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Lower bounds on the minimum eigenvalue of the Fan product of several M-matrices

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Abstract

The concept of the Fan product of several M-matrices is introduced. Furthermore, two new lower bounds of the minimum eigenvalue of the Fan product of several M-matrices are proposed. These obtained new lower bounds generalize and improve some earlier findings. One example is presented to illustrate the precision of the given lower bounds.

Keywords: Fan product; M-matrix; Minimum eigenvalue; Irreducible

1 Introduction

Many issues in the social, physical, and biological sciences can be reduced to problems using matrices that possess a unique structure owing to limitations. One of the most common situations is when the matrix K has nonpositive off-diagonal entries. The matrix K can be written as follows:

$$K = sI - P, \quad P \geq 0. \tag{1.1}$$

Here, $P \geq 0$ means that the matrix P is nonnegative.

Let $R^{n\times n}(C^{n\times n})$ denote the union of n-by-n real (complex) matrices. Here, the conventional notation is employed by setting

$$Z_n = \left\{K = (k_{ij}) \in R^{n\times n}, k_{ij} \leq 0, i \neq j\right\}.$$

The aim is to study a special subclass of matrices in Z_n called M-matrices.

For any matrix K of the form in Eq. (1.1), if $s > \rho(P)$, the spectral radius of P, then K is defined to be a nonsingular M-matrix. The set of nonsingular M-matrices is denoted by M_n. Let $K \in M_n$ and assume $K = sI - P$ with $s > \rho(P)$ and $P \geq 0$. It is known that $q(K) = s - \rho(P)$ is an eigenvalue of the matrix K with the minimum module [1], and $q(K)$ is considered to be the minimum eigenvalue of the M-matrix K.

M-matrices have been widely investigated and possess many appealing qualities [2, 3]. Research on the minimum eigenvalue is particularly important for an M-matrix and has produced many novel findings. In practice, the minimum eigenvalues of the M-matrices play an important role in evaluating the stability of a power system. Potential issues with the power system can be identified early by tracking and examining the minimum eigenvalues of the M-matrices, making it easier to obtain the proper solutions and increase the stability and reliability of the system.

For two matrices $A_1 = (a_{ij}) \in R^{n\times n}$ and $A_2 = (b_{ij}) \in R^{n\times n}$, the Fan product of A_1 and A_2 is denoted by $A_1 \star A_2 = (s_{ij})$, where

$$s_{ij} = \begin{cases} a_{ii}b_{ii}, & i = j, \\ -a_{ij}b_{ij}, & i \neq j. \end{cases}$$

The Fan product is a fundamental operation in the study of M-matrices. It plays a crucial role in understanding the properties and characteristics of M-matrices. It is used to analyze the interplay between the elements of two M-matrices and study the properties of the resulting matrix, such as eigenvalues, spectral radius, and invertibility. In previous studies, the computation and estimation of the minimum eigenvalue of the Fan product has become a popular research topic, and many results have been obtained [4–6].

Let $A_1, A_2 \in M_n$. The following classical result is proposed by Horn and Johnson [1]:

$$q(A_1 \star A_2) \geq q(A_1)q(A_2). \tag{1.2}$$

The above inequality shows that the minimum eigenvalue of the Fan product $A_1 \star A_2$ is more than the product of the minimum eigenvalues of A_1 and A_2.

As the class of M-matrices is closed under the Fan product, one can generalize the definition of the Fan product from two to several M-matrices. For $A_1 = (a_{ij})$, $A_2 = (b_{ij}), \dots,$ $A_m = (m_{ij}) \in M_n$, the Fan product of $A_1, A_2, \dots, A_m$ is denoted by $A_1 \star A_2 \star \cdots \star A_m = (p_{ij})$, where

$$p_{ij} = \begin{cases} a_{ii}b_{ii}\cdots m_{ii}, & i = j, \\ -|a_{ij}b_{ij}\cdots m_{ij}|, & i \neq j. \end{cases}$$

From the inequality in Eq. (1.2), one can observe that

$$q(A_1 \star A_2 \star \cdots \star A_m) \geq q(A_1)q(A_2)\cdots q(A_m). \tag{1.3}$$

Motivated by previous work [4–9], in this study, the lower bound of $q(A_1 \star A_2 \star \cdots \star A_m)$ was investigated further. The structure of the article is as follows.

In Sect. 2, a new lower bound on the minimum eigenvalue involving the Fan product of several M-matrices is introduced. In Sect. 3, this result is further improved. These new lower bounds generalize some earlier findings.

To verify the conclusions, a numerical test is described in Sect. 4, and these lower bounds are compared.

2 New lower bound for $q(A_1 \star A_2 \star \cdots \star A_m)$

This section begins with a basic definition.

Definition 1 *Let $A \in R^{n\times n}$ with $n \geq 2$. If there exists a permutation matrix P that satisfies*

$$PAP^{\mathrm{T}} = \begin{pmatrix} B & C \\ O & D \end{pmatrix},$$

where B and D are square matrices, A is considered reducible; if such a permutation matrix P does not exist, A is irreducible.

