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On an m-dimensional system of quantum inclusions by a new computational approach and heatmap

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Abstract

Recent research indicates the need for improved models of physical phenomena with multiple shocks. One of the newest methods is to use differential inclusions instead of differential equations. In this work, we intend to investigate the existence of solutions for an m-dimensional system of quantum differential inclusions. To ensure the existence of the solution of inclusions, researchers typically rely on the Arzela–Ascoli and Nadler's fixed point theorems. However, we have taken a different approach and utilized the endpoint technique of the fixed point theory to guarantee the solution's existence. This sets us apart from other researchers who have used different methods. For a better understanding of the issue and validation of the results, we presented numerical algorithms, tables, and some figures. The paper ends with an example.

Mathematics Subject Classification: 34A08; 34A40; 34D20; 33E30

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1 Introduction

It can be said that the study and extension of fractional calculus (FC) is pioneering the research in the field of nonlinear analysis in the 21st century. In other words, there is a general consensus among researchers to use fractional differential equations to model different natural and physical phenomena. Of course, this is not only due to the capabilities of fractional operators but also experimental data, and evidence confirms the capabilities of these operators in better interpretation of problems in engineering, biology, physics, etc. [1]. FC is a branch of mathematics in which derivative and integral operators are defined for arbitrary fractional orders. This fractional order causes the most significant feature of fractional operators, i.e., nonlocality. A detailed report on the properties of these operators can be found in the books of Podlubny and Kilbas [2, 3]. Viscosity, heat flow, and hereditary polarization in dielectrics are among the results of physicists' approach to the issue of nonlocality [4]. However, other areas have different approaches to this issue. Mathematicians also focused on generalizing, expanding, and introducing new fractional operators for use in modeling various phenomena. The most famous of these are the Riemann–Liouville, Caputo, Hadamard, Caputo–Hadamard, q-Caputo, ψ-Caputo, Atangana–Baleanu (AB), and Hilfer operators. For the latest results of these collaborative efforts, see [5–17]. Regardless of the type of fractional operator used, it is a more powerful model that has higher interpretation accuracy and less error, so the use of computers and software packages in presenting models has been of interest to researchers. Therefore, it is momentous to work in a field where it is possible to use computer algorithms for complex calculations. In the sequel, we also provide this space with the help of quantum calculus and time scales.

In 1910, Frank Hilton Jackson laid the foundations of quantum calculus (QC) by introducing two new operators, q- and h-derivatives [18, 19]. In his definition of a derivative, he removed the concept of a limit, and this caused a discrete space to be prepared for the analysis of problems. The characteristics of these two operators have been investigated in detail by Kac and Cheung in [20]. The q-derivative was more popular than the other, and it did not take long for it to be noticed by researchers in the fields of mathematics and physics. In recent years, the quantum derivative operator has been used due to the simultaneous use of FC capabilities and the possibility of providing algorithms for calculations in fractional modeling. For example, the most recent results can be seen in [21–24].

It is clear that finding the solution and its exclusivity for inclusions is a fundamental issue, that is why the theoretical investigation and knowledge of the properties of multifunctions is momentous. Here the fixed point theory with the help of inequalities plays a significant role in finding the solutions. The most recent results related to the fixed point of set-valued mappings are: a fixed point theorem for set-valued quasicontractions in b-metric spaces [25], existence of fixed points of generalized contractive multivalued mappings of integral type [26], the generic existence and approximation of fixed points [27], existence principles for integral inclusions [28], endpoint of closed set-valued contractions [29], and endpoint properties of multifunctions [30]. Various researchers have studied initial and boundary value problems related to multifunctions. These multivalued problems (inclusions) are present in the mathematical modeling of diverse issues in economics, optimal control, and other fields [31–33]. The growing interest in fractional-order inclusion problems is apparent in recent works [34–39]. For the first time in 2013, Ahmad and Ntouyas raised the issue of quantum differential inclusion [40]. They examined the existence of solutions for nonlinear fractional q-difference inclusions as follows:

$$\begin{cases} {}^{C}\mathcal{D}_q^{\eta}\mathbf{z}(\kappa) \in \mathcal{T}(\kappa, \mathbf{z}(\kappa)), \quad \kappa \in \mathcal{K} = [0,1], \\ a_1\mathbf{z}(0) - b_1\mathcal{D}_q\mathbf{z}(0) = c_1\mathbf{z}(r_1), \qquad a_2\mathbf{z}(0) - b_2\mathcal{D}_q\mathbf{z}(0) = c_2\mathbf{z}(r_2), \end{cases}$$

where ${}^{C}\mathcal{D}_q^{\eta}$ is the fractional q-derivative of the Caputo type, $\mathcal{T} : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ is a multifunction, $P(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, and $a_i, b_i, c_i, r_i \in \mathbb{R}$ $(i = 1, 2)$. A year later, Ahmed and his colleagues in [41] investigated the existence of solutions for fractional q-integro-difference inclusions with fractional q-integral boundary conditions as follows:

$$\begin{cases} {}^{C}\mathcal{D}_q^{\beta}({}^{C}\mathcal{D}_q^{\gamma} + \lambda)\mathbf{z}(\kappa) \in A\mathcal{T}(\kappa, \mathbf{z}(\kappa)) + BI_q^{\delta}\mathcal{F}(\kappa, \mathbf{z}(\kappa)), \quad \kappa \in \mathcal{K} = [0,1], \\ \mathbf{z}(0) = aI_q^{\alpha-1}\mathbf{z}(\eta), \qquad \mathbf{z}(1) = bI_q^{\alpha-1}\mathbf{z}(\sigma), \quad \alpha > 2, 0 < \eta, \sigma < 1, \end{cases}$$

where ${}^{C}\mathcal{D}_q^{\beta}$ is the fractional q-derivative of the Caputo type and $0 < \beta \leq 1$, $0 < \gamma \leq 1$, $0 < \delta < 1$, $\mathcal{T}, \mathcal{F} : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ are multifunctions, $a, b, A, B, \alpha, \sigma \in \mathbb{R}$, and

$$I_q^{\alpha}\mathbf{z}(\rho) = \int_0^{\rho} \frac{(\rho - qs)^{\alpha-1}}{\Gamma_q(\alpha)}\mathbf{z}(s)\,\mathrm{d}_q s \quad (\rho = \eta, \sigma).$$

After the aforementioned works (i.e., [40, 41]), further studies were conducted on quantum inclusions. For example, in [42] a fractional hybrid q-difference inclusion was examined. In [43], α–ψ-contraction and solutions of a q-fractional differential inclusion were explored. The existence of analytical and numerical results for a fractional q-differential inclusion with double integral boundary conditions was studied in [44]. Additionally, a computational method for investigating a quantum integro-differential inclusion was explored in [45].

To the best of our knowledge, the endpoint property has not been used to investigate the existence of the solution for an m-dimensional inclusion system, and this is part of the novelty of our work. Therefore, considering the topics discussed above and getting motivation from previous works, we want here to examine the existence of a solution for the following fractional quantum integro-differential inclusion problem:

$$\begin{cases} {}^{C}\mathcal{D}_q^{\eta_1}\mathbf{z}_1(\kappa) \in \mathcal{T}_1(\kappa, \mathbf{z}_1(\kappa), \mathbf{z}_1'(\kappa), {}^{C}\mathcal{D}_q^{\sigma_1}\mathbf{z}_1(\kappa), \mathcal{I}_q^{\delta_1}\mathbf{z}_1(\kappa)), \\ {}^{C}\mathcal{D}_q^{\eta_2}\mathbf{z}_2(\kappa) \in \mathcal{T}_2(\kappa, \mathbf{z}_2(\kappa), \mathbf{z}_2'(\kappa), {}^{C}\mathcal{D}_q^{\sigma_2}\mathbf{z}_2(\kappa), \mathcal{I}_q^{\delta_2}\mathbf{z}_2(\kappa)), \\ \vdots \\ {}^{C}\mathcal{D}_q^{\eta_m}\mathbf{z}_m(\kappa) \in \mathcal{T}_m(\kappa, \mathbf{z}_m(\kappa), \mathbf{z}_m'(\kappa), {}^{C}\mathcal{D}_q^{\sigma_m}\mathbf{z}_m(\kappa), \mathcal{I}_q^{\delta_m}\mathbf{z}_m(\kappa)). \end{cases} \tag{1}$$

