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η-Stability for stochastic functional differential equation driven by time-changed Brownian motion

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Abstract

This manuscript focuses on a class of stochastic functional differential equations driven by time-changed Brownian motion. By utilizing the Lyapunov method, we capture some sufficient conditions to ensure that the solution for the considered equation is η-stable in the pth moment sense. Subsequently, we present some new criteria of the η-stability in mean square by using time-changed Itô formula and proof by contradiction. Finally, we provide some examples to demonstrate the effectiveness of our main results.

Mathematics Subject Classification: 60H15; 60G15; 60H05

Keywords: *h*-stability; Time-changed Brownian motion; Lyapunov method; Time-changed Itô formula

1 Introduction

At present, the time-changed semimartingale theory has attracted much attention because of its widespread applications in cell biology, hydrology, physics, and economics [19]. Since Kobayashi [10] investigated stochastic calculus of the time-changed semimartingales, there have been a lot of authors working on the stochastic differential equations with the time-changed Brownian motion or Lévy processes. For example, we refer to [7, 9, 13, 20] for the numerical approximation scheme and to [2, 18] for the averaging principle. Particularly, an increasing number of experts devoted themselves to research the stability in significant senses for various SDEs with the time-changed semimartingales. For example, see [21, 22] for the stability in probability; [16, 17] for the moment stability and path stability; [25–27] for the asymptotic stability; and [12, 28] for the exponential stability.

Meanwhile, the η-stability is a valuable extension of certain well-known stability types such as polynomial, exponential, and logarithmic stability, etc. The η-stability with respect to the deterministic systems has attracted much attention from experts within a short time because it leads to a new understanding on the long-time behavior of the solution. For instance, see Choi et al. [3] for the linear dynamic equations; Damak et al. [5] for the boundedness and η-stability of the perturbed equations; Ghanmi [8] for the practical η-

stability; Xu and Liu [23] and Xu et al. [24] for the η-stability of the numerical solutions of the pantograph equations; Damak et al. [6] for the converse theorem on practical η-stability of nonlinear differential equations. Aslo Damak [4] and Mihit [15] worked on the η-stability of some evolution equations in Banach spaces by using some Gronwall-type inequalities.

However, according to the literature we reviewed, there is little literature on the η-stability for stochastic systems. Employing the Lyapunov's method, Caraballo et al. [1] studied the η-stability for neutral stochastic pantograph differential equations driven by Lévy noise, and Li et al. [11] explored the η-stability for stochastic Volterra–Levin equations. In our paper, we try to make a study of the h-stability of the following functional SDE:

$$\begin{cases} dy(t) = f(t,E_t,y_t)\,dt + u(t,E_t,y_t)\,dE_t + g(t,E_t,y_t)\,dB_{E_t}, \quad t \geq 0, \\ y_0(\cdot) = \varphi \in C([-r,0];\mathbb{R}^d), \end{cases} \tag{1.1}$$

where E_t is defined as the inverse of the β-stable subordinator with index $0 < \beta < 1$, $y_t = \{y(t+\vartheta) : \vartheta \in [-r,0]\}$ is treated as a $C([-r,0];\mathbb{R}^d)$-valued process. Giving some coefficient conditions ensuring that the solution of (1.1) is h-stable in the pth moment by using Lyapunov's technique is our first major research aim.

Effectively, it is difficult to look for a Lyapunov's function (functional) for time changed stochastic systems. Meanwhile, the obtained conditions captured by making use of Lyapunov's function are generally shown on the basis of some differential inequalities, matrix inequalities, and so on. There calculations are complicated and difficult to test. The second aim of our paper is to study some new explicit conditions to ensure that the solution of (1.1) possesses the η-stability in mean square under some hypotheses. In the proof, our method takes advantage of the Itô formula and involves a proof by contradiction.

2 Preliminary

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Assume that $\{D(t), t \geq 0\}$ is a càdlàg nondecreasing Lévy process, which is named a subordinator, that starts at 0. Particularly, $D(t)$ is called the β-stable subordinator denoted by $D_\beta(t)$ if it is strictly increasing with the following Laplace transform:

$$\mathbb{E}\bigl(e^{-\lambda D_\beta(t)}\bigr) = e^{-t\lambda^\beta}, \quad \lambda > 0, \beta \in (0,1).$$

Define the generalized inverse of $D_\beta(t)$ as

$$E_t := E_t^\beta = \inf\bigl\{u > 0 : D_\beta(u) > t\bigr\},$$

which is well known as the initial hitting time process. The time-change process E_t is nondecreasing and continuous. Define the special filtration as

$$\mathcal{F}_t = \bigcap_{u>t}\bigl\{\sigma(B_\nu : 0 \leq \nu \leq u) \vee \sigma(E_\nu : \nu \geq 0)\bigr\},$$

where B_ν is the standard Brownian motion and the notation $\sigma_1 \vee \sigma_2$ denotes the σ-algebra generated by the union of σ-algebras σ_1 and σ_2. By the results in [14], we can deduce that B_{E_t} is a square integrable martingale with respect to the filtration $\mathcal{G}_t = \mathcal{F}_{E_t}$.

Let $r > 0$ and $C := C([-r,0];\mathbb{R}^d)$ denote the family of all continuous functions ϕ from $[-r,0]$ to $\mathbb{R}^d$ with the norm $\|\phi\|_C = \sup_{-r\leq s\leq 0}|\phi(s)|$.