For the M-matrices $A_1=(a_{ij})$, $A_2=(b_{ij}),\ldots,A_m=(m_{ij})$ of order n and $k=1,2$, one can write

$$N_1=D_1-A_1^{(k)},\qquad N_2=D_2-A_2^{(k)},\qquad \ldots,\qquad N_m=D_m-A_m^{(k)},$$

where

$$D_1=\operatorname{diag}\bigl(a_{11}^k,a_{22}^k,\ldots,a_{nn}^k\bigr),\qquad D_2=\operatorname{diag}\bigl(b_{11}^k,b_{22}^k,\ldots,b_{nn}^k\bigr),\qquad \ldots,$$
$$D_m=\operatorname{diag}\bigl(m_{11}^k,m_{22}^k,\ldots,m_{nn}^k\bigr),$$

and

$$A_1^{(k)}=\begin{cases}A_1, & k=1,\\ A_1\star A_1, & k=2,\end{cases}\qquad A_2^{(k)}=\begin{cases}A_2, & k=1,\\ A_2\star A_2, & k=2,\end{cases}\qquad \ldots,$$
$$A_m^{(k)}=\begin{cases}A_m, & k=1,\\ A_m\star A_m, & k=2.\end{cases}$$

In addition, it is noted that

$$a_{ii}>0,\qquad b_{ii}>0,\qquad \ldots,\qquad m_{ii}>0,\quad i=1,2,\ldots,n.$$

Thus, $D_1,D_2,\ldots,D_m$ are all nonsingular. One can define

$$J_{A_1}^{(k)}=D_1^{-1}N_1,\qquad J_{A_2}^{(k)}=D_2^{-1}N_2,\qquad \ldots,\qquad J_{A_m}^{(k)}=D_m^{-1}N_m.$$

It is obvious that $J_{A_1}^{(k)},J_{A_2}^{(k)},\ldots,J_{A_m}^{(k)}$ are nonnegative.

The following important lemmas must be remembered to arrive at the primary conclusions of this work.

Lemma 1 [10] *Let $A\in R^{n\times n}$ be an irreducible nonnegative matrix. The following facts apply*:

(1) *There is a positive real eigenvalue that equals its spectral radius $\rho(A)$.*

(2) *There is an eigenvector $u>0$ satisfying $Au=\rho(A)u$.*

Lemma 2 [10] *If an irreducible M-matrix $A\in R^{n\times n}$ and a nonnegative nonzero vector z satisfy $Az\ge kz$, then $q(A)\ge k$.*

Lemma 3 [11] *Let $\beta_j=(\beta_j(1),\beta_j(2),\ldots,\beta_j(n))^{\mathrm T}\ge 0$, $j=1,2,\ldots,m$. If $\alpha_j>0$ such that $\sum_{j=1}^m\frac{1}{\alpha_j}\ge 1$, then*

$$\sum_{i=1}^n\prod_{j=1}^m\beta_j(i)\le\prod_{j=1}^m\left\{\sum_{i=1}^n\bigl[\beta_j(i)\bigr]^{\alpha_j}\right\}^{\frac{1}{\alpha_j}}.$$

Lemma 4 [1] *Let* $A = (a_{ij}) \in R^{n\times n}$ *be a nonnegative matrix. For* $\alpha \geq 0$, $A^{(\alpha)} = (a_{ij}^{\alpha})$. *If* $\alpha \geq 1$, *then*

$$\rho\left(A^{(\alpha)}\right) \leq \rho^{\alpha}(A).$$

Lemma 5 [10] *Let* $A = (a_{ij}) \in C^{n\times n}$ $(n \geq 2)$. *For any eigenvalue* λ *of the matrix* A, *there must exist two unequal positive integers* i, j *satisfying the inequality*

$$|\lambda - a_{ii}||\lambda - a_{jj}| \leq R_i(A)R_j(A),$$

where $R_i(A) = \sum_{k\neq i}^{n} |a_{ik}|$, $R_j(A) = \sum_{k\neq j}^{n} |a_{jk}|$.

The first result of the lower bound for $q(A_1 \star A_2 \star \cdots \star A_m)$ is shown below.

Theorem 1 *Let* $A_1 = (a_{ij}), A_2 = (b_{ij}), \ldots, A_m = (m_{ij}) \in M_n$. *For* $k = 1, 2$, *one obtains*

$$q(A_1 \star A_2 \star \cdots \star A_m) \geq \left[1 - \rho^{\frac{1}{k}}\left(J_{A_1}^{(k)}\right)\rho^{\frac{1}{k}}\left(J_{A_2}^{(k)}\right)\cdots\rho^{\frac{1}{k}}\left(J_{A_m}^{(k)}\right)\right] \min_{1\leq i\leq n}(a_{ii}b_{ii}\cdots m_{ii}). \tag{2.1}$$