under double integral boundary conditions

$$\begin{cases} \mathbf{z}_j(0) + \mathbf{z}_j'(0) + \mathcal{I}_q^{\delta_j}\mathbf{z}_j(0) = \int_0^{\theta} \mathbf{z}(p)\mathrm{d}p, \\ \mathbf{z}_j(1) + \mathbf{z}_j'(1) + {}^{C}\mathcal{D}_q^{\sigma_j}\mathbf{z}_j(1) = \int_0^{\lambda} \mathbf{z}(p)\,\mathrm{d}p, \end{cases} \tag{2}$$

where in our problem $\kappa \in \mathcal{K} = [0,1]$, ${}^{C}\mathcal{D}_q^{\eta}$ is the Caputo quantum operator of fractional order $1 \leq \eta_j < 2$, and $\theta, \lambda, \sigma_j, \delta_j \in (0,1)$ are such that $\mathcal{T}_j : \mathcal{K} \times \mathbb{R}^4 \to \mathcal{P}(\mathbb{R})$, for $j = 1, \dots, m$, is a multifunction where $\mathcal{P}(\mathbb{R})$ denotes the set of all subsets of real numbers. The dominant approach of researchers to guarantee the existence of the solution of inclusions is to use Arzela–Ascoli and Nadler's fixed point theorems. But these two methods, in addition to being long and complicated, are based on the convexity of the inclusion in the problem. But in the endpoint method, we do not have the limitation of convexity, so we will reach the desired result with this method. Note that we will continue to do all our calculations on the time scales, namely $TS_{\kappa_0} = \{\kappa_0, \kappa_0 q, \kappa_0 q^2, \dots\} \cup \{0\}$, where $\kappa_0 \in \mathbb{R}$ and $q \in (0,1)$.

2 Basic preliminaries

Definition 2.1 ([18]) Assume that $\nu, p \in \mathbb{R}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then the quantum-analogue of ν and power function $(\nu - p)^n$ are respectively defined as follows:

$$[\nu]_q = \frac{1 - q^{\nu}}{1 - q} = 1 + q + \cdots + q^{\nu-1}$$

Algorithm 1 The proposed procedure to calculate $\Gamma_q(\nu)$

```
function quantum gamma = qG(q,v,r)
t = 1;
for j = 0 : r
t = t * (1 - q^(j+1))/(1 - q^(v+j));
end
qG = t/(1 - q)^(v-1);
end
```

and

$$\begin{cases} (\nu - p)_q^{(n)} = \prod_{j=0}^{n-1}(\nu - pq^i) & \text{for } n \geq 1, \\ (\nu - p)_q^{(0)} = 1. \end{cases}$$

The above power function for any real number λ is expressed as follows:

$$(\nu - p)_q^{(\lambda)} = \nu^\lambda \prod_{n=0}^{\infty} \frac{1 - (\frac{p}{\nu})q^n}{1 - (\frac{p}{\nu})q^{\lambda+n}}, \quad \nu \neq 0.$$

It is obvious that if $p = 0$ then $\nu^{(\lambda)} = \nu^\lambda$.

Definition 2.2 ([19]) Let $\nu \in \mathbb{R} - \{0, -1, -2, \dots\}$, then the quantum gamma function is formulated as follows:

$$\Gamma_q(\nu) = \frac{(1-q)^{(\nu-1)}}{(1-q)^{\nu-1}}.$$

Also, it is worth mentioning that $\Gamma_q(\nu + 1) = [\nu]_q \Gamma_q(\nu)$ holds true.

