y,z) |a_j(z)| d\mu(z) d\mu(y) \le ||a_j||_{L^\infty(\mu)}, \tag{3.4}$$

which implies that $\mathcal{M}_\Phi(S_k(a_j))(x) \le \|a_j\|_{L^\infty(\mu)}$. This together with (2.7) further yields

$$I_1 \le \sum_{j=1}^{2} |\lambda_j| ||a_j||_{L^\infty(\mu)} \mu(2Q_j) \lesssim \sum_{j=1}^{2} |\lambda_j|. \tag{3.5}$$

On the other hand, for any $x \in 8R \setminus 2Q_j$ and $z \in Q_j$, $j = 1, 2$, $|x - z| \sim |x - x_j|$, where x_j denotes the center of Q_j. This observation together with the fact that for any $x, y, z \in \mathbb{R}^d$, if $|y - z| < (1/2)|x - z|$, then $|x - z| < 2|x - y|$. The properties (A-2) and (A-4) imply that for any $x \in 8R \setminus 2Q_j$, $\varphi \sim x$ and $z \in Q_j$,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(y) S_k(y,z) d\mu(y) &\lesssim \int_{|y-z| \ge (1/2)|x-z|} \frac{\varphi(y)}{|y - z|^n} d\mu(y) + \int_{|y-z| < (1/2)|x-z|} \frac{S_k(y,z)}{|x - y|^n} d\mu(y) \\ &\lesssim \int_{|y-z| \ge (1/2)|x-z|} \frac{\varphi(y)}{|x - z|^n} d\mu(y) + \int_{|y-z| < (1/2)|x-z|} \frac{S_k(y,z)}{|x - z|^n} d\mu(y) \\ &\lesssim \frac{1}{|x - x_j|^n}. \end{aligned} \tag{3.6}$$

From this fact and (2.7), it then follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\varphi(y)S_k(a_j)(y)d\mu(y)\right| &\le \int_{Q_j}|a_j(z)|\int_{\mathbb{R}^d}\varphi(y)S_k(y,z)d\mu(y)d\mu(z)\\ &\lesssim \frac{1}{|x-x_j|^n}\|a_j\|_{L^\infty(\mu)}\mu(Q_j)\lesssim\frac{1}{|x-x_j|^n}\frac{1}{1+\delta(Q_j,R)}.\end{aligned}\tag{3.7}$$

Thus, for any $x\in 8R\setminus 2Q_j$,

$$\mathcal{M}_\Phi(S_k(a_j))(x)\lesssim\frac{1}{|x-x_j|^n}\frac{1}{1+\delta(Q_j,R)}.\tag{3.8}$$

Moreover, by [8, Lemma 3.1 (a) and (d)], we obtain

$$\delta(2Q_j,8R)\le\delta(Q_j,8R)\lesssim 1+\delta(Q_j,R)+\delta(R,8R)\lesssim 1+\delta(Q_j,R).\tag{3.9}$$

Therefore, it follows that

$$I_2\lesssim\sum_{j=1}^{2}|\lambda_j|\frac{\delta(2Q_j,8R)}{1+\delta(Q_j,R)}\lesssim\sum_{j=1}^{2}|\lambda_j|.\tag{3.10}$$

To estimate II, by the observation that $\int_{\mathbb{R}^d}S_k(b)(x)d\mu(x)=0$, we write

$$\begin{aligned}II &\le\int_{\mathbb{R}^d\setminus 8R}\sup_{\varphi\sim x}\left|\int_{\mathbb{R}^d}S_k(b)(y)[\varphi(y)-\varphi(x_0)]d\mu(y)\right|d\mu(x)\\ &\le\int_{\mathbb{R}^d\setminus 8R}\sup_{\varphi\sim x}\int_{2R}|S_k(b)(y)||\varphi(y)-\varphi(x_0)|d\mu(y)d\mu(x)\\ &\quad+\int_{\mathbb{R}^d\setminus 8R}\sup_{\varphi\sim x}\left|\int_{\mathbb{R}^d\setminus 2R}S_k(b)(y)[\varphi(y)-\varphi(x_0)]d\mu(y)\right|d\mu(x)\equiv II_1+II_2.\end{aligned}\tag{3.11}$$

Notice that for any $y\in 2R$ and $x\in 2^{m+1}R\setminus 2^mR$ with $m\ge 3$, $|x-x_0|\ge l(2^{m-2}R)$, and $|x_0-y|\le 2\sqrt{d}l(R)$, which implies that $|y-x_0|\lesssim|x_0-x|$. This fact together with the mean value theorem yields that for any $\varphi\sim x$,

$$|\varphi(y)-\varphi(x_0)|\lesssim\frac{|y-x_0|}{|x_0-x|^{n+1}}.\tag{3.12}$$

Moreover, let N_j be the smallest integer k such that $2R\subset 2^kQ_j$. Because $\{S_k\}_k$ are bounded on $L^2(\mu)$ uniformly, (A-4) together with the Hölder inequality, [8, Lemma 3.1], (3.12), and (2.7) leads to

$$\begin{aligned}
II_1 \leq & \sum_{j=1}^{2} |\lambda_j| \sum_{m=3}^{\infty} \int_{2^{m+1}R\setminus 2^m R} \Big\{ \sup_{\varphi\sim x} \int_{2R\setminus 2Q_j} |S_k(a_j)(y)|\,|\varphi(y)-\varphi(x_0)|\,d\mu(y) \\
& \qquad + \sup_{\varphi\sim x} \int_{2Q_j} |S_k(a_j)(y)|\,|\varphi(y)-\varphi(x_0)|\,d\mu(y) \Big\} d\mu(x) \\
\lesssim & \sum_{j=1}^{2} |\lambda_j| \sum_{m=3}^{\infty} \int_{2^{m+1}R\setminus 2^m R} \frac{l(R)}{[l(2^m R)]^{n+1}} \Big\{ \int_{2R\setminus 2Q_j} \int_{Q_j} \frac{|a_j(z)|}{|y-z|^n} d\mu(z) d\mu(y) \\
& \qquad + [\mu(2Q_j)]^{1/2} \Big[\int_{2Q_j} |S_k(a_j)(y)|^2 d\mu(y) \Big]^{1/2} \Big\} d\mu(x) \\
\lesssim & \, l(R) \sum_{j=1}^{2} |\lambda_j| \sum_{m=3}^{\infty} \frac{\mu(2^{m+1}R)}{[l(2^m R)]^{n+1}} \Big\{ \sum_{i=1}^{N_j-1} \int_{2^{i+1}Q_j\setminus 2^i Q_j} \int_{Q_j} \frac{\|a_j\|_{L^\infty(\mu)}}{|y-z|^n} d\mu(z) d\mu(y) \\
& \qquad + [\mu(2Q_j)]^{1/2} \Big[\int_{Q_j} |a_j(y)|^2 d\mu(y) \Big]^{1/2} \Big\} \\
\lesssim & \sum_{j=1}^{2} |\lambda_j|\,\|a_j\|_{L^\infty(\mu)} \Big\{ \sum_{i=1}^{N_j-1} \frac{\mu(2^{i+1}Q_j)}{[l(2^i Q_j)]^n} \mu(Q_j) + \mu(2Q_j) \Big\} \\
\lesssim & \sum_{j=1}^{2} |\lambda_j| \left(\frac{1+\delta(2Q_j,2R)}{1+\delta(Q_j,R)} + 1 \right) \lesssim \sum_{j=1}^{2} |\lambda_j|.
\end{aligned} \tag{3.13}$$