Proof Define $A = A_1 \star A_2 \star \cdots \star A_m$. This problem can be solved in two cases. First, A is considered irreducible. One can then see that $A_1, A_2, \ldots, A_m$ are all irreducible. Therefore, $J_{A_1}^{(k)}, J_{A_2}^{(k)}, \ldots, J_{A_m}^{(k)}$ are all irreducible and nonnegative for $k = 1, 2$. From Lemma 1, there exist m vectors

$$x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} > 0, \qquad y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} > 0, \qquad \ldots, \qquad z = (z_1, z_2, \ldots, z_n)^{\mathrm{T}} > 0$$

that satisfy

$$J_{A_1}^{(k)}x^{(k)} = \rho\left(J_{A_1}^{(k)}\right)x^{(k)}, \tag{2.2}$$

$$J_{A_2}^{(k)}y^{(k)} = \rho\left(J_{A_2}^{(k)}\right)y^{(k)}, \tag{2.3}$$

$$\cdots\cdots$$

$$J_{A_m}^{(k)}z^{(k)} = \rho\left(J_{A_m}^{(k)}\right)z^{(k)}, \tag{2.4}$$

where

$$x^{(k)} = \left(x_1^k, x_2^k, \ldots, x_n^k\right)^{\mathrm{T}}, \qquad y^{(k)} = \left(y_1^k, y_2^k, \ldots, y_n^k\right)^{\mathrm{T}}, \qquad \ldots, \qquad z^{(k)} = \left(z_1^k, z_2^k, \ldots, z_n^k\right)^{\mathrm{T}}.$$

According to Eqs. (2.2)–(2.4),

$$\sum_{j\neq i}^{n} |a_{ij}|^k x_j^k = \rho\left(J_{A_1}^{(k)}\right)a_{ii}^k x_i^k,$$

$$\sum_{j\neq i}^{n} |b_{ij}|^k y_j^k = \rho\left(J_{A_2}^{(k)}\right)b_{ii}^k y_i^k,$$

$$\cdots\cdots$$

$$\sum_{j\neq i}^{n} |m_{ij}|^k z_j^k = \rho\left(J_{A_m}^{(k)}\right)m_{ii}^k z_i^k, \quad i = 1, 2, \ldots, n.$$

Now, let $w = (w_1, w_2, \dots, w_n) \in R^n$, where $w_i = x_i y_i \cdots z_i > 0$ for all $i = 1, 2, \dots, n$. For the irreducible M-matrix A, according to Lemma 3,

$$\begin{aligned}
(Aw)_i &= a_{ii}b_{ii}\cdots m_{ii}w_i - \sum_{j\neq i}^{n} |a_{ij}b_{ij}\cdots m_{ij}|w_j \\
&= a_{ii}b_{ii}\cdots m_{ii}w_i - \sum_{j\neq i}^{n} \bigl(|a_{ij}|x_j\bigr)\bigl(|b_{ij}|y_j\bigr)\cdots\bigl(|m_{ij}|z_j\bigr) \\
&\geq a_{ii}b_{ii}\cdots m_{ii}w_i - \left(\sum_{j\neq i}^{n} |a_{ij}|^k x_j^k\right)^{\frac{1}{k}} \left(\sum_{j\neq i}^{n} |b_{ij}|^k y_j^k\right)^{\frac{1}{k}} \cdots \left(\sum_{j\neq i}^{n} |m_{ij}|^k z_j^k\right)^{\frac{1}{k}} \\
&= a_{ii}b_{ii}\cdots m_{ii}w_i - \rho^{\frac{1}{k}}\bigl(J_{A_1}^{(k)}\bigr)a_{ii}x_i\rho^{\frac{1}{k}}\bigl(J_{A_2}^{(k)}\bigr)b_{ii}y_i\cdots\rho^{\frac{1}{k}}\bigl(J_{A_m}^{(k)}\bigr)m_{ii}z_i \\
&= a_{ii}b_{ii}\cdots m_{ii}\bigl[1 - \rho^{\frac{1}{k}}\bigl(J_{A_1}^{(k)}\bigr)\rho^{\frac{1}{k}}\bigl(J_{A_2}^{(k)}\bigr)\cdots\rho^{\frac{1}{k}}\bigl(J_{A_m}^{(k)}\bigr)\bigr]w_i.
\end{aligned} \tag{2.5}$$

By Lemma 2 and the inequality in Eq. (2.5),

$$\begin{aligned}
q(A_1 \star A_2 \star \cdots \star A_m) &\geq \bigl[1 - \rho^{\frac{1}{k}}\bigl(J_{A_1}^{(k)}\bigr)\rho^{\frac{1}{k}}\bigl(J_{A_2}^{(k)}\bigr)\cdots\rho^{\frac{1}{k}}\bigl(J_{A_m}^{(k)}\bigr)\bigr]a_{ii}b_{ii}\cdots m_{ii} \\
&\geq \bigl[1 - \rho^{\frac{1}{k}}\bigl(J_{A_1}^{(k)}\bigr)\rho^{\frac{1}{k}}\bigl(J_{A_2}^{(k)}\bigr)\cdots\rho^{\frac{1}{k}}\bigl(J_{A_m}^{(k)}\bigr)\bigr]\min_{1\leq i\leq n}(a_{ii}b_{ii}\cdots m_{ii}).
\end{aligned}$$

In the following, it is assumed that the matrix A is reducible. Let $H = (h_{ij})$ be the n-by-n permutation matrix with

$$h_{12} = h_{23} = \cdots = h_{n-1,n} = h_{n1} = 1,$$

the remaining $h_{ij} = 0$. A sufficiently small positive number ε is chosen such that $A_1 - \varepsilon H$, $A_2 - \varepsilon H, \dots, A_m - \varepsilon H$ are irreducible nonsingular M-matrices. Substituting $A_1 - \varepsilon H, A_2 - \varepsilon H, \dots, A_m - \varepsilon H$ for $A_1, A_2, \dots, A_m$ in the irreducible case, and then by setting $\varepsilon \to 0$, the conclusion holds by continuity theory. □

Next, two special cases are considered. By setting $m = 2$, $k = 1$ in Theorem 1, the conclusion is obtained as follows.