Based on [26], we put forward the following hypotheses for ensuring the existence and uniqueness of a solution for (1.1):

(H1) $f,u:\mathbb{R}_+\times\mathbb{R}_+\times C\to\mathbb{R}^d$ and $g:\mathbb{R}_+\times\mathbb{R}_+\times C\to\mathbb{R}^{d\times k}$ are some measurable functions and there is a positive constant K such that for all $t_1,t_2\geq 0$ and $x,y\in C$,

$$\begin{aligned}&\bigl|f(t_1,t_2,x)-f(t_1,t_2,y)\bigr|\vee\bigl|u(t_1,t_2,x)-u(t_1,t_2,y)\bigr|\vee\bigl\|g(t_1,t_2,x)-g(t_1,t_2,y)\bigr\|\\&\quad\leq K\|x-y\|_C.\end{aligned}$$

(H2) If $y(t)$ is a càdlàg $\mathcal{G}_t$-adapted process, then

$$f(t,E_t,y_t),u(t,E_t,y_t),g(t,E_t,y_t)\in\mathcal{L}(\mathcal{G}_t),$$

where $\mathcal{L}(\mathcal{G}_t)$ denotes the class of càdlàg $\mathcal{G}_t$-adapted processes.

To establish η-stability, we also demand the following assumption:

$$f(t_1,t_2,0)\equiv 0,\qquad h(t_1,t_2,0)\equiv 0,\qquad g(t_1,t_2,0)\equiv 0. \tag{2.1}$$

Referring to [26], we conclude that (1.1) has a unique $\mathcal{G}_t$-adapted solution process $y(t)$ under the assumptions (H1) and (H2). Furthermore, equation (1.1) has a trivial solution when the initial value is $\xi\equiv 0$.

Definition 2.1 A positive function η on $\mathbb{R}_+$ is called an η-type function if the following assumptions are fulfilled:

(i) It is nondecreasing and continuously differentiable in $\mathbb{R}_+$.
(ii) $\eta(0)=1$, $\lim_{t\to\infty}\eta(t)=\infty$, and $J=\sup_{t>0}|\frac{\eta'(t)}{\eta(t)}|<\infty$.
(iii) For all $u\geq 0$ and $v\geq 0$, one has $\eta(u+v)\leq\eta(u)\eta(v)$.

Definition 2.2 A solution $y(t,\varphi)$ of (1.1) is called η-stable in the pth moment sense if, for any initial data φ, there are positive constants $\delta>0$ and $K>0$ such that for each $t\geq 0$,

$$\mathbb{E}\bigl|y(t,\varphi)\bigr|^p\leq K\mathbb{E}\|\varphi\|_C\eta^{-\delta}(t). \tag{2.2}$$

In particular, if p is equal to 2, $y(t,\varphi)$ is said to be η-stable in mean square.

Remark 2.1 We remark that h-stability coincides with some known stability types when h are some special functions. In fact, if $h(t)=e^t$, then η-stability is consistent with exponential stability; if $\eta(t)=1+t$, then η-stability is consistent with polynomial stability, and if $\eta(t)=\ln(e+t)$, then η-stability is consistent with logarithmic stability.

Remark 2.2 There exists an η-type function which tends to infinity faster than e^t. For instance, $\eta(t)=(1+t)e^t$ is an η-type function and $\lim_{t\to+\infty}\frac{(1+t)e^t}{e^t}=+\infty$.

3 Main results

Let $V \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$, $V_{t_1}(t_1,t_2,y) = \frac{\partial V(t_1,t_2,y)}{\partial t_1}$, $V_{t_2}(t_1,t_2,y) = \frac{\partial V(t_1,t_2,y)}{\partial t_2}$, $V_y(t_1, t_2,y) = (\frac{\partial V(t_1,t_2,y)}{\partial y_1},\dots,\frac{\partial V(t_1,t_2,y)}{\partial y_d})$ and $V_{yy}(t_1,t_2,y) = (\frac{\partial^2 V(t_1,t_2,y)}{\partial y_i \partial y_j})_{d\times d}$ be continuous for all $(t_1,t_2,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$. By Itô formula (see [21]), from (1.1) we have

$$dV(t_1,t_2,y) = J_1 V(t_1,t_2,y)\,dt + J_2 V(t_1,t_2,y)\,dE_t + V_y(t_1,t_2,y)g(t_1,t_2,y)\,dB_{E_t}, \tag{3.1}$$

where

$$J_1 V(t_1,t_2,y) = V_{t_1}(t_1,t_2,y) + V_x(t_1,t_2,y)f(t_1,t_2,y) \tag{3.2}$$

and

$$\begin{aligned} J_2 V(t_1,t_2,y) &= V_{t_2}(t_1,t_2,y) + V_x(t_1,t_2,y)u(t_1,t_2,y) \\ &\quad + \frac{1}{2}\operatorname{Tr}\bigl[g^T(t_1,t_2,y)V_{yy}(t_1,t_2,y)g(t_1,t_2,y)\bigr]. \end{aligned} \tag{3.3}$$

Theorem 3.1 *Let the hypotheses (H1) and (H2) hold. Assume that there is* $V \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)$ *such that for any* $(t, E_t, y(t)) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$,

(i) $c_1|y(t)|^p \le V(t,E_t,y(t)) \le c_2|y(t)|^p$,
(ii) $J_1 V(t,E_t,y(t)) \le -\lambda V(t,E_t,y(t))$,
(iii) $J_2 V(t,E_t,y(t)) \le 0$,

hold for the solution $y(t)$ *of* (1.1), *where* p, c_1, c_2, *and* λ *are some positive constants. If for* $\delta \in (0,\lambda/J)$, *we can show*

$$\mathbb{E}\bigl(\bigl|y(t)\bigr|^p\bigr) \le \frac{c_2}{c_1}\eta^{-\delta}(t)\mathbb{E}\bigl(\|\varphi\|_C^p\bigr),$$

then the trivial solution of (1.1) *is* η*-stable in mean square.*

Proof Let $\delta \in (0,\lambda/J)$. Applying Itô formula to $\eta^\delta(t)V(t_1,t_2,y(t))$, for every $t \ge 0$, we obtain

$$\begin{aligned} &\eta^\delta(t)V\bigl(t_1,t_2,y(t)\bigr) \\ &\quad = V\bigl(0,0,y(0)\bigr) + \int_0^t \eta^\delta(s)\biggl[\delta\frac{\eta'(s)}{\eta(s)}V\bigl(s,E_s,y(s)\bigr) + V_{t_1}\bigl(s,E_s,y(s)\bigr) \\ &\qquad + V_y\bigl(s,E_s,y(s)\bigr)f(s,E_s,y_s)\biggr]\,ds \\ &\qquad + \int_0^t h^\delta(s)\biggl[V_{t_2}\bigl(s,E_s,y(s)\bigr) + V_y\bigl(s,E_s,y(s)\bigr)u(s,E_s,y_s) \\ &\qquad + \frac{1}{2}\operatorname{Tr}\bigl[g^T(s,E_s,y_s)V_{yy}\bigl(s,E_s,y(s)\bigr)g(s,E_s,y_s)\bigr]\biggr]\,dE_s \\ &\qquad + \int_0^t \eta^\delta(s)V_y\bigl(s,E_s,y(s)\bigr)g(s,E_s,y_s)\,dB_{E_s}. \end{aligned}$$