To estimate II_2, we write

$$\begin{aligned}
II_2 \leq & \sum_{m=3}^{\infty} \int_{2^{m+1}R\setminus 2^m R} \mathcal{M}_\Phi \left(S_k(b)\chi_{2^{m+2}R\setminus 2^{m-1}R}\right)(x)\, d\mu(x) \\
& + \sum_{m=3}^{\infty} \int_{2^{m+1}R\setminus 2^m R} \sup_{\varphi\sim x} \int_{2^{m+2}R\setminus 2^{m-1}R} |S_k(b)(y)|\,\varphi(x_0)\, d\mu(y) d\mu(x) \\
& + \sum_{m=3}^{\infty} \int_{2^{m+1}R\setminus 2^m R} \sup_{\varphi\sim x} \int_{\mathbb{R}^d\setminus 2^{m+2}R} |S_k(b)(y)|\,|\varphi(y)-\varphi(x_0)|\, d\mu(y) d\mu(x) \\
& + \sum_{m=3}^{\infty} \int_{2^{m+1}R\setminus 2^m R} \sup_{\varphi\sim x} \int_{2^{m-1}R\setminus 2R} |S_k(b)(y)|\,|\varphi(y)-\varphi(x_0)|\, d\mu(y) d\mu(x) \\
\equiv & \, E_1 + E_2 + E_3 + E_4.
\end{aligned} \tag{3.14}$$

Since $\mathcal{M}_\Phi$ is bounded from $H^1(\mu)$ to $L^1(\mu)$ (see [9, Lemma 3.1]) and bounded on $L^\infty(\mu)$, then it is bounded on $L^p(\mu)$ for any $p \in (1,\infty)$ by an argument similar to the proof of [7, Theorem 7.2]. The only difference is that in the current case, we do not need to invoke the sharp operator $\mathcal{M}^\sharp$ in [7, equation (6.4)]. On the other hand, by (A-3) and (A-1), we have $\mathrm{supp}(S_k(b)) \subset \cup_{y\in R} Q_{y,k-1}$, which together with $k \le H_R^{x_0}$ and [8, Lemma 4.2 (c)] further implies that $\mathrm{supp}(S_k(b)) \subset Q_{x_0,k-2}$. These facts together with the Hölder inequality lead to

$$\begin{aligned} E_1 &\le \sum_{m=3}^{\infty} \left\{ \int_{2^{m+1}R\setminus 2^m R} \left[\mathcal{M}_\Phi\left(S_k(b)\chi_{2^{m+2}R\setminus 2^{m-1}R}\right)(x)\right]^2 d\mu(x) \right\}^{1/2} \left[\mu(2^{m+1}R)\right]^{1/2} \\ &\lesssim \sum_{m=3}^{\infty} \left\{ \int_{(2^{m+2}R\setminus 2^{m-1}R)\cap(Q_{x_0,k-2})} \left[S_k(b)(x)\right]^2 d\mu(x) \right\}^{1/2} \left[\mu(2^{m+1}R)\right]^{1/2}. \end{aligned} \tag{3.15}$$

Let m_0 be the largest integer and m_1 be the smallest integer satisfying

$$2^{m_0}R \subset 2Q_{x_0,k} \subset Q_{x_0,k-2} \subset 2^{m_1}R. \tag{3.16}$$

Then [8, Lemma 3.1] along with the facts that $l(2^{m_0}R) \sim l(2Q_{x_0,k})$ and that $l(2^{m_1}R) \sim l(Q_{x_0,k-2})$ yields

$$\delta(2^{m_0}R, 2^{m_1}R) \lesssim 1 + \delta(2Q_{x_0,k}, Q_{x_0,k-2}) \lesssim 1. \tag{3.17}$$

If $m \ge m_1 + 1$, then $Q_{x_0,k-2} \cap (2^{m+2}R \setminus 2^{m-1}R) = \varnothing$, and if $m \le m_0 - 2$, then

$$(Q_{x_0,k-2} \setminus 2Q_{x_0,k}) \cap (2^{m+2}R \setminus 2^{m-1}R) = \varnothing. \tag{3.18}$$

It then follows that

$$\begin{aligned} E_1 &\lesssim \sum_{m=3}^{m_1} \left\{ \int_{(2^{m+2}R\setminus 2^{m-1}R)\cap(2Q_{x_0,k})} \left[S_k(b)(x)\right]^2 d\mu(x) \right\}^{1/2} \left[\mu(2^{m+1}R)\right]^{1/2} \\ &\quad + \sum_{m=m_0-1}^{m_1} \left\{ \int_{(2^{m+2}R\setminus 2^{m-1}R)\cap(Q_{x_0,k-2}\setminus 2Q_{x_0,k})} \left[S_k(b)(x)\right]^2 d\mu(x) \right\}^{1/2} \left[\mu(2^{m+1}R)\right]^{1/2}. \end{aligned} \tag{3.19}$$

Let us estimate the first term. By the vanishing moment of b together with (A-5), (A-1), and $R \subset Q_{x_0,k}$ for $k \le H_R^{x_0}$,

$$\begin{aligned} |S_k(b)(x)| &\le \int_R |S_k(x,z) - S_k(x,x_0)|\,|b(z)|\,d\mu(z) \\ &\lesssim \int_R \frac{|x_0 - z|\,|b(z)|}{l(Q_{x_0,k})\left[l(Q_{x_0,k}) + |x_0 - x|\right]^n}\,d\mu(z) \\ &\lesssim \frac{l(R)\|b\|_{L^1(\mu)}}{l(Q_{x_0,k})\left[l(Q_{x_0,k}) + |x_0 - x|\right]^n}. \end{aligned} \tag{3.20}$$

For any $x \in 2^{m+2}R \setminus 2^{m-1}R$ with $m \ge 3$, if $x \in 2Q_{x_0,k}$, then $|x - x_0| \lesssim l(Q_{x_0,k})$. This observation together with (3.20) implies that

$$\begin{aligned} &\left\{\int_{(2^{m+2}R\setminus 2^{m-1}R)\cap 2Q_{x_0,k}} \left[S_k(b)(x)\right]^2 d\mu(x)\right\}^{1/2} \\ &\quad\lesssim l(R)\|b\|_{L^1(\mu)}\left\{\int_{2^{m+2}R\setminus 2^{m-1}R} \frac{1}{|x_0 - x|^{2(n+1)}}\,d\mu(x)\right\}^{1/2} \\ &\quad\lesssim l(R)\|b\|_{L^1(\mu)}\frac{\left[\mu(2^{m+2}R)\right]^{1/2}}{\left[l(2^mR)\right]^{n+1}}. \end{aligned} \tag{3.21}$$

Moreover, another application of (3.20) leads to that

$$\begin{aligned} &\left\{\int_{(2^{m+2}R\setminus 2^{m-1}R)\cap\left(Q_{x_0,k-2}\setminus 2Q_{x_0,k}\right)} \left[S_k(b)(x)\right]^2 d\mu(x)\right\}^{1/2} \\ &\quad\lesssim \|b\|_{L^1(\mu)}\left\{\int_{2^{m+2}R\setminus 2^{m-1}R} \frac{1}{|x_0 - x|^{2n}}\,d\mu(x)\right\}^{1/2} \lesssim \|b\|_{L^1(\mu)}\frac{\left[\mu(2^{m+2}R)\right]^{1/2}}{\left[l(2^mR)\right]^{n}}. \end{aligned} \tag{3.22}$$

Combining these estimates above, by (1.1), we obtain that

$$\begin{aligned} E_1 &\lesssim \|b\|_{L^1(\mu)}\left\{\sum_{m=3}^{m_1} \frac{l(R)\mu(2^{m+2}R)}{\left[l(2^mR)\right]^{n+1}} + \sum_{m=m_0-1}^{m_1} \frac{\mu(2^{m+2}R)}{\left[l(2^mR)\right]^{n}}\right\} \\ &\lesssim \left[1 + \delta(2Q_{x_0,k}, Q_{x_0,k-2})\right]\|b\|_{L^1(\mu)} \lesssim \sum_{j=1}^{2}|\lambda_j|, \end{aligned} \tag{3.23}$$

where in the last-to-second inequality, we use the following fact that for any cube R,

$$\sum_{m=m_0-1}^{m_1} \frac{\mu(2^{m+1}R)}{\left[l(2^mR)\right]^{n}} \sim 1 + \delta(2^{m_0}R, 2^{m_1}R). \tag{3.24}$$