Corollary 1 *Let $A_1 = (a_{ij})$, $A_2 = (b_{ij}) \in M_n$, then*

$$q(A_1 \star A_2) \geq \bigl[1 - \rho(J_{A_1})\rho(J_{A_2})\bigr]\min_{1\leq i\leq n}(a_{ii}b_{ii}). \tag{2.6}$$

This is the result of Theorem 4 in a previous report [4]. Let $m = k = 2$, then, Theorem 1 yields the following corollary, which is the conclusion of Theorem 2.7 of Li [5].

Corollary 2 *Let $A_1 = (a_{ij})$, $A_2 = (b_{ij}) \in M_n$, then*

$$q(A_1 \star A_2) \geq \bigl[1 - \rho^{\frac{1}{2}}\bigl(J_{A_1}^{(2)}\bigr)\rho^{\frac{1}{2}}\bigl(J_{A_2}^{(2)}\bigr)\bigr]\min_{1\leq i\leq n}(a_{ii}b_{ii}). \tag{2.7}$$

As a result, the conclusions of Theorem 4 in an earlier report [4] and Theorem 2.7 in other work [5] are contained in Theorem 1 of this study.

Remark 1 *From Lemma 4, one can get*

$$\rho^{\frac{1}{2}}\big(J_{A_1}^{(2)}\big)\rho^{\frac{1}{2}}\big(J_{A_2}^{(2)}\big) \le \rho(J_{A_1})\rho(J_{A_2}).$$

This shows that

$$\big[1-\rho^{\frac{1}{2}}\big(J_{A_1}^{(2)}\big)\rho^{\frac{1}{2}}\big(J_{A_2}^{(2)}\big)\big]\min_{1\le i\le n}(a_{ii}b_{ii}) \ge \big[1-\rho(J_{A_1})\rho(J_{A_2})\big]\min_{1\le i\le n}(a_{ii}b_{ii}).$$

Therefore, the lower bound in the inequality in Eq. (2.7) *is superior to the lower bound in the inequality in Eq.* (2.6).

3 Improved lower bound for $q(A_1 \star A_2 \star \cdots \star A_m)$

In this section, a second lower bound is proposed for $q(A_1 \star A_2 \star \cdots \star A_m)$, which is an improvement of the lower bound in Sect. 2.

Theorem 2 *Let* $A_1 = (a_{ij}), A_2 = (b_{ij}), \ldots, A_m = (m_{ij}) \in M_n$. *For* $k = 1, 2$, *one obtains*

$$\begin{aligned} &q(A_1 \star A_2 \star \cdots \star A_m) \\ &\quad \ge \min_{i\ne j}\frac{1}{2}\big\{a_{ii}b_{ii}\cdots m_{ii} + a_{jj}b_{jj}\cdots m_{jj} - \big[(a_{ii}b_{ii}\cdots m_{ii} - a_{jj}b_{jj}\cdots m_{jj})^2 \\ &\qquad + 4(a_{ii}b_{ii}\cdots m_{ii})(a_{jj}b_{jj}\cdots m_{jj})\rho^{\frac{2}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{2}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{2}{k}}\big(J_{A_m}^{(k)}\big)\big]^{\frac{1}{2}}\big\}. \end{aligned} \tag{3.1}$$

Proof Define $A = A_1 \star A_2 \star \cdots \star A_m$. To illustrate this issue, two aspects are considered. First, it is assumed that A is irreducible. One can see that $A_1, A_2, \ldots, A_m$ are all irreducible. In addition, $J_{A_1}^{(k)}, J_{A_2}^{(k)}, \ldots, J_{A_m}^{(k)}$ are all irreducible and nonnegative for $k = 1, 2$. In terms of Lemma 1, for $k = 1, 2$, there exist

$$u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} > 0, \qquad v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}} > 0, \qquad \ldots, \qquad t = (t_1, t_2, \ldots, t_n)^{\mathrm{T}} > 0$$

that satisfy

$$J_{A_1}^{(k)}u^{(k)} = \rho\big(J_{A_1}^{(k)}\big)u^{(k)}, \tag{3.2}$$

$$J_{A_2}^{(k)}v^{(k)} = \rho\big(J_{A_2}^{(k)}\big)v^{(k)}, \tag{3.3}$$

$$\cdots\cdots$$

$$J_{A_m}^{(k)}t^{(k)} = \rho\big(J_{A_m}^{(k)}\big)t^{(k)}, \tag{3.4}$$

where

$$u^{(k)} = \big(u_1^k, u_2^k, \ldots, u_n^k\big)^{\mathrm{T}}, \qquad v^{(k)} = \big(v_1^k, v_2^k, \ldots, v_n^k\big)^{\mathrm{T}}, \qquad \ldots, \qquad t^{(k)} = \big(t_1^k, t_2^k, \ldots, t_n^k\big)^{\mathrm{T}}.$$