In the following, we present an Algorithm 1 for calculating the quantum gamma function, which we illustrated in Tables 1 and 2 for some values of q.

Definition 2.3 ([46]) The quantum derivative of a continuous function $\mathbf{z}(\kappa)$ is defined as follows:

$$(\mathcal{D}_q \mathbf{z})(\kappa) = \frac{\mathbf{z}(\kappa) - \mathbf{z}(q\kappa)}{(1-q)\kappa},$$

in addition, $(\mathcal{D}_q \mathbf{z})(0) = \lim_{\kappa \to 0} (\mathcal{D}_q \mathbf{z})(\kappa)$. Furthermore, for all $n \in \mathbb{N}$, the relation $(\mathcal{D}_q^n \mathbf{z})(\kappa) = \mathcal{D}_q(\mathcal{D}_q^{n-1} \mathbf{z})(\kappa)$ holds true.

Definition 2.4 ([46]) The Jackson integral of $\mathbf{z} \in \mathcal{C}([0,a], \mathbb{R})$ is given as follows:

$$\mathcal{I}_q \mathbf{z}(\kappa) = \int_0^\kappa \mathbf{z}(p)\, \mathrm{d}_q p = \kappa(1-q) \sum_{j=0}^{\infty} q^j \mathbf{z}(q^j \kappa), \quad \kappa \in [0,a],$$

provided the right-hand side is absolutely convergent. Furthermore, for all $n \in \mathbb{N}$, the relation $\mathcal{I}_q^n \mathbf{z}(\kappa) = \mathcal{I}(\mathcal{I}_q^{n-1} \mathbf{z}(\kappa))$ holds true.

Table 1 Numerical results for $\Gamma_q(2.375)$ for different value of q

r	$q = 0.13$	$q = 0.25$	$q = 0.39$	$q = 0.58$	$q = 0.72$	$q = 0.89$
	$v = 2.25$					
1	1.0451	1.0948	1.1924	1.5053	2.1388	6.0508
2	1.0429	1.0802	1.1402	1.3348	1.7582	4.4696
3	<u>1.0426</u>	1.0766	1.1209	1.2506	1.5511	3.5776
4	1.0426	1.0757	1.1135	1.2060	1.4267	3.0134
5	1.0426	1.0755	1.1107	1.1813	1.3474	2.6289
6	1.0426	<u>1.0754</u>	1.1096	1.1674	1.2949	2.3526
7	1.0426	1.0754	1.1091	1.1595	1.2593	2.1463
8	1.0426	1.0754	1.1090	1.1550	1.2347	1.9876
9	1.0426	1.0754	<u>1.1089</u>	1.1523	1.2175	1.8626
..	…	…	…	…	…	…
18	1.0426	1.0754	1.1138	<u>1.1487</u>	1.1772	1.3821
19	1.0426	1.0754	1.1105	1.1487	1.1766	1.3604
...	…	…	…	…	…	…
29	1.0426	1.0754	1.1031	1.1487	<u>1.1751</u>	1.2496
30	1.0426	1.0754	1.1031	1.1751	1.1751	1.2445
...	…	…	…	…	…	…
42	1.0426	1.0754	1.1031	1.1487	1.1751	1.2142
43	1.0426	1.0754	1.1031	1.1487	1.1751	1.2131
..	…	…	…	…	...	…
85	1.0426	1.0754	1.1031	1.1487	1.1751	1.2046
86	1.0426	1.0754	1.1031	1.1487	1.1751	<u>1.2045</u>