By condition (iii), we get

$$\begin{aligned} & h^{\delta}(t)V\big(t_1,t_2,y(t)\big) \\ & \quad \le V\big(0,0,y(0)\big) + \int_0^t h^{\delta}(s)\bigg[\delta\frac{h'(s)}{h(s)}V\big(s,E_s,y(s)\big) + V_{t_1}\big(s,E_s,y(s)\big) \\ & \qquad + V_y\big(s,E_s,y(s)\big)f(s,E_s,y_s)\bigg]\,ds \\ & \qquad + \int_0^t h^{\delta}(s)V_y\big(s,E_s,y(s)\big)g(s,E_s,y_s)\,dB_{E_s}. \end{aligned} \tag{3.4}$$

Notice that

$$\mathbb{E}\int_0^t h^{\delta}(s)V_y\big(s,E_s,y(s)\big)g(s,E_s,y_s)\,dB_{E_s} = 0.$$

Then, taking expectations on both sides of (3.4), we have

$$\begin{aligned} \mathbb{E}\big[h^{\delta}(t)V\big(t,E_t,y(t)\big)\big] &= \mathbb{E}\big[V\big(0,0,y(0)\big)\big] \\ & \quad + \mathbb{E}\bigg(\int_0^t \eta^{\delta}(s)\bigg[\delta\frac{\eta'(s)}{\eta(s)}V\big(s,E_s,y(s)\big) + J_1V\big(s,E_s,y(s)\big)\bigg]\,ds\bigg). \end{aligned}$$

By using Definition 2.1(ii), one has

$$\begin{aligned} \mathbb{E}\big[\eta^{\delta}(t)V\big(t,E_t,y(t)\big)\big] &\le \mathbb{E}\big[V\big(0,0,y(0)\big)\big] \\ & \quad + \mathbb{E}\bigg(\int_0^t \eta^{\delta}(s)\big[\delta JV\big(s,E_s,y(s)\big) + J_1V\big(s,E_s,y(s)\big)\big]\,ds\bigg). \end{aligned}$$

Since $\delta \in (0,\lambda/J)$, according to condition (ii), one has

$$\mathbb{E}\big[\eta^{\delta}(t)V\big(t,E_t,y(t)\big)\big] \le \mathbb{E}\big[V\big(0,0,y(0)\big)\big].$$

Furthermore, from condition (i), we can obtain

$$c_1\mathbb{E}\big[\eta^{\delta}(t)\big|y(t)\big|^p\big] \le \mathbb{E}\big[\eta^{\delta}(t)V\big(t,E_t,y(t)\big)\big] \le \mathbb{E}\big[V\big(0,0,y(0)\big)\big] \le c_2\mathbb{E}\big[\|\varphi\|_C^p\big].$$

Hence

$$\mathbb{E}\big[\big|y(t)\big|^p\big] \le \frac{c_2}{c_1}\eta^{-\delta}(t)\mathbb{E}\big[\|\varphi\|_C^p\big].$$

The proof is complete. □

Remark 3.1 The authors of [28] showed that the solution of (1.1) without time delay is the pth moment exponentially stable under (H1) and (H2) when the conditions (i)–(iii) are satisfied. Thus, it follows from Remark 2.1 that our Theorem 3.1 generalizes Theorem 4.1 of [28].

Remark 3.2 The authors of [26] showed that the solution for (1.1) with Markovian switching is the pth moment exponentially stable under (H1) and (H2) when the conditions (i)–(iii) are satisfied. We remark that the solution of (1.1) with Markovian switching is h-stable in the pth moment sense under (H1) and (H2) when the corresponding conditions (i)–(iii) hold, which implies that our Theorem 3.1 expands Theorem 3.1 of [26] by using Remark 2.1.

Next, we want to make use of Theorem 3.1 to establish the following corollary.

Corollary 3.1 Let the assumptions (H1)–(H2) hold. If there is a positive constant $\lambda > 0$ such that, for all $t \geq 0$ and the solution $x(t)$ of (1.1), one has

$$\langle y(t), f(t, E_t, y_t)\rangle \leq -\lambda \left|y(t)\right|^2 \tag{3.5}$$

and

$$2\langle y(t), u(t, E_t, y_t)\rangle + \mathrm{Tr}\left[g^T(t, E_t, y_t) g(t, E_t, y_t)\right] \leq 0, \tag{3.6}$$

then the trivial solution of (1.1) is η-stable in mean square.

Proof Let $V(t, E_t, y(t)) = |y(t)|^2$. One can obviously check that condition (i) in Theorem 3.1 holds for $p = 2$, $c_1 = c_2 = 1$, and

$$\begin{aligned} J_1 V\big(t, E_t, y(t)\big) &= 2\langle y(t), f(t, E_t, y_t)\rangle, \\ J_2 V\big(t, E_t, y(t)\big) &= 2\langle y(t), u(t, E_t, y_t)\rangle + Tr\left[g^T(t, E_t, y_t) g(t, E_t, y_t)\right]. \end{aligned}$$

Thus, (3.5) and (3.6) respectively imply that the conditions (ii) and (iii) hold. The proof is complete. □

And now we are going to study some new conditions ensuring the η-stability for (1.1). At this time, we need to introduce some functions. Let $\kappa(\vartheta, t) : [-r, 0] \times \mathbb{R}_+ \to \mathbb{R}^d$ be increasing in ϑ for all $t \in \mathbb{R}_+$. Besides, we also assume that $\kappa(\theta, t)$ is normalized to be continuous from the left in ϑ on $[-r, 0]$. Let

$$L(\varphi, t) := \int_{-r}^{0} \varphi(\vartheta)\, d\big[\kappa(\vartheta, t)\big], \quad t \geq 0, \tag{3.7}$$

be a locally bounded Borel-measurable function in t for each $\varphi \in C([-r, 0]; \mathbb{R}^d)$. In our case, the integral in (3.4) is the Riemann–Stieltjes integral.