Similarly, it follows from (3.17), (3.20), (3.24), (1.1), and $\sup_{\varphi\sim x}\varphi(x_0)\le 1/|x-x_0|^n$ that

$$
\begin{aligned}
E_2 &\lesssim \sum_{m=3}^{m_1}\int_{2^{m+1}R\setminus 2^mR}\sup_{\varphi\sim x}\varphi(x_0)\int_{2^{m+2}R\setminus 2^{m-1}R}\frac{l(R)\|b\|_{L^1(\mu)}}{l(Q_{x_0,k})\,|x_0-y|^n}d\mu(y)d\mu(x)\\
&\lesssim \|b\|_{L^1(\mu)}\Bigg\{\sum_{m=3}^{m_1}\int_{2^{m+1}R\setminus 2^mR}\frac{l(R)}{|x_0-x|^n}\int_{(2^{m+2}R\setminus 2^{m-1}R)\cap 2Q_{x_0,k}}\frac{1}{|x_0-y|^{n+1}}d\mu(y)d\mu(x)\\
&\qquad+\sum_{m=m_0-1}^{m_1}\int_{2^{m+1}R\setminus 2^mR}\frac{1}{|x_0-x|^n}\\
&\qquad\times\int_{(2^{m+2}R\setminus 2^{m-1}R)\cap\left(Q_{x_0,k-2}\setminus 2Q_{x_0,k}\right)}\frac{1}{|x_0-y|^n}d\mu(y)d\mu(x)\Bigg\}\\
&\lesssim \|b\|_{L^1(\mu)}\Bigg\{\sum_{m=3}^{m_1}\frac{l(R)\mu(2^{m+2}R)}{[l(2^mR)]^{n+1}}+\sum_{m=m_0-1}^{m_1}\frac{\mu(2^{m+1}R)}{[l(2^mR)]^n}\delta(2Q_{x_0,k},Q_{x_0,k-2})\Bigg\}\lesssim\sum_{j=1}^{2}|\lambda_j|.
\end{aligned}
\tag{3.25}
$$

Now we estimate E_3. Recalling that $\operatorname{supp}(S_k(b))\subset Q_{x_0,k-2}\subset 2^{m_1}R$, we see

$$
E_3=\sum_{m=3}^{m_1-3}\int_{2^{m+1}R\setminus 2^mR}\sup_{\varphi\sim x}\int_{\mathbb{R}^d\setminus 2^{m+2}R}|S_k(b)(y)|\,|\varphi(y)-\varphi(x_0)|\,d\mu(y)d\mu(x).\tag{3.26}
$$

For any $m\le m_1-3$, any $x\in 2^{m+1}R\setminus 2^mR$ and $y\in 2^{i+1}R\setminus 2^iR$ with $i\ge m+2$, it is easy to see that

$$
|x_0-x|\gtrsim 2^m l(R),\qquad |y-x|\gtrsim 2^m l(R).\tag{3.27}
$$

Using (3.20) again, we have

$$
\begin{aligned}
&\sup_{\varphi\sim x}\int_{\mathbb{R}^d\setminus 2^{m+2}R}|S_k(b)(y)|\,|\varphi(y)-\varphi(x_0)|\,d\mu(y)\\
&\quad\lesssim\sum_{i=m+2}^{\infty}\int_{(2^{i+1}R\setminus 2^iR)\cap Q_{x_0,k-2}}\frac{l(R)\|b\|_{L^1(\mu)}}{l(Q_{x_0,k})\,|x_0-y|^n}\left(\frac{1}{|y-x|^n}+\frac{1}{|x_0-x|^n}\right)d\mu(y)\\
&\quad\lesssim\frac{\|b\|_{L^1(\mu)}}{[l(2^mR)]^n}\sum_{i=m+2}^{m_1-3}\int_{(2^{i+1}R\setminus 2^iR)\cap Q_{x_0,k-2}}\frac{l(R)}{l(Q_{x_0,k})\,|x_0-y|^n}d\mu(y)\\
&\quad\lesssim\frac{\|b\|_{L^1(\mu)}}{[l(2^mR)]^n}\sum_{i=m+2}^{m_1-3}\Bigg\{\int_{(2^{i+1}R\setminus 2^iR)\cap 2Q_{x_0,k}}\frac{l(R)}{|x_0-y|^{n+1}}d\mu(y).\\
&\qquad+\int_{(2^{i+1}R\setminus 2^iR)\cap(Q_{x_0,k-2}\setminus 2Q_{x_0,k})}\frac{l(R)}{l(Q_{x_0,k})\,|x_0-y|^n}d\mu(y)\Bigg\}.
\end{aligned}
\tag{3.28}
$$

Therefore, from (3.17), (3.20), (3.24), and (1.1), it follows that

$$
\begin{aligned}
E_3 &\lesssim \|b\|_{L^1(\mu)}\Bigg\{\sum_{m=3}^{m_1-3}\frac{\mu(2^{m+1}R)}{[l(2^mR)]^n}\sum_{i=m+2}^{m_1-3}\int_{(2^{i+1}R\setminus 2^iR)\cap 2Q_{x_0,k}}\frac{l(R)}{|x_0-y|^{n+1}}d\mu(y)\\
&\quad+\sum_{m=m_0-1}^{m_1-3}\frac{\mu(2^{m+1}R)}{[l(2^mR)]^n}\sum_{i=m+2}^{m_1-3}\int_{(2^{i+1}R\setminus 2^iR)\cap(Q_{x_0,k-2}\setminus 2Q_{x_0,k})}\frac{1}{|x_0-y|^n}d\mu(y)\\
&\quad+\sum_{m=3}^{m_0-2}\frac{\mu(2^{m+1}R)}{[l(2^mR)]^n}\sum_{i=m+2}^{m_1-3}\int_{(2^{i+1}R\setminus 2^iR)\cap(Q_{x_0,k-2}\setminus 2Q_{x_0,k})}\frac{l(R)}{l(Q_{x_0,k})|x_0-y|^n}d\mu(y)\Bigg\}\\
&\lesssim \|b\|_{L^1(\mu)}\Bigg\{\sum_{m=3}^{m_1-3}\sum_{i=m+2}^{m_1-3}\frac{\mu(2^{i+1}R)l(R)}{[l(2^iR)]^{n+1}}+\sum_{m=m_0-1}^{m_1-3}\frac{\mu(2^{m+1}R)}{[l(2^mR)]^n}\sum_{i=m_0+1}^{m_1-3}\frac{\mu(2^{i+1}R)}{[l(2^iR)]^n}\\
&\quad+\sum_{m=3}^{m_0-2}\sum_{i=m+2}^{m_0}\frac{\mu(2^{i+1}R)l(R)}{[l(2^iR)]^{n+1}}+\sum_{m=3}^{m_0-2}\sum_{i=m_0}^{m_1-3}\frac{\mu(2^{i+1}R)}{[l(2^iR)]^n}\frac{l(R)}{l(2^mR)}\Bigg\}\\
&\lesssim \|b\|_{L^1(\mu)}\big[1+\delta(2Q_{x_0,k},Q_{x_0,k-2})\big]^2\lesssim\sum_{j=1}^{2}|\lambda_j|,
\end{aligned}
\tag{3.29}
$$

where in the third-to-last inequality, we used the facts that if $i\le m_0$, then $l(2^iR)\le l(Q_{x_0,k})$ and that if $m\le m_0-2$, then $l(2^mR)\le l(Q_{x_0,k})$.