Table 2 Numerical results for $\Gamma_q(1.25)$ for different value of q

r	$q = 0.2$	$q = 0.35$	$q = 0.59$	$q = 0.7$	$q = 0.83$	$q = 0.98$
	$v = 1.25$					
1	0.9632	0.9597	0.9953	1.0415	1.1575	1.9005
2	0.9606	0.9498	0.9645	0.9971	1.0912	1.7587
3	0.9601	0.9465	0.9484	0.9709	1.0481	1.6593
4	<u>0.9600</u>	0.9454	0.9394	0.9545	1.0180	1.5843
5	0.9600	0.9450	0.9343	0.9438	0.9960	1.5247
6	0.9600	0.9448	0.9314	0.9366	0.9794	1.4757
7	0.9600	0.9448	0.9297	0.9317	0.9666	1.4345
8	0.9600	<u>0.9447</u>	0.9287	0.9284	0.9566	1.3992
9	0.9600	0.9447	0.9281	0.9261	0.9487	1.3684
...	…	…	…	…	…	…
13	0.9600	0.9447	0.9274	0.9221	0.9297	1.2756
14	0.9600	0.9447	<u>0.9273</u>	0.9217	0.9270	1.2577
...	…	…	…	…	…	…
29	0.9600	0.9447	0.9273	0.9209	0.9149	1.1008
30	0.9600	0.9447	0.9273	<u>0.9208</u>	0.9148	1.0945
..	…	…	…	…	...	…
47	0.9600	0.9447	0.9273	0.9208	0.9142	1.0201
48	0.9600	0.9447	0.9273	0.9208	<u>0.9141</u>	1.0171
...	…	…	…	…	…	…
385	0.9600	0.9447	0.9273	0.9208	0.9141	0.9074
386	0.9600	0.9447	0.9273	0.9208	0.9141	<u>0.9073</u>

Remark 2.5 ([46]) For a continuous function $\mathbf{z}$, the following relations hold true at $\kappa = 0$:

$$\begin{cases} \mathcal{I}_q(\mathcal{D}_q\mathbf{z}(\kappa)) = \mathbf{z}(\kappa) - \mathbf{z}(0), \\ \mathcal{D}_q(\mathcal{I}_q\mathbf{z}(\kappa)) = \mathbf{z}(\kappa), \quad \text{for all } \kappa. \end{cases}$$

Definition 2.6 ([47]) Suppose that $\mathbf{z}(\kappa) : [0, \infty] \to \mathbb{R}$ is a continuous function, then its fractional Riemann–Liouville quantum integral and its fractional Caputo quantum derivative are respectively expressed by

$$\mathcal{I}_q^{\eta}\mathbf{z}(\kappa) = \frac{1}{\Gamma_q(\eta)}\int_0^{\kappa}(\kappa - qp)^{\eta-1}\mathbf{z}(p)\,\mathrm{d}_q p$$

and

$${}^{c}\mathcal{D}^{\eta}\mathbf{z}(\kappa) = \frac{1}{\Gamma_q(n-\eta)}\int_0^{\kappa}(\kappa - qp)^{n-\eta-1}\mathcal{D}_q^{n}\mathbf{z}(p)\,\mathrm{d}_q p, \quad n = [\eta] + 1.$$

Lemma 2.7 ([48]) *Assume that $n = [\eta] + 1$, then the following relation holds true*:

$${}^{C}\mathcal{I}_q^{\eta}{}^{C}\mathcal{D}_q^{\eta}\mathbf{z}(\kappa) = \mathbf{z}(\kappa) - \sum_{j=0}^{n-1}\frac{\kappa^j}{\Gamma_q(j+1)}\big(\mathcal{D}_q^{j}\mathbf{z}\big)(0),$$

which is deduced from the general solution for ${}^{C}\mathcal{D}_q^{\eta}\mathbf{z}(\kappa) = 0$, expressed by

$$\mathbf{z}(\kappa) = t_0 + t_1\kappa + t_2\kappa^2 + \cdots + t_{n-1}\kappa^{n-1},$$

where $t_0, \dots, t_{n-1} \in \mathbb{R}$.

Notation 2.8 Here, we introduce some symbols that are used in the topology of the used space. Let $(\mathcal{G}, d_{\mathcal{G}})$ be a metric space, also suppose that $\mathcal{P}(\mathcal{G})$ and $2^{\mathcal{G}}$ represent the set of all subsets of $\mathcal{G}$ and the set of all nonempty subsets of $\mathcal{G}$, respectively. In the sequel, we use the symbols $\mathcal{P}_{cl}(\mathcal{G})$, $\mathcal{P}_{bd}(\mathcal{G})$, $\mathcal{P}_{cx}(\mathcal{G})$, and $\mathcal{P}_{ct}(\mathcal{G})$, respectively, to denote the classes of all closed, bounded, convex, and compact subsets of $\mathcal{G}$.