According to Eqs. (3.2)–(3.4), we arrive at

$$\sum_{p\ne i}^{n}\frac{|a_{ip}|^k u_p^k}{u_i^k} = \rho\big(J_{A_1}^{(k)}\big)a_{ii}^k,$$

$$\sum_{p\neq i}^{n}\frac{|b_{ip}|^k v_p^k}{v_i^k}=\rho\left(J_{A_2}^{(k)}\right)b_{ii}^k,$$

$$\cdots\cdots$$

$$\sum_{p\neq i}^{n}\frac{|m_{ip}|^k t_p^k}{t_i^k}=\rho\left(J_{A_m}^{(k)}\right)m_{ii}^k,\quad i=1,2,\ldots,n.$$

Now, the following is defined:

$$P_1=\operatorname{diag}(u_1,u_2,\ldots,u_n),\qquad P_2=\operatorname{diag}(v_1,v_2,\ldots,v_n),\qquad \ldots,$$
$$P_m=\operatorname{diag}(t_1,t_2,\ldots,t_n).$$

Clearly, $P_1, P_2, \ldots, P_m$ are nonsingular. Let

$$\tilde{A}_1=P_1^{-1}A_1P_1=\left(\frac{a_{ij}u_j}{u_i}\right),\qquad \tilde{A}_2=P_2^{-1}A_2P_2=\left(\frac{b_{ij}v_j}{v_i}\right),\qquad \ldots,$$
$$\tilde{A}_m=P_m^{-1}A_mP_m=\left(\frac{m_{ij}t_j}{t_i}\right),$$

and let

$$\tilde{A}_1\star\tilde{A}_2\star\cdots\star\tilde{A}_m=K=(k_{ij}).$$

From the definition of the Fan product of several M-matrices, one obtains

$$k_{ij}=\begin{cases}a_{ii}b_{ii}\cdots m_{ii}, & i=j,\\ -|\frac{a_{ij}u_j}{u_i}\frac{b_{ij}v_j}{v_i}\cdots\frac{m_{ij}t_j}{t_i}|, & i\neq j.\end{cases}$$

It is assumed that $P=P_1P_2\cdots P_m$ and $P^{-1}(A_1\ \star A_2\star\cdots\star A_m)P=K'=(k'_{ij})$. One obtains

$$k'_{ij}=\begin{cases}\frac{1}{u_iv_i\cdots t_i}(a_{ii}b_{ii}\cdots m_{ii})u_iv_i\cdots t_i=a_{ii}b_{ii}\cdots m_{ii}, & i=j,\\ \frac{1}{u_iv_i\cdots t_i}(-|a_{ij}b_{ij}\cdots m_{ij}|)u_jv_j\cdots t_j=-|\frac{a_{ij}u_j}{u_i}\frac{b_{ij}v_j}{v_i}\cdots\frac{m_{ij}t_j}{t_i}|, & i\neq j.\end{cases}$$

Thus, we have

$$P^{-1}(A_1\star A_2\star\cdots\star A_m)P=\tilde{A}_1\star\tilde{A}_2\star\cdots\star\tilde{A}_m.$$

This implies that

$$q(A_1\star A_2\star\cdots\star A_m)=q(\tilde{A}_1\star\tilde{A}_2\star\cdots\star\tilde{A}_m).$$

In addition, according to Lemma 3, one obtains

$$R_i(\tilde{A}_1\star\tilde{A}_2\star\cdots\star\tilde{A}_m)=\sum_{p\neq i}^{n}\left|\frac{a_{ip}u_p}{u_i}\frac{b_{ip}v_p}{v_i}\cdots\frac{m_{ip}t_p}{t_i}\right|$$

$$\leq \left(\sum_{p\neq i}^{n}\frac{|a_{ip}|^k u_p^k}{u_i^k}\right)^{\frac{1}{k}}\left(\sum_{p\neq i}^{n}\frac{|b_{ip}|^k v_p^k}{v_i^k}\right)^{\frac{1}{k}}\cdots\left(\sum_{p\neq i}^{n}\frac{|m_{ip}|^k t_p^k}{t_i^k}\right)^{\frac{1}{k}}$$
$$= a_{ii}b_{ii}\cdots m_{ii}\rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big). \tag{3.5}$$

Similarly, one obtains

$$R_j(\tilde{A}_1\star\tilde{A}_2\star\cdots\star\tilde{A}_m)\leq a_{jj}b_{jj}\cdots m_{jj}\rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big). \tag{3.6}$$

As $q(\tilde{A}_1\star\tilde{A}_2\star\cdots\star\tilde{A}_m)$ is an eigenvalue of $\tilde{A}_1\star\tilde{A}_2\star\cdots\star\tilde{A}_m$, it follows from Lemma 5 that there exist two unequal positive integers i, j that satisfy

$$\begin{aligned}&\big|q(A_1\star A_2\star\cdots\star A_m)-a_{ii}b_{ii}\cdots m_{ii}\big|\big|q(A_1\star A_2\star\cdots\star A_m)-a_{jj}b_{jj}\cdots m_{jj}\big|\\&\quad\leq R_i(\tilde{A}_1\star\tilde{A}_2\star\cdots\star\tilde{A}_m)R_j(\tilde{A}_1\star\tilde{A}_2\star\cdots\star\tilde{A}_m).\end{aligned}$$