Now we estimate E_4. Notice that if $m\le m_0+1$, then $(2^{m-1}R\setminus 2R)\cap(Q_{x_0,k-2}\setminus 2Q_{x_0,k})=\emptyset$. Therefore, by $\operatorname{supp}(S_k(b))\subset Q_{x_0,k-2}$, we have

$$
\begin{aligned}
E_4 &\le \sum_{m=3}^{\infty}\int_{2^{m+1}R\setminus 2^mR}\sup_{\varphi\sim x}\int_{(2^{m-1}R\setminus 2R)\cap 2Q_{x_0,k}}|S_k(b)(y)||\varphi(y)-\varphi(x_0)|d\mu(y)d\mu(x)\\
&\quad+\sum_{m=m_0+2}^{m_1-1}\int_{2^{m+1}R\setminus 2^mR}\sup_{\varphi\sim x}\int_{(2^{m-1}R\setminus 2R)\cap(Q_{x_0,k-2}\setminus 2Q_{x_0,k})}\cdots\\
&\quad+\sum_{m=m_1}^{\infty}\int_{2^{m+1}R\setminus 2^mR}\sup_{\varphi\sim x}\int_{(2^{m-1}R\setminus 2R)\cap(Q_{x_0,k-2}\setminus 2Q_{x_0,k})}\cdots\equiv J_1+J_2+J_3.
\end{aligned}
\tag{3.30}
$$

Observing that (3.12) holds for any $y\in 2^{m-1}R\setminus 2R$ and $x\in 2^{m+1}R\setminus 2^mR$ with $m\ge 3$, by (3.12), (3.20), and (1.1), we see that

$$
\begin{aligned}
&\sup_{\varphi\sim x}\int_{(2^{m-1}R\setminus 2R)\cap 2Q_{x_0,k}}|S_k(b)(y)||\varphi(y)-\varphi(x_0)|d\mu(y)\\
&\quad\lesssim\int_{(2^{m-1}R\setminus 2R)\cap 2Q_{x_0,k}}|S_k(b)(y)|\frac{l(Q_{x_0,k})}{|x_0-x|^{n+1}}d\mu(y)\\
&\quad\lesssim\frac{l(R)\|b\|_{L^1(\mu)}}{|x_0-x|^{n+1}}\int_{(2^{m-1}R\setminus 2R)\cap 2Q_{x_0,k}}\frac{1}{[l(Q_{x_0,k})+|x_0-y|]^n}d\mu(y)\lesssim\frac{l(R)\|b\|_{L^1(\mu)}}{|x_0-x|^{n+1}}.
\end{aligned}
\tag{3.31}
$$

From this fact and (1.1), it follows that

$$J_1 \lesssim \|b\|_{L^1(\mu)} l(R) \sum_{m=3}^{\infty} \int_{2^{m+1}R\setminus 2^m R} \frac{1}{|x_0 - x|^{n+1}} d\mu(x) \lesssim \sum_{j=1}^{2} |\lambda_j|. \tag{3.32}$$

On the other hand, since (3.27) holds for any $x \in 2^{m+1}R \setminus 2^m R$ and $y \in 2^{m-1}R \setminus 2R$ with $m \geq 3$, by (3.17), (3.20), and (3.24) together with Definition 2.7 (ii),

$$\begin{aligned} J_2 &\lesssim \sum_{m=m_0+2}^{m_1-1} \int_{2^{m+1}R\setminus 2^m R} \int_{(2^{m-1}R\setminus 2R)\cap(Q_{x_0,k-2}\setminus 2Q_{x_0,k})} \frac{\|b\|_{L^1(\mu)} l(R)}{l(Q_{x_0,k})\,|x_0 - y|^n} \\ &\quad \times \left(\frac{1}{|y-x|^n} + \frac{1}{|x_0 - x|^n} \right) d\mu(y)\, d\mu(x) \\ &\lesssim \|b\|_{L^1(\mu)} \sum_{m=m_0+2}^{m_1-1} \frac{\mu(2^{m+1}R)}{[l(2^m R)]^n} \int_{Q_{x_0,k-2}\setminus 2Q_{x_0,k}} \frac{1}{|x_0 - y|^n} d\mu(y) \lesssim \sum_{j=1}^{2} |\lambda_j|. \end{aligned} \tag{3.33}$$

Finally, using (3.27), (3.12), (3.17), (3.20), (1.1), and the fact that for any $y \in Q_{x_0,k-2}$, $|x_0 - y| \lesssim l(2^{m_1}R)$, we have

$$\begin{aligned} J_3 &\lesssim \sum_{m=m_1}^{\infty} \int_{2^{m+1}R\setminus 2^m R} \int_{Q_{x_0,k-2}\setminus 2Q_{x_0,k}} \frac{\|b\|_{L^1(\mu)}}{|x_0 - y|^n} \frac{l(2^{m_1}R)}{|x_0 - x|^{n+1}} d\mu(y)\, d\mu(x) \\ &\lesssim \|b\|_{L^1(\mu)} \sum_{m=m_1}^{\infty} \frac{l(2^{m_1}R)\mu(2^{m+1}R)}{[l(2^m R)]^{n+1}} \lesssim \sum_{j=1}^{2} |\lambda_j|. \end{aligned} \tag{3.34}$$

Combining the estimates for J_1, J_2, and J_3 completes the proof of Theorem 3.1 in case (1).

In case (2), we further consider the following two subcases. Subcase (i) $k \geq H_R^{x_0} + 1$ and for all $y \in R \cap \operatorname{supp}(\mu)$, $R \not\subset Q_{y,k-1}$. In this subcase, it is easy to see that for any $y \in R$, $Q_{y,k-1} \subset 4R$, which together with $\operatorname{supp}(S_k(b)) \subset \cup_{y\in R} Q_{y,k-1}$ implies that $\operatorname{supp}(S_k(b)) \subset 4R$. Let I and II be as in case (1). We also have $\|\mathcal{M}_\Phi(S_k(b))\|_{L^1(\mu)} \leq I + II$ and $I \lesssim \sum_{j=1}^{2} |\lambda_j|$. On the other hand, since $\operatorname{supp}(S_k(b)) \subset 4R$, similar to the estimate for II_1 in case (1) with $2R$ replaced by $4R$, we obtain

$$II \leq \int_{\mathbb{R}^d\setminus 8R} \sup_{\varphi\sim x} \int_{4R} |S_k(b)(y)|\,|\varphi(y) - \varphi(x_0)|\, d\mu(y)\, d\mu(x) \lesssim \sum_{j=1}^{2} |\lambda_j|. \tag{3.35}$$

Subcase (ii) $k \geq H_R^{x_0} + 1$ and there exists some $y_0 \in R \cap \operatorname{supp}(\mu)$ such that $R \subset Q_{y_0,k-1}$. In this subcase, by applying [8, Lemma 4.2], we see that $\operatorname{supp}(S_k(b)) \subset \cup_{y\in R} Q_{y,k-1} \subset Q_{y_0,k-2} \subset Q_{x_0,k-3}$. Then

$$\|\mathcal{M}_\Phi(S_k(b))\|_{L^1(\mu)} = \int_{4Q_{x_0,k-3}} \mathcal{M}_\Phi(S_k(b))(x)\, d\mu(x) + \int_{\mathbb{R}^d\setminus 4Q_{x_0,k-3}} \cdots \equiv F_1 + F_2. \tag{3.36}$$

Arguing as in the estimate for II_1 in case (1) with $2R$ replaced by $Q_{x_0,k-3}$ again, we have $F_2 \lesssim \sum_{j=1}^2 |\lambda_j|$. On the other hand, by the fact that $\mathcal{M}_\Phi$ is sublinear, we obtain

$$F_1 \leq \sum_{j=1}^{2} |\lambda_j| \int_{2Q_j} \mathcal{M}_\Phi(S_k(a_j))(x)d\mu(x) + \sum_{j=1}^{2} |\lambda_j| \int_{4Q_{x_0,k-3}\setminus 2Q_j} \cdots \equiv L_1 + L_2. \tag{3.37}$$