Theorem 3.2 Let $\zeta(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ be a locally bounded Borel-measurable function. If for any $t \in \mathbb{R}_+$ and $\varphi \in C([-r, 0]; \mathbb{R}^d)$, one has

$$2\langle \varphi(0), f(t, E_t, \varphi)\rangle \leq \zeta(t)\left|\varphi(0)\right|^2 + \int_{-r}^{0} \left|\varphi(\vartheta)\right|^2 d\big[\kappa(t, \vartheta)\big] \tag{3.8}$$

and

$$2\langle \varphi(0), u(t, E_t, \varphi)\rangle + \mathrm{Tr}\left[g^T(t, E_t, \varphi) g(t, E_t, \varphi)\right] \leq 0, \tag{3.9}$$

then the solution of (1.1) is η-stable in the mean square sense if there exists $\beta > 0$ such that for any $t \in \mathbb{R}_+$,

$$\zeta(t) + \int_{-r}^{0} \eta^{\beta}(-s)\, d\big[\kappa(t,s)\big] \le -\beta. \tag{3.10}$$

Proof To prove the above conclusion, we will divide the value range of J into two, namely $J \in (0,1]$ and $J \in (1,+\infty)$. The specific proof process is as follows:

Case 1. Suppose $J \in (0,1]$. Fix $K > 1$ and let $\varphi \in C([-r,0];\mathbb{R}^d)$ be such that $\mathbb{E}\|\varphi\|_C^2 > 0$. We denote $y(t) := y(t,\varphi)$, $t \ge -r$, where $y(t,\varphi)$ is the solution of (2.1). Denote $Y(t) := \mathbb{E}|y(t)|^2$, $t \in \mathbb{R}_+$ and $Z(t) := K\mathbb{E}\|\varphi\|_C^2 \eta^{-\beta}(t)$, $t \ge 0$. For convenience, we define $\eta(t) = \eta(0) = 1$ for $t \in [-r,0]$. Next, we can conclude that $Y(t) < Z(t)$ for $t \in [-r,0]$ since $K > 1$ and $\mathbb{E}\|\varphi\|_C^2 > 0$. We shall show

$$Y(t) \le Z(t), \quad \forall t \in \mathbb{R}_+. \tag{3.11}$$

To the contrary, we assume that there exists $t_1 > 0$ such that $Y(t_1) > Z(t_1)$. Let $t^* := \inf\{t > 0 : Y(t) > Z(t)\}$. By continuity of $Y(t)$ and $Z(t)$, we have

$$Y(t) \le Z(t), \quad t \in \big[0,t^*\big], \qquad Y\big(t^*\big) = Z\big(t^*\big) \tag{3.12}$$

and

$$\mathbb{E}\big|y(t_n)\big|^2 > K\mathbb{E}\|\varphi\|_C^2 \eta^{-\beta}(t_n),$$

for some $t_n \in (t^*, t^* + \frac{1}{n})$, $n \in \mathbb{N}$.

Choosing $0 < \delta < \beta$ and applying Itô formula to $V(t,y) = \eta^{\delta}(t)|y(t)|^2$, one has

$$\begin{aligned}\eta^{\delta}(t)\big|y(t)\big|^2 = {}& \big|\varphi(0)\big|^2 + \int_0^t \delta\eta^{\delta}(s)\frac{\eta'(s)}{\eta(s)}\big|y(s)\big|^2\, ds + 2\int_0^t \eta^{\delta}(s)\big\langle y(s), f(s,E_s,y_s)\big\rangle\, ds \\ & + 2\int_0^t \eta^{\delta}(s)\big\langle y(s), u(s,E_s,y_s)\big\rangle\, dE_s + 2\int_0^t \eta^{\delta}(s)\big\langle y(s), u(s,E_s,y_s)\big\rangle\, dB_{E_s} \\ & + \int_0^t \eta^{\delta}(s)\operatorname{Tr}\big[g^T(s,E_s,y_s)g\big((s,E_s,y_s)\big)\big]\, dE_s.\end{aligned}$$

Utilizing the standard property of the Itô integral, one has

$$\mathbb{E}\left(\int_0^t h^{\delta}(s)\big\langle y(s), g(s,E_s,y_s)\big\rangle\, dB_{E_s}\right) = 0.$$

From (3.8), (3.9), and the Fubini theorem, we get

$$\begin{aligned}\eta^{\delta}(t)\mathbb{E}\big|y(t)\big|^2 \le {}& \big|\varphi(0)\big|^2 + \int_0^t \eta^{\delta}(u)\mathbb{E}\big|y(u)\big|^2\big(J\delta + \zeta(u)\big)\, ds \\ & + \int_0^t \eta^{\delta}(u)\left(\int_{-r}^{0} \mathbb{E}\big|y(u+\vartheta)\big|^2\, d\big[\kappa(u,\vartheta)\big]\right) du.\end{aligned} \tag{3.13}$$

Let $K_1 := K\mathbb{E}\|\xi\|^2$. Since $\kappa(u,\vartheta)$ is nondecreasing in ϑ on $[-r,0]$, from (3.11) we know that

$$\int_{-r}^{0}\mathbb{E}\big|y(u+\vartheta)\big|^2\,d\big[\kappa(u,\vartheta)\big]\le K_1\int_{-r}^{0}\eta^{-\beta}(u+\vartheta)\,d\big[\kappa(u,\vartheta)\big],$$

for any $u\le t^*$.