Combining the inequalities in Eqs. (3.5) and (3.6), one obtains

$$\begin{aligned}&\big|q(A_1\star A_2\star\cdots\star A_m)-a_{ii}b_{ii}\cdots m_{ii}\big|\big|q(A_1\star A_2\star\cdots\star A_m)-a_{jj}b_{jj}\cdots m_{jj}\big|\\&\quad\leq (a_{ii}b_{ii}\cdots m_{ii})(a_{jj}b_{jj}\cdots m_{jj})\rho^{\frac{2}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{2}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{2}{k}}\big(J_{A_m}^{(k)}\big).\end{aligned} \tag{3.7}$$

As $0 < q(A_1\star A_2\star\cdots\star A_m) < a_{ii}b_{ii}\cdots m_{ii}$, one can obtain

$$\begin{aligned}&\big[q(A_1\star A_2\star\cdots\star A_m)-a_{ii}b_{ii}\cdots m_{ii}\big]\big[q(A_1\star A_2\star\cdots\star A_m)-a_{jj}b_{jj}\cdots m_{jj}\big]\\&\quad\leq (a_{ii}b_{ii}\cdots m_{ii})(a_{jj}b_{jj}\cdots m_{jj})\rho^{\frac{2}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{2}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{2}{k}}\big(J_{A_m}^{(k)}\big).\end{aligned} \tag{3.8}$$

From the inequality in Eq. (3.8), we acquire

$$\begin{aligned}&q(A_1\star A_2\star\cdots\star A_m)\\&\quad\geq\frac{1}{2}\Big\{a_{ii}b_{ii}\cdots m_{ii}+a_{jj}b_{jj}\cdots m_{jj}-\big[(a_{ii}b_{ii}\cdots m_{ii}-a_{jj}b_{jj}\cdots m_{jj})^2\\&\qquad+4(a_{ii}b_{ii}\cdots m_{ii})(a_{jj}b_{jj}\cdots m_{jj})\rho^{\frac{2}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{2}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{2}{k}}\big(J_{A_m}^{(k)}\big)\big]^{\frac{1}{2}}\Big\}\\&\quad\geq\min_{i\neq j}\frac{1}{2}\Big\{a_{ii}b_{ii}\cdots m_{ii}+a_{jj}b_{jj}\cdots m_{jj}-\big[(a_{ii}b_{ii}\cdots m_{ii}-a_{jj}b_{jj}\cdots m_{jj})^2\\&\qquad+4(a_{ii}b_{ii}\cdots m_{ii})(a_{jj}b_{jj}\cdots m_{jj})\rho^{\frac{2}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{2}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{2}{k}}\big(J_{A_m}^{(k)}\big)\big]^{\frac{1}{2}}\Big\}.\end{aligned}$$

Now, considering that the matrix $A_1\star A_2\star\cdots\star A_m$ is reducible, one can prove similarly by following the proof of Theorem 1. □

Remark 2 *A novel proof of Theorem 1 is introduced. According to the Gerschgorin theorem* [10],

$$\big|q(A_1\star A_2\star\cdots\star A_m)-a_{ii}b_{ii}\cdots m_{ii}\big|\leq R_i(\tilde{A}_1\star\tilde{A}_2\star\cdots\star\tilde{A}_m).$$

Combining the inequalities in Eq. (3.5) *and*

$$0 < q(A_1 \star A_2 \star \cdots \star A_m) \le a_{ii} b_{ii} \cdots m_{ii},$$

one obtains

$$a_{ii} b_{ii} \cdots m_{ii} - q(A_1 \star A_2 \star \cdots \star A_m) \le a_{ii} b_{ii} \cdots m_{ii} \rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big) \rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big) \cdots \rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big).$$

This indicates

$$\begin{aligned} q(A_1 \star A_2 \star \cdots \star A_m) &\ge a_{ii} b_{ii} \cdots m_{ii}\big[1 - \rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big) \rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big) \cdots \rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)\big] \\ &\ge \big[1 - \rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big) \rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big) \cdots \rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)\big] \min_{1 \le i \le n} (a_{ii} b_{ii} \cdots m_{ii}). \end{aligned}$$

The following corollary is a special case of Theorem 2 by setting $m = 2$, $k = 1$.

Corollary 3 *Let $A_1 = (a_{ij})$, $A_2 = (b_{ij}) \in M_n$, then*

$$q(A_1 \star A_2) \ge \min_{i \ne j} \frac{1}{2}\big\{a_{ii} b_{ii} + a_{jj} b_{jj} - \big[(a_{ii} b_{ii} - a_{jj} b_{jj})^2 + 4 a_{ii} b_{ii} a_{jj} b_{jj} \rho^2(J_{A_1}) \rho^2(J_{A_2})\big]^{\frac{1}{2}}\big\}. \quad (3.9)$$

This is the direct result of Theorem 2 of Liu [6]. Setting $m = k = 2$ in Theorem 2, one can obtain the following conclusion.