Since the argument of I_1 in case (1) still works for L_1, it suffices to show $L_2 \lesssim \sum_{j=1}^2 |\lambda_j|$. However, because $R \subset Q_{y_0,k-1}$, we obtain that $k \leq H_R^{y_0} + 1$. This fact together with Lemma 2.18(c) leads to that $k \leq H_R^{x_0} + 2$. Then by the assumption that $H_R^{x_0} + 1 \leq k$ together with [8, Lemma 3.1] and Lemma 2.18(e) implies $\delta(R, Q_{x_0,k-2}) \lesssim 1 + \delta(R, Q_{x_0,H_R^{x_0}}) + \delta(Q_{x_0,H_R^{x_0}}, Q_{x_0,k-2}) \lesssim 1$. Moreover, another application of [8, Lemma 3.1] yields

$$\begin{aligned}\delta(2Q_j, 4Q_{x_0,k-2}) &\leq \delta(Q_j, 4Q_{x_0,k-2}) \\ &\lesssim 1 + \delta(Q_j, R) + \delta(R, Q_{x_0,k-2}) + \delta(Q_{x_0,k-2}, 4Q_{x_0,k-2}) \\ &\lesssim 1 + \delta(Q_j, R).\end{aligned} \tag{3.38}$$

Therefore, arguing as in case (1), we have

$$L_2 \lesssim \sum_{j=1}^{2} |\lambda_j| \frac{\delta(2Q_j, 4Q_{x_0,k-2})}{1 + \delta(Q_j, R)} \lesssim \sum_{j=1}^{2} |\lambda_j|, \tag{3.39}$$

which completes the proof of Theorem 3.1. □

For any $k \in \mathbb{Z}$, from Theorem 3.1, the linearity of S_k, the fact that $(H^1(\mu))^* = \mathrm{RBMO}(\mu)$, and a dual argument, it is easy to deduce the uniform boundedness of S_k in $\mathrm{RBMO}(\mu)$. We omit the details.

Corollary 3.2. *For any $k \in \mathbb{Z}$, let S_k be as in Section 2. Then there exists a constant $C > 0$ independent of k such that for all $f \in \mathrm{RBMO}(\mu)$,*

$$\|S_k(f)\|_{\mathrm{RBMO}(\mu)} \leq C\|f\|_{\mathrm{RBMO}(\mu)}. \tag{3.40}$$

We now consider the uniform boundedness of S_k in $\mathrm{RBLO}(\mu)$. To this end, we first establish the following lemma, which is a version of [18, Lemma 3.1] for $\mathrm{RBLO}(\mu)$.

Lemma 3.3. *There exists a constant $C > 0$ such that for any two cubes $Q \subset R$ and $f \in \mathrm{RBLO}(\mu)$,*

$$\int_R \frac{\left| f(y) - \operatorname{ess\,inf}_{y\in\widetilde{Q}} f(y)\right|}{[|y - x_Q| + l(Q)]^n} d\mu(y) \leq C[1 + \delta(Q, R)]^2 \|f\|_{\mathrm{RBLO}(\mu)}. \tag{3.41}$$

Proof. The proof of this lemma can be conducted as that of [18, Lemma 3.1]. Alternatively, since $\mathrm{RBLO}(\mu) \subset \mathrm{RBMO}(\mu)$, we can also deduce it from [18, Lemma 3.1] as below. From Definition 2.13, it is easy to see that for any $f \in \mathrm{RBLO}(\mu)$ and cube Q,

$$m_{\widetilde{Q}}(f) - \operatorname*{ess\,inf}_{y\in\widetilde{Q}} f(y) \leq \|f\|_{\mathrm{RBLO}(\mu)}. \tag{3.42}$$

Therefore, an easy computation involving [18, Lemma 3.1] and (1.1) yields

$$\begin{aligned}
&\int_R \frac{\left|f(y)-\operatorname{ess\,inf}_{y\in\widetilde{Q}} f(y)\right|}{[|y-x_Q|+l(Q)]^n}\,d\mu(y)\\
&\quad\le \int_R \frac{|f(y)-m_{\widetilde{Q}}(f)|}{[|y-x_Q|+l(Q)]^n}\,d\mu(y)+\int_R \frac{m_{\widetilde{Q}}(f)-\operatorname{ess\,inf}_{y\in\widetilde{Q}} f(y)}{[|y-x_Q|+l(Q)]^n}\,d\mu(y)\\
&\quad\lesssim [1+\delta(Q,R)]^2\|f\|_{\mathrm{RBLO}(\mu)},
\end{aligned}\tag{3.43}$$

which completes the proof of Lemma 3.3. □

Theorem 3.4. *For any $k\in\mathbb{Z}$, let S_k be as in Section 2. Then S_k is uniformly bounded on* $\mathrm{RBLO}(\mu)$*, namely, there exists a nonnegative constant C independent of k such that for all* $f\in\mathrm{RBLO}(\mu)$,

$$\|S_k(f)\|_{\mathrm{RBLO}(\mu)}\le C\|f\|_{\mathrm{RBLO}(\mu)}.\tag{3.44}$$

Proof. Without loss of generality, we may assume that $\|f\|_{\mathrm{RBLO}(\mu)}=1$. We only need to consider the case that $\mathbb{R}^d$ is not an initial cube, since if $\mathbb{R}^d$ is an initial cube, then for any $k\in\mathbb{N}$, the argument is similar; and for any $k\le 0$, $S_k=0$, and Theorem 3.4 holds automatically in this case. To this end, it suffices to show that for any doubling Q,

$$\frac{1}{\mu(Q)}\int_Q\left[S_k(f)(x)-\operatorname*{ess\,inf}_Q S_k(f)(y)\right]d\mu(x)\lesssim 1,\tag{3.45}$$

and for any two doubling cubes $Q\subset R$,

$$m_Q(S_k(f))-m_R(S_k(f))\lesssim 1+\delta(Q,R).\tag{3.46}$$

To show (3.45), let us consider the following two cases:

(i) there exists some $x_0\in Q\cap\operatorname{supp}(\mu)$ such that $Q\subset Q_{x_0,k-2}$;

(ii) for any $x\in Q\cap\operatorname{supp}(\mu)$, $Q\not\subset Q_{x,k-2}$.

In case (i), for each $x\in Q$,

$$\begin{aligned}
S_k(f)(x)-\operatorname*{ess\,inf}_Q S_k(f)(y)&=\left[S_k(f)(x)-\operatorname*{ess\,inf}_{Q_{x,k}} f(y)\right]+\left[\operatorname*{ess\,inf}_{Q_{x,k}} f(y)-\operatorname*{ess\,inf}_Q S_k(f)(y)\right]\\
&\equiv I_1+I_2.
\end{aligned}\tag{3.47}$$

It then follows from (A-3), (A-4), and Lemma 3.3 that

$$I_1\lesssim\int_{Q_{x,k-1}}\frac{\left|f(y)-\operatorname{ess\,inf}_{Q_{x,k}} f(y)\right|}{[|x-y|+l(Q_{x,k})]^n}\,d\mu(y)\lesssim 1.\tag{3.48}$$

On the other hand, in this case, for any $x,y\in Q\cap\operatorname{supp}(\mu)$, we have that $Q_{x,k}$ and $Q_{y,k}$ are contained in $Q_{x,k-4}$ by [8, Lemma 4.2], which together with (2.13) and [8, Lemma 3.1] further yields

$$\begin{aligned}
&\Big|\operatorname*{ess\,inf}_{Q_{x,k}} f(y)-\operatorname*{ess\,inf}_{Q_{y,k}} f(y)\Big|\\
&\quad\le \Big|\operatorname*{ess\,inf}_{Q_{x,k}} f(y)-\operatorname*{ess\,inf}_{Q_{x,k-4}} f(y)\Big|+\Big|\operatorname*{ess\,inf}_{Q_{x,k-4}} f(y)-\operatorname*{ess\,inf}_{Q_{y,k}} f(y)\Big|\\
&\quad\lesssim 1+\delta(Q_{x,k},Q_{x,k-4})+\delta(Q_{y,k},Q_{x,k-4})\\
&\quad\lesssim 1+\delta(Q_{y,k},Q_{y,k-3})+\delta(Q_{y,k-3},Q_{x,k-4})\\
&\quad\lesssim 1+\delta(Q_{y,k-3},Q_{y,k-5})\lesssim 1.
\end{aligned}\tag{3.49}$$

By this observation, (A-2) through (A-4) and Lemma 3.3, similar to the proof of (3.48), we see that for any $y\in Q\cap\operatorname{supp}(\mu)$,

$$\begin{aligned}
&S_k(f)(y)-\operatorname*{ess\,inf}_{Q_{x,k}} f(z)\\
&\quad\le\int_{Q_{y,k-1}} S_k(y,w)\Big|f(w)-\operatorname*{ess\,inf}_{Q_{x,k}} f(z)\Big|\,d\mu(w)\\
&\quad\le\int_{Q_{y,k-1}} S_k(y,w)\Big|f(w)-\operatorname*{ess\,inf}_{Q_{y,k}} f(z)\Big|\,d\mu(w)+\Big|\operatorname*{ess\,inf}_{Q_{x,k}} f(z)-\operatorname*{ess\,inf}_{Q_{y,k}} f(z)\Big|\lesssim 1.
\end{aligned}\tag{3.50}$$

Taking the infimum over all doubling cubes containing y, we have $I_2\lesssim 1$, which completes the proof of case (i).