If $u+\vartheta\le 0$, then $u\le-\vartheta$. Since η is increasing in $\mathbb{R}_+$, we have

$$\eta^{-\beta}(u)\eta^{\beta}(-\vartheta)\ge 1=\eta^{-\beta}(u+\vartheta).$$

If $u+\vartheta\ge 0$, by Definition 2.1(iii), we also have

$$\eta^{-\beta}(u+\vartheta)\le\eta^{-\beta}(u)\eta^{\beta}(-\vartheta).$$

So, we get for each $t^*\ge s$ that

$$\int_{-r}^{0}\mathbb{E}\big|y(s+\vartheta)\big|^2\,d\big[\kappa(u,\vartheta)\big]\le K_1\int_{-r}^{0}\eta^{-\beta}(u)\eta^{\beta}(-\vartheta)\,d\big[\kappa(u,\vartheta)\big].$$

Then, combining (3.10) and (3.13), we get for any $u\le t^*$,

$$\begin{aligned}\eta^{\delta}(t)\mathbb{E}\big|y(t)\big|^2&\le\big|\varphi(0)\big|^2+K_1\int_0^t\eta^{\delta-\beta}(u)\big(\delta+\zeta(u)\big)\,du\\&\quad+\int_0^t\eta^{\delta-\beta}(u)\bigg(\int_{-r}^{0}\eta^{\beta}(-\vartheta)d\big[\kappa(u,\vartheta)\big]\bigg)\,du\\&\le\big|\xi(0)\big|^2+K_1\int_0^t h^{\delta-\beta}(s)(\delta-\beta)\,ds.\end{aligned}$$

Noticing that $J\in(0,1]$, we have $-\eta(u)\le\eta'(u)$. Since $\delta-\beta<0$, we obtain

$$\eta^{\delta}(t)\mathbb{E}\big|y(t)\big|^2\le\mathbb{E}\big|\varphi(0)\big|^2+K_1\int_0^t(\delta-\beta)\eta^{\delta-\beta-1}(u)\eta'(u)\,du.$$

Due to the fact that $K>1$, we have

$$\begin{aligned}\eta^{\delta}\big(t^*\big)\mathbb{E}\big|y\big(t^*\big)\big|^2&\le\big|\varphi(0)\big|^2+K_1\big[\eta^{\delta-\beta}\big(t^*\big)-1\big]\\&=\big(\big|\varphi(0)\big|^2-K\|\varphi\|_C^2\big)+K\mathbb{E}\|\varphi\|_C^2\eta^{\delta-\beta}\big(t^*\big)\\&<K\mathbb{E}\|\varphi\|_C^2\eta^{\delta-\beta}\big(t^*\big).\end{aligned}$$

Thus, $\mathbb{E}|y(t^*)|^2<K\mathbb{E}\|\varphi\|_C^2\eta^{-\beta}(t^*)$, which conflicts with (3.12). Hence,

$$\mathbb{E}\big|y(t)\big|^2\le K\mathbb{E}\|\varphi\|_C^2\eta^{-\beta}(t),\quad\text{for each } t\ge 0.$$

Consequently, the solution of (1.1) is η-stable in the mean square sense.

Case 2. Suppose $J\in(1,+\infty)$. Choose $0<\delta<\frac{\beta}{J}$. By a simple calculation, we know that $0<\frac{J\delta-\beta}{J(\delta-\beta)}<1$. Fix $K>\frac{J(\delta-\beta)}{J\delta-\beta}>1$ and let $\varphi\in C([-r,0];\mathbb{R}^d)$ be such that $\mathbb{E}\|\varphi\|_C^2>0$. In a similar manner, we can also show that for each $t\in\mathbb{R}_+$,

$$Y(t)\le Z(t).\tag{3.14}$$

To the contrary, we assume that there exists $t_1 > 0$ such that $X(t_1) > Z(t_1)$. Let $t^* := \inf\{t > 0 : Z(t) < Y(t)\}$. By continuity of $Y(t)$ and $Z(t)$,

$$Y(t) \leq Z(t), \quad t \in [0, t^*], \qquad Y(t^*) = Z(t^*), \tag{3.15}$$

and

$$\mathbb{E}|y(t_n)|^2 > K\mathbb{E}\|\varphi\|_C^2 \eta^{-\beta}(t_n),$$

for some $t_n \in (t^*, t^* + \frac{1}{n})$, $n \in \mathbb{N}$.

Applying the Itô formula to $V(t,y) = \eta^\delta(t)|y(t)|^2$, we get for all $t \leq t^*$,

$$\begin{aligned}\eta^\delta(t)\mathbb{E}|y(t)|^2 &\leq \mathbb{E}|\varphi(0)|^2 + K_1 \int_0^t \eta^{\delta-\beta}(u)\big(J\delta + \zeta(u)\big)\,du \\ &\quad + \int_0^t \eta^{\delta-\beta}(u)\bigg(\int_{-r}^0 \eta^\beta(-\vartheta)d\big[\kappa(u,\vartheta)\big]\bigg)\,du \\ &\leq \mathbb{E}|\varphi(0)|^2 + K_1 \int_0^t \eta^{\delta-\beta}(u)(J\delta - \beta)\,du.\end{aligned}$$

Noticing that $J \in (1, +\infty)$, we have $-\eta(s) \leq \frac{1}{J}\eta'(s)$. Since $J\delta - \beta < 0$, we get

$$\begin{aligned}\eta^\delta(t)\mathbb{E}|y(t)|^2 &\leq \mathbb{E}|\varphi(0)|^2 + K_1 \int_0^t \frac{J\delta - \beta}{J}\eta^{\delta-\beta-1}(u)h'(u)\,du \\ &= \mathbb{E}|\varphi(0)|^2 + K_1 \frac{J\delta - \beta}{J(\delta - \beta)}\big[\eta^{\delta-\beta}(t^*) - 1\big].\end{aligned}$$