Definition 2.9 ([30]) A fixed point of a multifunction (set-valued map) such as $\mathcal{E} : \mathcal{G} \to 2^{\mathcal{G}}$ is an element $\kappa \in \mathcal{K}$ such that $\kappa \in \mathcal{E}(\kappa)$. Moreover, if we have $\mathcal{E}(\kappa) = \{\kappa\}$, then this element, namely κ, is called an endpoint of $\mathcal{E}$.

Definition 2.10 ([30]) Let $(\mathcal{G}, d_{\mathcal{G}})$ be a metric space and $\mathcal{E} : \mathcal{G} \to 2^{\mathcal{G}}$ a multifunction. Then $\mathcal{E}$ has an approximative property if $\inf_{\kappa\in\mathcal{G}}\sup_{r\in\mathcal{E}(\kappa)} \mathrm{d}_{\mathcal{G}}(\kappa, r) = 0$.

Definition 2.11 ([49]) If $(\mathcal{G}, d_{\mathcal{G}})$ is a metric space, then the Pompeiu–Hausdorff metric, namely $\mathcal{HM} : 2^{\mathcal{G}} \times 2^{\mathcal{G}} \to [0, \infty]$, is defined as follows:

$$\mathcal{HM}(\mathcal{Z}, \mathcal{W}) = \max\Big\{\sup_{z\in\mathcal{Z}} \mathrm{d}_{\mathcal{G}}(z, \mathcal{W}), \sup_{w\in\mathcal{W}} \mathrm{d}_{\mathcal{G}}(\mathcal{Z}, w)\Big\},$$

where $\mathcal{HM}(\mathcal{Z}, w) = \inf_{z\in\mathcal{Z}} \mathrm{d}_{\mathcal{G}}(z, w)$. Then $(\mathcal{P}_{bd,cl}(\mathcal{G}), \mathcal{HM})$ and $(\mathcal{P}_{cl}(\mathcal{G}), \mathcal{HM})$ represent a metric space and a generalized metric space, respectively.

Definition 2.12 ([49]) Assume that $\mathcal{V} = \mathcal{C}(\mathcal{K}, \mathbb{R})$. Then define the space

$$\mathcal{G}_j = \big\{\mathbf{z}_j(\kappa) : \mathbf{z}_j(\kappa), \mathbf{z}_j'(\kappa), {}^{C}\mathcal{D}_q^{\sigma_j}\mathbf{z}_j(\kappa), \mathcal{I}_q^{\delta_j}\mathbf{z}_j(\kappa) \in \mathcal{V}\big\}$$

equipped with the norm

$$\|\mathbf{z}\|_j = \sup_{\kappa\in\mathcal{K}}\big|\mathbf{z}(\kappa)\big| + \sup_{\kappa\in\mathcal{K}}\big|\mathbf{z}_j'(\kappa)\big| + \sup_{\kappa\in\mathcal{K}}\big|{}^{C}\mathcal{D}_q^{\sigma_j}\mathbf{z}_j(\kappa)\big| + \sup_{\kappa\in\mathcal{K}}\big|\mathcal{I}_q^{\delta_j}\mathbf{z}_j(\kappa)\big|,$$

for all $j = 1, \dots, m$. Therefore the product space $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, equipped with the norm $\|(z_1, \dots, z_m)\| = \sum_{j=1}^{m} \|\mathbf{z}\|_j$, namely $(\mathcal{G}, \|\cdot\|)$, is a Banach space.