In case (ii), it easy to see that for any $y\in Q\cap\operatorname{supp}(\mu)$, $k\ge H_Q^y+3$. Then by Lemma 2.18(b), for any $y\in Q\cap\operatorname{supp}(\mu)$, $Q_{y,k-1}\subset(7/5)Q$. Therefore, for any $x,y\in Q$,

$$S_k(f)(x)-S_k(f)(y)\le\Big[S_k(f)(x)-\operatorname*{ess\,inf}_{\widetilde{(7/5)Q}} f(y)\Big]+\Big[\operatorname*{ess\,inf}_{Q_{y,k}} f(y)-S_k(f)(y)\Big]\equiv J_1+J_2.\tag{3.51}$$

From the Tonelli theorem, (A-1), (A-2), (2.12), and the doubling property of Q, it follows that

$$\frac{1}{\mu(Q)}\int_Q J_1\,d\mu(x)\le\frac{1}{\mu(Q)}\int_{(7/5)Q}\Big|f(w)-\operatorname*{ess\,inf}_{\widetilde{(7/5)Q}} f(y)\Big|\,d\mu(w)\lesssim 1.\tag{3.52}$$

On the other hand, (3.48) implies that $J_2\lesssim 1$, which verifies (3.45).

Now we estimate (3.46). As in the proof of (3.45), we consider the following three cases:

(i) there exists some $x_0\in Q\cap\operatorname{supp}(\mu)$ such that $R\subset Q_{x_0,k-2}$;

(ii) for any $x\in Q\cap\operatorname{supp}(\mu)$, $Q\not\subset Q_{x,k-2}$;

(iii) for any $x\in Q\cap\operatorname{supp}(\mu)$, $R\not\subset Q_{x,k-2}$, and there exists some $x_0\in Q\cap\operatorname{supp}(\mu)$ such that $Q\subset Q_{x_0,k-2}$.

In case (i), (3.49) together with (3.48) leads to

$$\begin{aligned} & m_Q(S_k(f)) - m_R(S_k(f)) \\ & \quad = \frac{1}{\mu(Q)}\frac{1}{\mu(R)}\int_Q\int_R [S_k(f)(x) - S_k(f)(y)]\,d\mu(x)d\mu(y) \\ & \quad \le \frac{1}{\mu(Q)}\frac{1}{\mu(R)}\int_Q\int_R \Big\{ \Big| S_k(f)(x) - \operatorname*{ess\,inf}_{z\in Q_{x,k}} f(z)\Big| + \Big|\operatorname*{ess\,inf}_{z\in Q_{x,k}} f(z) - \operatorname*{ess\,inf}_{z\in Q_{y,k}} f(z)\Big| \\ & \qquad + \Big| S_k(f)(y) - \operatorname*{ess\,inf}_{z\in Q_{y,k}} f(z)\Big| \Big\} d\mu(x)d\mu(y) \lesssim 1. \end{aligned} \tag{3.53}$$

In case (ii), Lemma 2.18(b) implies that for any $x \in Q \cap \operatorname{supp}(\mu)$, $Q_{x,k-1} \subset \frac{7}{5}Q$. By [8, Lemma 3.1] and Remark 2.14,

$$\begin{aligned} \Big| \operatorname*{ess\,inf}_{z\in \widetilde{(7/5)Q}} f(z) - \operatorname*{ess\,inf}_{z\in \widetilde{(7/5)R}} f(z)\Big| & \le \Big| \operatorname*{ess\,inf}_{z\in \widetilde{(7/5)Q}} f(z) - \operatorname*{ess\,inf}_{z\in Q} f(z)\Big| + \Big| \operatorname*{ess\,inf}_{z\in Q} f(z) - \operatorname*{ess\,inf}_{z\in \widetilde{(7/5)R}} f(z)\Big| \\ & \lesssim 1 + \delta(Q,R). \end{aligned} \tag{3.54}$$

This fact and the Tonelli theorem yield

$$\begin{aligned} & m_Q(S_k(f)) - m_R(S_k(f)) \\ & \quad \le \frac{1}{\mu(Q)}\frac{1}{\mu(R)}\int_Q\int_R |S_k(f)(x) - S_k(f)(y)|\,d\mu(x)d\mu(y) \\ & \quad \le \frac{1}{\mu(Q)}\frac{1}{\mu(R)}\int_Q\int_R \Big\{ \Big| S_k(f)(x) - \operatorname*{ess\,inf}_{z\in \widetilde{(7/5)Q}} f(z)\Big| + \Big|\operatorname*{ess\,inf}_{z\in \widetilde{(7/5)Q}} f(z) - \operatorname*{ess\,inf}_{z\in \widetilde{(7/5)R}} f(z)\Big| \\ & \qquad + \Big| S_k(f)(y) - \operatorname*{ess\,inf}_{z\in \widetilde{(7/5)R}} f(z)\Big| \Big\} d\mu(x)d\mu(y) \lesssim 1 + \delta(Q,R). \end{aligned} \tag{3.55}$$

Finally, in case (iii), by [8, Lemma 3.1(e)] and the fact that for any $x \in Q \cap \operatorname{supp}(\mu)$, $Q_{x,k-1} \subset (7/5)R$, and $Q_{x_0,k-2} \subset Q_{x,k-3}$, we have that for any $x \in Q \cap \operatorname{supp}(\mu)$,

$$\begin{aligned} \Big| \operatorname*{ess\,inf}_{z\in Q_{x,k}} f(z) - \operatorname*{ess\,inf}_{z\in \widetilde{(7/5)R}} f(z)\Big| & \le 1 + \delta\Big(Q_{x,k}, \widetilde{\frac{7}{5}R}\Big) \\ & \lesssim 1 + \delta(Q_{x,k}, Q_{x_0,k-2}) + \delta\Big(Q_{x_0,k-2}, \widetilde{\frac{7}{5}R}\Big) \\ & \lesssim 1 + \delta(Q_{x,k}, Q_{x,k-3}) + \delta\Big(Q, \widetilde{\frac{7}{5}R}\Big) \lesssim 1 + \delta(Q,R). \end{aligned} \tag{3.56}$$

From this, the Tonelli theorem, and (3.48), we deduce that

$$
\begin{aligned}
& m_Q(S_k(f)) - m_R(S_k(f)) \\
& \quad \le \frac{1}{\mu(Q)} \frac{1}{\mu(R)} \int_Q \int_R \bigg\{ \bigg| S_k(f)(x) - \operatorname*{ess\,inf}_{z \in Q_{x,k}} f(z) \bigg| + \bigg| \operatorname*{ess\,inf}_{z \in Q_{x,k}} f(z) - \operatorname*{ess\,inf}_{z \in \widetilde{(7/5)R}} f(z) \bigg| \\
& \qquad\qquad + \bigg| \operatorname*{ess\,inf}_{z \in \widetilde{(7/5)R}} f(z) - S_k(f)(y) \bigg| \bigg\} d\mu(x) d\mu(y) \lesssim 1 + \delta(Q,R),
\end{aligned}
\tag{3.57}
$$

which completes the proof of Theorem 3.4. $\square$

4. Maximal operators in $H^1(\mu)$ and $h^{1,\infty}_{\mathrm{atb}}(\mu)$

In this section, let $S = \{S_k\}_{k \in \mathbb{Z}}$ be an approximation of the identity as in Section 2. We then consider the following maximal operators: for any locally integrable function f, define

$$
\begin{aligned}
\dot{\mathcal{M}}_S(f)(x) &\equiv \sup_{k \in \mathbb{Z}} |S_k(f)(x)|, \\
\mathcal{M}_S(f)(x) &\equiv \sup_{k \in \mathbb{N}} |S_k(f)(x)|.
\end{aligned}
\tag{4.1}
$$

Obviously, $\mathcal{M}_S(f)(x) \le \dot{\mathcal{M}}_S(f)(x)$ for all $x \in \mathbb{R}^d$, which together with [8, Remark 8.1] further implies the following lemma.