Noticing that $0 < \frac{J\delta-\beta}{J(\delta-\beta)} < 1$ and using the fact that $K > \frac{J(\delta-\beta)}{J\delta-\beta}$, one has

$$\begin{aligned}\eta^\delta(t^*)\mathbb{E}|y(t^*)|^2 &\leq \mathbb{E}|\varphi(0)|^2 + K_1\eta^{\delta-\beta}(t^*) - K\frac{J\delta - \beta}{J(\delta - \beta)}\mathbb{E}\|\varphi\|_C^2 \\ &= \bigg(\mathbb{E}|\varphi(0)|^2 - K\frac{J\delta - \beta}{J(\delta - \beta)}\mathbb{E}\|\varphi\|_C^2\bigg) + K\mathbb{E}\|\varphi\|_C^2\eta^{\delta-\beta}(t^*) \\ &< K\mathbb{E}\|\varphi\|_C^2\eta^{\delta-\beta}(t^*).\end{aligned}$$

Thus, $\mathbb{E}|y(t^*)|^2 < K\mathbb{E}\|\varphi\|_C^2\eta^{-\beta}(t^*)$, which conflicts with (3.15). Hence,

$$\mathbb{E}|y(t)|^2 \leq K\mathbb{E}\|\varphi\|_C^2\eta^{-\beta}(t), \quad \text{for each } t \geq 0.$$

Consequently, the solution of (1.1) is h-stable in the mean square sense.

The proof is complete. □

Corollary 3.2 Let $\Gamma(\cdot,\cdot) : [-r,0] \times \mathbb{R}_+ \to \mathbb{R}_+$, $\zeta_i(\cdot), r_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, $i = 0,1,2,\ldots,m$ with $0 := r_0(t) \leq r_1(t) \leq r_2(t) \leq \cdots \leq r_m(t) \leq r$, $t \in \mathbb{R}_+$, be locally bounded Borel-measurable functions. If for each $t \in \mathbb{R}_+$ and $\varphi \in C([-r,0];\mathbb{R}^d)$ one has

$$2\langle\varphi(0), f(t,E_t,\varphi)\rangle \leq \sum_{i=1}^m \zeta_i(t)\big|\varphi\big(-r_i(t)\big)\big|^2 + \int_{-r}^0 \Gamma(t,s)\big|\varphi(s)\big|^2\,ds \tag{3.16}$$

and

$$2\langle\varphi(0),u(t,E_t,\varphi)\rangle+\mathrm{Tr}\big[g^T(t,E_t,\varphi)g(t,E_t,\varphi)\big]\le 0, \tag{3.17}$$

then the solution of (1.1) is η-stable in the mean square sense if there exists $\beta>0$ such that for each $t\in\mathbb{R}_+$,

$$\sum_{i=1}^{m}\eta^{\beta}\big(r_i(t)\big)\gamma_i(t)+\int_{-r}^{0}\eta^{\beta}(-u)\Gamma(t,u)\,du\le-\beta. \tag{3.18}$$

Proof Define the following functions for $t\ge 0$, $u\in[-r,0]$:

$$u_i(u,t):=\begin{cases}0, & \text{if } u\in[-r,-r_i(t)],\\ \gamma_i(t), & \text{if } u\in(-r_i(t),0],\end{cases}$$

$$\kappa(u,t):=\sum_{i=1}^{n}u_i(u,t)+\int_{-r}^{u}\Gamma(r,t)\,dr.$$

By the properties of the Riemann–Stieltjes integrals, for any $\varphi(\cdot)\in C([-r,0];\mathbb{R}^d)$, we have

$$\int_{-r}^{0}\varphi(u)\,d\bigg[\int_{-r}^{u}\Gamma(r,t)\,dr\bigg]=\int_{-r}^{0}\varphi(u)\Gamma(u,t)\,du,\quad t\in\mathbb{R}_+,$$

Then for any $t\in\mathbb{R}_+$, $\varphi(\cdot)\in C([-r,0];\mathbb{R}^d)$,

$$\int_{-r}^{0}\phi(u)\,d\big[\kappa(u,t)\big]=\sum_{i=1}^{m}\zeta_i(t)\varphi\big(-r_i(t)\big)+\int_{-r}^{0}\varphi(u)\Gamma(u,t)\,du.$$

Hence, (3.16) implies that (3.6) holds, and (3.18) implies that (3.10) holds. According to Theorem 3.1, we can immediately derive our desired result. The proof is complete. □

Corollary 3.3 Let ζ be a constant and $\nu(\cdot):[-r,0]\to\mathbb{R}_+$ an increasing function. If for any $t\in\mathbb{R}_+$ and $\varphi\in C([-r,0];\mathbb{R}^d)$, one has

$$2\langle\varphi(0),f(t,E_t,\varphi)\rangle\le\zeta\,|\varphi(0)|^2+\int_{-r}^{0}|\varphi(\vartheta)|^2\,d\big[\nu(\vartheta)\big] \tag{3.19}$$

and

$$2\langle\varphi(0),u(t,E_t,\varphi)\rangle+\mathrm{Tr}\big[g^T(t,E_t,\varphi)g(t,E_t,\varphi)\big]\le 0, \tag{3.20}$$

then the solution of (1.1) is η-stable in the mean square sense if

$$\zeta+\nu(0)-\nu(-r)<0. \tag{3.21}$$

Proof By (3.21) and the continuity of $\eta(t)$, we can show that for a sufficiently small $\beta>0$ one has the following inequality:

$$\zeta+\eta^{\beta}(r)\big[\nu(0)-\nu(-r)\big]<-\beta.$$

Since $\nu(\cdot)$ is increasing, we obtain

$$\zeta + \int_{-r}^{0} \eta^{\beta}(-\vartheta)\, d\big[\nu(\vartheta)\big] \le \zeta + \eta^{\beta}(r)\big[\nu(0) - \nu(-r)\big] < -\beta,$$

which implies that (3.10) holds. The proof is complete. □

From Corollaries 3.2 and 3.3, we can immediately obtain the following Corollary 3.4.