Corollary 4 *Let $A_1 = (a_{ij})$, $A_2 = (b_{ij}) \in M_n$, then*

$$q(A_1 \star A_2) \ge \min_{i \ne j} \frac{1}{2}\big\{a_{ii} b_{ii} + a_{jj} b_{jj} - \big[(a_{ii} b_{ii} - a_{jj} b_{jj})^2 + 4 a_{ii} b_{ii} a_{jj} b_{jj} \rho\big(J_{A_1}^{(2)}\big) \rho\big(J_{A_2}^{(2)}\big)\big]^{\frac{1}{2}}\big\}. \quad (3.10)$$

This happens to be the conclusion of Theorem 2.8 in an earlier report [5]. Therefore, the results of Theorem 2 in another report [6] and Theorem 2.8 in the earlier report [5] are contained in Theorem 2 of this study.

Remark 3 *According to Lemma 4,*

$$\rho\big(J_{A_1}^{(2)}\big) \rho\big(J_{A_2}^{(2)}\big) \le \rho^2(J_{A_1}) \rho^2(J_{A_2}).$$

This implies that the lower bound in the inequality in Eq. (3.10) *is superior to the lower bound in the inequality in Eq.* (3.9).

Next, the two lower bounds for $q(A_1 \star A_2 \star \cdots \star A_m)$ in Theorem 1 and Theorem 2 are compared.

Theorem 3 *Let $A_1 = (a_{ij})$, $A_2 = (b_{ij}), \ldots, A_m = (m_{ij}) \in M_n$, then*

$$\begin{aligned} &q(A_1 \star A_2 \star \cdots \star A_m) \\ &\quad \ge \min_{i \ne j} \frac{1}{2}\big\{a_{ii} b_{ii} \cdots m_{ii} + a_{jj} b_{jj} \cdots m_{jj} - \big[(a_{ii} b_{ii} \cdots m_{ii} - a_{jj} b_{jj} \cdots m_{jj})^2 \end{aligned}$$

$$
\begin{aligned}
&+ 4(a_{ii}b_{ii}\cdots m_{ii})(a_{jj}b_{jj}\cdots m_{jj})\rho^{\frac{2}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{2}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{2}{k}}\big(J_{A_m}^{(k)}\big)\big]^{\frac{1}{2}}\Big\} \\
&\geq \big[1-\rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)\big]\min_{1\leq i\leq n}(a_{ii}b_{ii}\cdots m_{ii}).
\end{aligned}
$$

Proof It can be assumed that

$$
\begin{aligned}
&\big[1-\rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)\big](a_{ii}b_{ii}\cdots m_{ii}) \\
&\quad\leq \big[1-\rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)\big](a_{jj}b_{jj}\cdots m_{jj}).
\end{aligned}
$$

As a result, the above inequality can be expressed as

$$
\begin{aligned}
&\rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)(a_{jj}b_{jj}\cdots m_{jj}) \\
&\quad\leq \rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)(a_{ii}b_{ii}\cdots m_{ii}) + (a_{jj}b_{jj}\cdots m_{jj} - a_{ii}b_{ii}\cdots m_{ii}).
\end{aligned}
$$

Then, one obtains

$$
\begin{aligned}
&(a_{ii}b_{ii}\cdots m_{ii} - a_{jj}b_{jj}\cdots m_{jj})^2 \\
&\qquad + 4(a_{ii}b_{ii}\cdots m_{ii})(a_{jj}b_{jj}\cdots m_{jj})\rho^{\frac{2}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{2}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{2}{k}}\big(J_{A_m}^{(k)}\big) \\
&\quad\leq (a_{ii}b_{ii}\cdots m_{ii} - a_{jj}b_{jj}\cdots m_{jj})^2 + 4(a_{ii}b_{ii}\cdots m_{ii})^2\rho^{\frac{2}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{2}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{2}{k}}\big(J_{A_m}^{(k)}\big) \\
&\qquad + 4(a_{ii}b_{ii}\cdots m_{ii})\rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)(a_{jj}b_{jj}\cdots m_{jj} - a_{ii}b_{ii}\cdots m_{ii}) \\
&\quad= \big[a_{jj}b_{jj}\cdots m_{jj} - a_{ii}b_{ii}\cdots m_{ii} + 2(a_{ii}b_{ii}\cdots m_{ii})\rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)\big]^2, \qquad (3.11)
\end{aligned}
$$

which, together with the inequalities in Eqs. (3.1) and (3.11), leads to

$$
\begin{aligned}
&q(A_1 \star A_2 \star \cdots \star A_m) \\
&\quad\geq \min_{i\neq j}\frac{1}{2}\Big\{a_{ii}b_{ii}\cdots m_{ii} + a_{jj}b_{jj}\cdots m_{jj} - \big[(a_{ii}b_{ii}\cdots m_{ii} - a_{jj}b_{jj}\cdots m_{jj})^2 \\
&\qquad + 4(a_{ii}b_{ii}\cdots m_{ii})(a_{jj}b_{jj}\cdots m_{jj})\rho^{\frac{2}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{2}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{2}{k}}\big(J_{A_m}^{(k)}\big)\big]^{\frac{1}{2}}\Big\} \\
&\quad\geq \min_{i\neq j}\frac{1}{2}\Big\{a_{ii}b_{ii}\cdots m_{ii} + a_{jj}b_{jj}\cdots m_{jj} - [a_{jj}b_{jj}\cdots m_{jj} - a_{ii}b_{ii}\cdots m_{ii}] \\
&\qquad - 2(a_{ii}b_{ii}\cdots m_{ii})\rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)\Big\} \\
&\quad= \big[1-\rho^{\frac{1}{k}}\big(J_{A_1}^{(k)}\big)\rho^{\frac{1}{k}}\big(J_{A_2}^{(k)}\big)\cdots\rho^{\frac{1}{k}}\big(J_{A_m}^{(k)}\big)\big]\min_{1\leq i\leq n}(a_{ii}b_{ii}\cdots m_{ii}).
\end{aligned}
$$