Lemma 4.1. *Let $p \in (1, \infty]$. Then there exists a constant $C_p > 0$ such that for all $f \in L^p(\mu)$,*

$$
\|\mathcal{M}_S(f)\|_{L^p(\mu)} \le \|\dot{\mathcal{M}}_S(f)\|_{L^p(\mu)} \le C_p \|f\|_{L^p(\mu)}
\tag{4.2}
$$

and there exists a constant $C > 0$ such that for all $f \in L^1(\mu)$ and all $\lambda > 0$,

$$
\mu(\{x \in \mathbb{R}^d : \mathcal{M}_S(f)(x) > \lambda\}) \le \mu(\{x \in \mathbb{R}^d : \dot{\mathcal{M}}_S(f)(x) > \lambda\}) \le \frac{C}{\lambda} \|f\|_{L^1(\mu)}.
\tag{4.3}
$$

The following result further shows that $\dot{\mathcal{M}}_S$ is bounded from $H^1(\mu)$ to $L^1(\mu)$.

Theorem 4.2. *There exists a nonnegative constant C such that for all $f \in H^1(\mu)$,*

$$
\|\dot{\mathcal{M}}_S(f)\|_{L^1(\mu)} \le C \|f\|_{H^1(\mu)}.
\tag{4.4}
$$

Proof. Let $b = \lambda_1 a_1 + \lambda_2 a_2$ be any ∞-atomic block as in Definition 2.9. By the Fatou lemma, to prove Theorem 4.2, it suffices to show that

$$
\|\dot{\mathcal{M}}_S(b)\|_{L^1(\mu)} \lesssim |\lambda_1| + |\lambda_2|.
\tag{4.5}
$$

Since $\dot{\mathcal{M}}_S$ is sublinear, we write

$$\begin{aligned}
&\int_{\mathbb{R}^d} \dot{\mathcal{M}}_S(b)(x)\,d\mu(x) \\
&\quad = \int_{4R} \dot{\mathcal{M}}_S(b)(x)\,d\mu(x) + \int_{\mathbb{R}^d\setminus 4R} \dot{\mathcal{M}}_S(b)(x)\,d\mu(x) \\
&\quad \le \sum_{j=1}^{2} |\lambda_j| \int_{2Q_j} \dot{\mathcal{M}}_S(a_j)(x)\,d\mu(x) + \sum_{j=1}^{2} |\lambda_j| \int_{4R\setminus 2Q_j} \cdots + \int_{\mathbb{R}^d\setminus 4R} \dot{\mathcal{M}}_S(b)(x)\,d\mu(x) \\
&\quad \equiv I_1 + I_2 + I_3.
\end{aligned} \tag{4.6}$$

Recall that $\dot{\mathcal{M}}_S$ is bounded on $L^2(\mu)$ by Lemma 4.1. From the Hölder inequality and (2.7), it then follows that

$$\begin{aligned}
I_1 &\le \sum_{j=1}^{2} |\lambda_j| \left\{ \int_{2Q_j} [\dot{\mathcal{M}}_S(a_j)(x)]^2\,d\mu(x) \right\}^{1/2} [\mu(2Q_j)]^{1/2} \\
&\lesssim \sum_{j=1}^{2} |\lambda_j| \left\{ \int_{Q_j} [a_j(x)]^2\,d\mu(x) \right\}^{1/2} [\mu(2Q_j)]^{1/2} \\
&\lesssim \sum_{j=1}^{2} |\lambda_j| \|a_j\|_{L^\infty(\mu)} \mu(2Q_j) \le \sum_{j=1}^{2} |\lambda_j|,
\end{aligned} \tag{4.7}$$

which is the desired result.

For $j = 1, 2$, let x_j be the center of Q_j. Notice that for any $x \notin 2Q_j$ and $y \in Q_j$, $|x-y| \sim |x-x_j|$. From this fact, the Hölder inequality, (A-4) and (2.7), it follows that

$$\dot{\mathcal{M}}_S(a_j)(x) \lesssim \int_{Q_j} \frac{|a_j(y)|}{|x-y|^n}\,d\mu(y) \lesssim \frac{\|a_j\|_{L^\infty(\mu)}\mu(Q_j)}{|x-x_j|^n} \lesssim \frac{1}{|x-x_j|^n} \frac{1}{1+\delta(Q_j,R)}. \tag{4.8}$$

Therefore, by (3.9),

$$I_2 \lesssim \sum_{j=1}^{2} \frac{|\lambda_j|\,\delta(2Q_j,4R)}{1+\delta(Q_j,R)} \lesssim \sum_{j=1}^{2} |\lambda_j|. \tag{4.9}$$

We now estimate I_3. Fix any $x_0 \in R \cap \operatorname{supp}(\mu)$. It follows from Lemma 2.18(a) that $4R \subset Q_{x_0,H_R^{x_0}-1}$. We then write

$$I_3 = \int_{\mathbb{R}^d\setminus Q_{x_0,H_R^{x_0}-1}} \dot{\mathcal{M}}_S(b)(x)\,d\mu(x) + \int_{Q_{x_0,H_R^{x_0}-1}\setminus 4R} \cdots \equiv F_1 + F_2. \tag{4.10}$$

By Lemma 2.18(a) again, we see that $Q_{x_0,H_R^{x_0}+1} \subset 4R$. From this fact, (A-4), (2.7), and the fact that for any $x \notin 4R$ and $y \in R$, $|x-x_0| \sim |x-y|$, it follows that

$$\begin{aligned} F_2 &\lesssim \sum_{j=1}^{2} |\lambda_j| \int_{Q_{x_0,H_R^{x_0}-1}\setminus 4R} \sup_{k\in\mathbb{Z}} \int_{Q_j} \frac{|a_j(y)|}{|x-x_0|^n} d\mu(y)d\mu(x) \\ &\lesssim \sum_{j=1}^{2} |\lambda_j| \int_{Q_{x_0,H_R^{x_0}-1}\setminus Q_{x_0,H_R^{x_0}+1}} \frac{\|a_j\|_{L^\infty(\mu)}\mu(Q_j)}{|x-x_0|^n} d\mu(x) \\ &\lesssim \sum_{j=1}^{2} |\lambda_j| \sum_{i=H_R^{x_0}-1}^{H_R^{x_0}} \delta(Q_{x_0,i+1}, Q_{x_0,i}) \lesssim \sum_{j=1}^{2} |\lambda_j|. \end{aligned} \tag{4.11}$$

By the vanishing moment of b, for any $x \in \mathbb{R}^d \setminus Q_{x_0,H_R^{x_0}-1}$ and any $k \in \mathbb{Z}$,

$$\begin{aligned} |S_k(b)(x)| &\le \int_R |S_k(x,y) - S_k(x,x_0)|\,|b(y)|\,d\mu(y) \\ &\le \sum_{j=1}^{2} |\lambda_j| \int_{Q_j} |S_k(x,y) - S_k(x,x_0)|\,|a_j(y)|\,d\mu(y). \end{aligned} \tag{4.12}$$

We claim that for any $y \in Q_j$, $j = 1,2$, for any integer $i \ge 2$ and $k \ge H_R^{x_0} - i + 3$,

$$\operatorname{supp}\left(S_k(\cdot,y) - S_k(\cdot,x_0)\right) \subset Q_{x_0,H_R^{x_0}-i+1}. \tag{4.13}$$

In fact, by (A-3) and the fact that $\{Q_{x,k}\}_k$ is decreasing in k, $\operatorname{supp}(S_k(\cdot,y) - S_k(\cdot,x_0)) \subset (Q_{y,k-1} \cup Q_{x_0,k-1}) \subset (Q_{y,H_R^{x_0}-i+2} \cup Q_{x_0,H_R^{x_0}-i+2})$. Since $i \ge 2$, then $y \in Q_j$ together with the decreasing property of $\{Q_{x_0,k}\}_k$ in k implies that $y \in Q_{x_0,H_R^{x_0}-i+2}$. From this fact and [8, Lemma 4.2 (c)], it follows that $Q_{y,H_R^{x_0}-i+2} \subset Q_{x_0,H_R^{x_0}-i+1}$. Thus, the above claim (4.13) holds.