Corollary 3.4 Let $r_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, $i = 0, 1, 2, \dots, m$ with $0 := r_0(t) \le r_1(t) \le r_2(t) \le \cdots \le r_m(t) \le r$, $t \in \mathbb{R}_+$, be locally bounded Borel-measurable functions. Assume that there exist constants ζ_i, $i = 0, 1, 2, \dots, m$ and a Borel-measurable function $\mu : [-r, 0] \to \mathbb{R}_+$ such that for any $t \in \mathbb{R}_+$ and $\varphi \in C([-r, 0]; \mathbb{R}^d)$,

$$2\big\langle \varphi(0), f(t, E_t, \varphi)\big\rangle \le \sum_{i=1}^{m} \zeta_i \big|\varphi\big(-r_i(t)\big)\big|^2 + \int_{-r}^{0} \mu(u)\big|\varphi(u)\big|^2\, du \tag{3.22}$$

and

$$2\big\langle \varphi(0), u(t, E_t, \varphi)\big\rangle + \operatorname{Tr}\big[g^T(t, E_t, \varphi) g(t, E_t, \varphi)\big] \le 0. \tag{3.23}$$

Then, the solution of (1.1) is h-stable in the mean square sense if

$$\sum_{i=0}^{m} \zeta_i + \int_{-r}^{0} \mu(u)\, du < 0. \tag{3.24}$$

Remark 3.3 In fact, the assumptions (3.8) and (3.9) are generalizations of some existing conditions. According to the information we have found in the reported literature, even for deterministic differential equations, the assumptions (3.8) and (3.9) have not been used to research the η-stability in mean square of stochastic systems. Our results are of innovative value and they provide advantage when studying applications of "mixed" delay time-changed SDEs, including the point, variable, and distributed delay.

Remark 3.4 The conditions of Theorems 3.1 and 3.2 highlight the dominant role of the drift term "dt" in the study of the η-stability for a time-changed system. In the meantime, it indicates that "dE_t" and "dB_{E_t}" are relatively less important.

Now, we intend to present an example to explain the statement in Remark 3.4. We consider the following two time-changed SDEs:

$$dy(t) = -y(t)\, dt + cy(t)\, dE_t + dy(t)\, dB_{E_t} \tag{3.25}$$

and

$$dy(t) = y(t)\, dt + cy(t)\, dE_t + dy(t)\, dB_{E_t}. \tag{3.26}$$

By Corollary 3.4, we conclude that the time-changed equation (3.25) is h-stable if $2c + d^2 \le 0$, while we cannot conclude that the time-changed equation (3.26) is h-stable no matter what c and d are.

4 Some examples

Example 4.1 Consider the following functional stochastic differential equation driven by the time-changed Brownian motion:

$$dy(t) = \big(f_0\big(t, y(t)\big) + f_1\big(t, y(t)\big)\big)\,dt + u(t, E_t)y(t)\,dE_t + g(t, E_t)y(t)\,dB_{E_t} \tag{4.1}$$

with $y_0(\cdot) = \xi \in C([-r, 0]; \mathbb{R}^d)$. Assume that there exists a continuous function $\zeta(\cdot) : [-r, 0] \to \mathbb{R}_+$ such that for all $t \geq 0$, $y \in \mathbb{R}^d$, and $\varphi \in C([-r, 0]; \mathbb{R}^d)$,

$$y^T f_0(t, y) \leq \alpha |y|^2, \tag{4.2}$$

$$\big|f_1(t, \varphi)\big| \leq \int_{-r}^{0} \zeta(\vartheta)\big|\varphi(t + \vartheta)\big|\,d\vartheta, \tag{4.3}$$

and for all $t_1, t_2 > 0$,

$$2u(t_1, t_2) + g^2(t_1, t_2) \leq 0. \tag{4.4}$$

According to [26], we can draw the conclusion that the null solution of (4.1) is mean square exponentially stable if

$$\alpha + \sqrt{r}\left(\int_{-r}^{0} \big(\zeta(u)\big)^2\,du\right)^{1/2} < 0. \tag{4.5}$$

Notice that (4.2) and (4.3) mean that

$$\varphi(0)^T f(t, \varphi) \leq \left(\alpha + \frac{1}{2}\int_{-r}^{0} \zeta(u)\,du\right)\big|\varphi(0)\big|^2 + \frac{1}{2}\int_{-r}^{0} \zeta(u)\big|\varphi(u)\big|^2\,du.$$

By Corollary 3.4, the null solution of (4.1) is η-stable in mean square if (4.4) holds and

$$\alpha + \int_{-r}^{0} \zeta(s)\,ds < 0. \tag{4.6}$$

Notice that

$$\int_{-r}^{0} \zeta(u)\,du \leq \sqrt{r}\left(\int_{-r}^{0} \big(\zeta(u)\big)^2\,du\right)^{1/2},$$

owing to Hölder inequality. So, (4.6) is weaker than (4.5).

Example 4.2 In order to further explain the applicability of our main conclusions, we carefully consider the following scalar linear stochastic differential equation with variable delay:

$$dy(t) = \big(-c(t)y(t) + d(t)y\big(t - h_1(t)\big)\big)\,dt + G(t, E_t)y(t)\,dE_t + H(t, E_t)y(t)\,dB_{E_t}, \tag{4.7}$$

where c, d, h_1 are continuous functions on $\mathbb{R}_+$ and $h_1(t) \leq r$ for some $r > 0$.