Definition 2.13 Let $\boldsymbol{w} \in \mathcal{G}$, then for all $\kappa \in \mathcal{K}$, define the set of selection of $\mathcal{S}^*$ as follows:

$$\mathcal{S}^*_{\mathcal{T}_j,\mathbf{z}} = \left\{ \mathfrak{g} \in \mathcal{L}^1(\mathcal{K}) : \mathfrak{g}(\kappa) \in \mathcal{T}_j\big(\kappa, \mathbf{z}_j(\kappa), \mathbf{z}'_j(\kappa), {}^C\mathcal{D}_q^{\sigma_j}\mathbf{z}_j(\kappa), \mathcal{I}_q^{\delta_j}\mathbf{z}_j(\kappa)\big)\right\}.$$

If $\dim(\mathcal{G}) < \infty$, then the above selection is nonempty, which is proved in [49].

In 2010, Amini–Harandi introduced the endpoint technique, which plays an essential role in proving our main result [30]. Now we will present it here.

Lemma 2.14 ([30]) *Suppose that $(\mathcal{G}, d_{\mathcal{G}})$ is a complete metric space, also consider two maps Ψ and $\mathcal{E}$ with the following properties*:

- $\Psi : [0,\infty) \to [0,\infty)$ *is upper semicontinuous (usc), i.e., $\forall \kappa > 0$ we have $\Psi(\kappa) < \kappa$ and* $\liminf_{\kappa\to\infty}(\kappa - \Psi(\kappa)) > 0$.
- $\forall w, z \in \mathcal{G}$, *for the set-valued map $\mathcal{E} : \mathcal{G} \to \mathcal{P}_{cl,bd}(\mathcal{G})$, the inequality* $\mathcal{HM}(\mathcal{E}(w), \mathcal{E}(r)) \le \Psi(d_{\mathcal{G}}(w, r))$ *holds true.*

Then the set-valued map $\mathcal{E}$ has a unique endpoint iff $\mathcal{E}$ has an approximative endpoint property.

3 Main results

Now we have provided the prerequisites necessary to express our main results, and only one lemma remains, which we prove here.

Lemma 3.1 *The unique solution to the fractional q-differential problem ${}^c\mathcal{D}_q^{\eta}\mathbf{z}(\kappa) = g(\kappa)$ under boundary conditions* (2) *is expressed by*

$$\begin{aligned}
\mathbf{z}(\kappa) &= \frac{1}{\Gamma_q(\eta)} \int_0^{\kappa} (\kappa - qp)^{\eta-1} g(p)\,\mathrm{d}_q p \\
&\quad + \frac{\mathbf{a}}{\mathbf{b}\Gamma_q(\eta)} \int_0^{\theta} \int_0^{p} (p - qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&\quad + \frac{(\theta^2 - 2)}{2\mathbf{b}\Gamma_q(\eta)} \int_0^{\lambda} \int_0^{p} (p - qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&\quad + \frac{2 - \theta^2}{2\mathbf{b}\Gamma_q(\eta)} \int_0^{1} (1 - qp)^{\eta-1} g(p)\,\mathrm{d}_q p \\
&\quad + \frac{2 - \theta^2}{2\mathbf{b}\Gamma_q(\eta - \sigma)} \int_0^{1} (1 - qp)^{\eta-\sigma-1} g(p)\,\mathrm{d}_q p \\
&\quad + \frac{2 - \theta^2}{2\mathbf{b}\Gamma_q(\eta - 1)} \int_0^{1} (1 - qp)^{\eta-2} g(p)\,\mathrm{d}_q p \\
&\quad + \frac{(2\mathbf{b} + 2\mathbf{a}(\theta - 1))\kappa}{\mathbf{b}(2 - \theta^2)\Gamma_q(\eta)} \int_0^{\theta} \int_0^{p} (p - qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&\quad + \frac{(1 - \theta)\kappa}{\mathbf{b}\Gamma_q(\eta)} \int_0^{\lambda} \int_0^{p} (p - qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&\quad + \frac{(\theta - 1)\kappa}{\mathbf{b}\Gamma_q(\eta)} \int_0^{1} (1 - qp)^{\eta-1} g(p)\,\mathrm{d}_q p
\end{aligned}$$

$$+\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}g(p)\,\mathrm{d}_qp$$
$$+\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta-1)}\int_0^1(1-qp)^{\eta-2}g(p)\,\mathrm{d}_qp$$

such that $\eta\in[1,2)$, $g(\kappa)\in\mathcal{V}$, *and*

$$\mathbf{a}=2+\frac{1}{\Gamma_q(2-\sigma)}-\frac{\lambda^2}{2},\qquad \mathbf{b}=\mathbf{a}(1-\theta)+\left(\frac{\theta^2}{2}-1\right)(1-\lambda).$$