Observe that $Q_j \subset Q_{x_0,k}$ for $k \le H_R^{x_0} - i + 2$, $j = 1,2$. Then (A-1) and (A-5) imply that for any $y \in Q_j$,

$$|S_k(x,y) - S_k(x,x_0)| \lesssim \frac{|x_0 - y|}{l(Q_{x_0,k})} \frac{1}{[l(Q_{x_0,k}) + |x-x_0|]^n} \le \frac{l(R)}{l(Q_{x_0,H_R^{x_0}-i+2})} \frac{1}{|x-x_0|^n}. \tag{4.14}$$

Therefore, from the fact that $\int_{\mathbb{R}^d} b(y)d\mu(y) = 0$, (4.13), and the last inequality above, it follows that

$$\begin{aligned}
F_1 &= \sum_{i=2}^{\infty} \int_{Q_{x_0,H_R^{x_0}-i} \setminus Q_{x_0,H_R^{x_0}-i+1}} \sup_{k\in\mathbb{Z}} |S_k(b)(x)|\, d\mu(x) \\
&\lesssim \sum_{j=1}^{2} |\lambda_j| \sum_{i=2}^{\infty} \int_{Q_{x_0,H_R^{x_0}-i} \setminus Q_{x_0,H_R^{x_0}-i+1}} \sup_{k\le H_R^{x_0}-i+2} \int_{Q_j} |S_k(x,y) - S_k(x,x_0)| \\
&\quad \times |a_j(y)|\, d\mu(y) d\mu(x) \\
&\lesssim \sum_{j=1}^{2} |\lambda_j| \sum_{i=2}^{\infty} \int_{Q_{x_0,H_R^{x_0}-i} \setminus Q_{x_0,H_R^{x_0}-i+1}} \frac{l(R)}{l(Q_{x_0,H_R^{x_0}-i+2})} \frac{1}{|x-x_0|^n} d\mu(x) \\
&\lesssim \sum_{j=1}^{2} |\lambda_j| \sum_{i=2}^{\infty} \frac{l(R)}{l(Q_{x_0,H_R^{x_0}-i+2})} \lesssim \sum_{j=1}^{2} |\lambda_j|.
\end{aligned} \tag{4.15}$$

Therefore, $I_3 \lesssim \sum_{j=1}^{2} |\lambda_j|$, which completes the proof of Theorem 4.2. □

We now establish the boundedness of $\mathcal{M}_S$ from $h_{\mathrm{atb}}^{1,\infty}(\mu)$ to $L^1(\mu)$.

THEOREM 4.3. *There exists a nonnegative constant C such that for all $f \in h_{\mathrm{atb}}^{1,\infty}(\mu)$,*

$$\|\mathcal{M}_S(f)\|_{L^1(\mu)} \le C\|f\|_{h_{\mathrm{atb}}^{1,\infty}(\mu)}. \tag{4.16}$$

Proof. By the Fatou lemma, to prove Theorem 4.3, it suffices to show that for any ∞-atomic block or ∞-block $b = \sum_{j=1}^{2} \lambda_j a_j$ as in Definition 2.16, we have

$$\|\mathcal{M}_S(b)\|_{L^1(\mu)} \lesssim \sum_{j=1}^{2} |\lambda_j|. \tag{4.17}$$

If b is ∞-atomic block as in Definition 2.16, then by the fact that $\mathcal{M}_S b(x) \le \dot{\mathcal{M}}_S b(x)$ for all $x \in \mathbb{R}^d$ and (4.5), we see

$$\|\mathcal{M}_S(b)\|_{L^1(\mu)} \lesssim \sum_{j=1}^{2} |\lambda_j|. \tag{4.18}$$

Let b be an ∞-block as in Definition 2.16. By Definition 2.16, there exists $R \in \mathcal{D}$ such that $\operatorname{supp}(b) \subset R$. Write

$$\begin{aligned}
&\int_{\mathbb{R}^d} \sup_{k\in\mathbb{N}} |S_k(b)(x)|\,d\mu(x)\\
&\qquad \le \sum_{j=1}^{2} |\lambda_j| \int_{2Q_j} \sup_{k\in\mathbb{N}} |S_k(a_j)(x)|\,d\mu(x) + \sum_{j=1}^{2} |\lambda_j| \int_{4R\setminus 2Q_j} \cdots + \sum_{j=1}^{2} |\lambda_j| \int_{\mathbb{R}^d\setminus 4R} \cdots\\
&\qquad \equiv J_1 + J_2 + J_3.
\end{aligned} \tag{4.19}$$

Since the argument of estimates for I_1 and I_2 in the proof of Theorem 4.2 also works in the current situation, we then have that $J_1 + J_2 \lesssim \sum_{j=1}^{2} |\lambda_j|$.

To estimate J_3, fix any $x_0 \in R \cap \operatorname{supp}(\mu)$. Notice that for any $x \in \mathbb{R}^d \setminus 4R$ and any $y \in Q_j$, $j = 1, 2$, $|x-y| \sim |x-x_0|$. From this fact, Definition 2.16, and (A-4), it follows that for $j = 1$, 2 and any $x \in \mathbb{R}^d \setminus 4R$,

$$\sup_{k\in\mathbb{N}} |S_k(a_j)(x)| \lesssim \sup_{k\in\mathbb{N}} \int_{Q_j} \frac{|a_j(y)|}{|x-y|^n}\,d\mu(y) \lesssim \frac{\|a_j\|_{L^\infty(\mu)}\mu(Q_j)}{|x-x_0|^n} \lesssim \frac{1}{|x-x_0|^n}. \tag{4.20}$$

On the other hand, since $R \in \mathcal{D}$, by Lemma 2.18(d), we obtain that $H_R^{x_0} \le 1$. This observation together with [8, Lemma 4.2] in turn implies that for any $k \in \mathbb{N}$ and $y \in R \cap \operatorname{supp}(\mu)$, $Q_{y,k-1} \subset Q_{y,H_R^{x_0}-1} \subset Q_{x_0,H_R^{x_0}-2}$. It then follows that $\operatorname{supp}(S_k(b)) \subset Q_{x_0,H_R^{x_0}-2}$ for any $k \in \mathbb{N}$. Moreover, Lemma 2.18(a) yields $Q_{x_0,H_R^{x_0}+1} \subset 4R$. Therefore, we obtain that

$$\begin{aligned}
J_3 &\le \sum_{j=1}^{2} |\lambda_j| \int_{\mathbb{R}^d\setminus 4R} \sup_{k\in\mathbb{N}} |S_k(a_j)(x)|\,d\mu(x)\\
&\lesssim \sum_{j=1}^{2} |\lambda_j| \int_{Q_{x_0,H_R^{x_0}-2}\setminus 4R} \frac{1}{|x-x_0|^n}\,d\mu(x) \lesssim \sum_{j=1}^{2} |\lambda_j|,
\end{aligned} \tag{4.21}$$

which completes the proof of Theorem 4.3. □

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