Let

$$f(t,\varphi) := -c(t)\varphi(0) + d(t)\varphi\bigl(-h_1(t)\bigr)$$

for $t \geq 0$, $\varphi \in C([-r,0];\mathbb{R})$. Then, for all $t \geq 0$, $\varphi \in C([-r,0];\mathbb{R})$, one has

$$\begin{aligned} 2\varphi(0)f(t,\varphi) &= -2c(t)\bigl|\varphi(0)\bigr|^2 + 2d(t)\varphi(0)\varphi\bigl(-h_1(t)\bigr) \\ &\leq -2c(t)\bigl|\varphi(0)\bigr|^2 + \bigl|d(t)\bigr|\bigl(\varphi^2(0) + \varphi^2\bigl(-h_1(t)\bigr)\bigr). \end{aligned} \tag{4.8}$$

Next, referring to Corollary 3.2, one can conclude that if there is a positive constant λ such that for each $t \geq 0$,

$$-2c(t) + \bigl|d(t)\bigr| + \eta^{\zeta}\bigl(h_1(t)\bigr)\bigl|d(t)\bigr| \leq -\lambda, \tag{4.9}$$

and for any $t_1, t_2 > 0$,

$$2G(t_1,t_2) + H^2(t_1,t_2) \leq 0, \tag{4.10}$$

then the null solution of (4.7) is η-stable in the mean square sense.

On the other hand, due to continuity,

$$-c(t) + \bigl|d(t)\bigr| \leq 0 \tag{4.11}$$

means that (4.9) holds with some sufficiently small $\zeta > 0$. Hence, as long as (4.10) and (4.11) hold, one can derive that the null solution of (4.7) is h-stable in the mean square sense.

Example 4.3 Lastly, we intend to investigate the following distributed delay equation for $t \geq 0$:

$$dy(t) = \left(-c(t)y(t) + \int_{-r}^{0} y(t+\vartheta)\,d\bigl[\kappa(\vartheta)\bigr]\right)dt - G(t,E_t)x(t)\,dE_t + H(t,E_t)x(t)\,dB_{E_t}, \tag{4.12}$$

where $\kappa(t)$ has bounded variation on $[-r,0]$ and $c(t)$ is a continuous function.

Denote $\zeta(t) := c(t) - \mathrm{Var}_{[-r,0]}\kappa(\cdot)$, $t \geq 0$. According to [26], the null solution of (4.12) is asymptotically stable in mean square provided that for all $t, s > 0$,

$$-2G(t,s) + H^2(t,s) \leq 0 \tag{4.13}$$

and

$$\zeta := \inf_{u \geq 0} \zeta(u) > 0. \tag{4.14}$$

In fact, one can also be certain that the null solution of (4.12) is h-stable in mean square if (4.13) and (4.14) hold. Let

$$f(t,\varphi) := -c(t)\varphi(0) + \int_{-r}^{0} \varphi(\vartheta)\,d\bigl[\kappa(\vartheta)\bigr],$$

for $t \geq 0$, $\varphi \in C([-r,0];\mathbb{R})$. Define $V(u) := \mathrm{Var}_{[-r,u]}\kappa(\cdot)$, $u \in [-r,0]$. Then $V(u)$ is increasing on $[-r,0]$. We obtain by the properties of the Riemann–Stieltjes integral

$$\left|\int_{-r}^{0} \varphi(0)\varphi(\vartheta)\, d\big[\kappa(\vartheta)\big]\right| \leq \int_{-r}^{0} \big|\varphi(0)\varphi(\vartheta)\big|\, d\big[V(\vartheta)\big].$$

Thus,

$$\begin{aligned}\varphi(0)f(t,\varphi) &\leq -c(t)\varphi^2(0) + \int_{-r}^{0} \big|\varphi(0)\varphi(\vartheta)\big|\, d\big[V(\vartheta)\big] \\ &\leq \left(-c(t) + \frac{1}{2}\int_{-r}^{0} d\big[V(\vartheta)\big]\right)\varphi^2(0) + \frac{1}{2}\int_{-r}^{0} \varphi^2(\vartheta)\, d\big[V(\vartheta)\big].\end{aligned}$$

According to Theorem 3.2, the null solution of (4.12) is η-stable in mean square if (4.13) is satisfied and there is a positive constant $\delta > 0$ such that for all $t \geq 0$,

$$-c(t) + \frac{1}{2}\int_{-r}^{0} d\big[V(\vartheta)\big] + \frac{1}{2}\int_{-r}^{0} \eta^{\delta}(-\vartheta)\, d\big[V(\vartheta)\big] \leq -\delta. \tag{4.15}$$

It follows from (4.14) that for each $t \in \mathbb{R}_+$,

$$-c(t) + V(0) \leq -\zeta.$$

Setting $\delta \in (0, \frac{\zeta}{2})$ sufficiently small, we can immediately derive that $\frac{1}{2}(\eta^{\delta}(r) - 1)V(0) < \frac{\zeta}{2}$, and, for each $t \in \mathbb{R}_+$, we get

$$-c(t) + \frac{1}{2}V(0) + \frac{1}{2}\eta^{\delta}(r)V(0) \leq -\frac{\zeta}{2} \leq -\delta.$$

Noticing that $V(t)$ is increasing, we can immediately know that $\int_{-r}^{0} \eta^{\delta}(-\vartheta)\, d[V(\vartheta)] \leq \eta^{\delta}(r)V(0)$. So, one obtains, for any $t \in \mathbb{R}_+$,

$$-c(t) + \frac{1}{2}\int_{-r}^{0} d\big[V(\vartheta)\big] + \frac{1}{2}\int_{-r}^{0} \eta^{\delta}(-\vartheta)\, d\big[V(\vartheta)\big] \leq -c(t) + \frac{1}{2}V(0) + \frac{1}{2}\eta^{\delta}(r)V(0) \leq -\delta.$$

5 Conclusion

In this paper, by using the time-changed Itô formula and proof by contradiction, we gained some new criteria of the η-stability in mean square for the stochastic functional differential equation driven by time-changed Brownian motion. Three concrete examples were given to illustrate the validity of our main conclusions. Hopefully, in the future, we can continue our study of the η-stability in mean square for other special stochastic equations.

Author contributions

Zhi Li: Writing Original draft preparation. Yaru Zhang: Resion Yue Wang: Stability. Liping Xu: is the corresponding author is responsible for ensuring that the descriptions are accurate and agreed by all authors. Xianping He: Writing- Reviewing and Editing.

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Not applicable to this paper.

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