Therefore, the conclusion is proved. □

4 Numerical example

In this section, a concrete example is used to verify the findings. Three M-matrices are considered:

$$A_1 = (a_{ij}) = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}, \qquad A_2 = (b_{ij}) = \begin{pmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ 0 & 0 & -0.5 & 5 \end{pmatrix},$$

$$A_3 = (c_{ij}) = \begin{pmatrix} 5 & -1 & -2 & -1 \\ -3 & 5 & -1 & -1 \\ 0 & -3 & 8 & -1 \\ 0 & 0 & -3 & 8 \end{pmatrix}.$$

By direct calculation, one obtains

$$\begin{aligned} &\rho(J_{A_1}) = 0.7652, \qquad \rho(J_{A_2}) = 0.8090, \qquad \rho(J_{A_3}) = 0.6666, \\ &\rho\big(J_{A_1}^{(2)}\big) = 0.2287, \qquad \rho\big(J_{A_2}^{(2)}\big) = 0.4045, \qquad \rho\big(J_{A_3}^{(2)}\big) = 0.2490, \\ &q(A_1) = 1, \qquad q(A_2) = 0.1910, \qquad q(A_3) = 1.9199. \end{aligned}$$

(1) In terms of Corollaries 1 and 3, one gets

$$q(A_1 \star A_2) \geq 1.5238.$$

According to Corollaries 2 and 4, we acquire

$$q(A_1 \star A_2) \geq 2.7834.$$

However, from the inequality in Eq. (1.2) in a previous study [1], one can only obtain

$$q(A_1 \star A_2) \geq q(A_1)q(A_2) = 0.1910.$$

In fact, $q(A_1 \star A_2) = 3.2296$.

(2) From calculation, $q(A_1 \star A_2 \star A_3) = 19.7097$. Applying Theorem 1, one obtains

$$q(A_1 \star A_2 \star A_3) \geq \big[1 - \rho(J_{A_1})\rho(J_{A_2})\rho(J_{A_3})\big] \min_{1\leq i\leq n} (a_{ii}b_{ii}c_{ii}) = 11.7469$$

and

$$q(A_1 \star A_2 \star A_3) \geq \big[1 - \rho^{\frac{1}{2}}\big(J_{A_1}^{(2)}\big)\rho^{\frac{1}{2}}\big(J_{A_2}^{(2)}\big)\rho^{\frac{1}{2}}\big(J_{A_3}^{(2)}\big)\big] \min_{1\leq i\leq n} (a_{ii}b_{ii}c_{ii}) = 16.9646.$$

Utilizing Theorem 2, one obtains

$$\begin{aligned} q(A_1 \star A_2 \star A_3) \geq \min_{i\neq j} \frac{1}{2}\Big\{ & a_{ii}b_{ii}c_{ii} + a_{jj}b_{jj}c_{jj} - \big[(a_{ii}b_{ii}c_{ii} - a_{jj}b_{jj}c_{jj})^2 \\ & + 4(a_{ii}b_{ii}c_{ii})(a_{jj}b_{jj}c_{jj})\rho^2(J_{A_1})\rho^2(J_{A_2})\rho^2(J_{A_3})\big]^{\frac{1}{2}}\Big\} = 12.9400 \end{aligned}$$

and

$$q(A_1 \star A_2 \star A_3) \geq \min_{i \neq j} \frac{1}{2} \Big\{ a_{ii}b_{ii}c_{ii} + a_{jj}b_{jj}c_{jj} - \big[(a_{ii}b_{ii}c_{ii} - a_{jj}b_{jj}c_{jj})^2 + 4(a_{ii}b_{ii}c_{ii})(a_{jj}b_{jj}c_{jj}) \rho\big(J_{A_1}^{(2)}\big) \rho\big(J_{A_2}^{(2)}\big) \rho\big(J_{A_3}^{(2)}\big) \big]^{\frac{1}{2}} \Big\} = 18.2849.$$

However, according to the inequality in Eq. (1.3), one only gets

$$q(A_1 \star A_2 \star A_3) \geq q(A_1)q(A_2)q(A_3) = 0.3667.$$

The result is trivial. One can see from the example provided that, in certain instances, the results are more accurate than earlier results.

5 Conclusions

For the Fan product of M-matrices $A_1, A_2, \ldots, A_m$, two new inequalities on the lower bound of $q(A_1 \star A_2 \star \cdots \star A_m)$ were proposed. The derived new lower bounds generalize some previous results.

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This article was done independently by the author.

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No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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