Proof In view of Lemma 2.7, the problem ${}^C\mathcal{D}_q^\eta\mathbf{z}(\kappa)=g(\kappa)$ has a unique solution which is given by

$$\mathbf{z}(\kappa)=\mathcal{I}_q^\eta g(\kappa)+t_0+t_1\kappa=\frac{1}{\Gamma_q(\eta)}\int_0^\kappa(\kappa-qp)^{\eta-1}g(p)\,\mathrm{d}_qp+t_0+t_1\kappa,\tag{3}$$

with $t_0,t_1\in\mathbb{R}$. To apply the boundary conditions, it is necessary to calculate the first-order derivative, namely $\mathbf{z}'(\kappa)=t_1+\mathcal{I}_q^{\eta-1}g(\kappa)$. Now with regard to boundary conditions (2), we get $\mathbf{z}(0)+\mathbf{z}'(0)+\mathcal{I}_q^\delta\mathbf{z}(0)=t_0+t_1$, and

$$\begin{aligned}\mathbf{z}_j(1)+\mathbf{z}'(1)+{}^C\mathcal{D}_q^\sigma\mathbf{z}(1)&=\frac{1}{\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}g(p)\,\mathrm{d}_qp\\&\quad+\frac{1}{\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}g(p)\,\mathrm{d}_qp\\&\quad+\frac{1}{\Gamma_q(\eta-1)}\int_0^1(1-qp)^{\eta-2}g(p)\,\mathrm{d}_qp\\&\quad+t_0+t_1\left(2+\frac{1}{\Gamma_q(2-\sigma)}\right).\end{aligned}$$

By performing simple calculations, we obtain

$$t_0(1-\theta)+t_1\left(1-\frac{\theta^2}{2}\right)=\frac{1}{\Gamma_q(\eta)}\int_0^\theta\int_0^p(p-qm)^{\eta-1}g(m)\,\mathrm{d}_qm\,\mathrm{d}p$$

and

$$\begin{aligned}t_0(1-\lambda)+t_1\left(2+\frac{1}{\Gamma_q(2-\sigma)}-\frac{\lambda^2}{2}\right)&=\frac{1}{\Gamma_q(\eta)}\int_0^\lambda\int_0^p(p-qm)^{\eta-1}g(m)\,\mathrm{d}_qm\,\mathrm{d}p\\&\quad-\frac{1}{\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}g(p)\,\mathrm{d}_qp\\&\quad-\frac{1}{\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}g(p)\,\mathrm{d}_qp\\&\quad-\frac{1}{\Gamma_q(\eta-1)}\int_0^1(1-qp)^{\eta-2}g(p)\,\mathrm{d}_qp.\end{aligned}$$

Just for simplicity in computations, we set

$$\mathbf{a}=2+\frac{1}{\Gamma_q(2-\sigma)}-\frac{\lambda^2}{2}\quad\text{and}\quad\mathbf{b}=\mathbf{a}(1-\theta)+\left(\frac{\theta^2}{2}-1\right)(1-\lambda).$$

Thus, the values of t_0 and t_1 will be as follows:

$$\begin{aligned}
t_0 &= \frac{\mathbf{a}}{\mathbf{b}\Gamma_q(\eta)} \int_0^\theta \int_0^p (p-qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&\quad + \frac{(\theta^2-2)}{2\mathbf{b}\Gamma_q(\eta)} \int_0^\lambda \int_0^p (p-qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&\quad + \frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta)} \int_0^1 (1-qp)^{\eta-1} g(p)\,\mathrm{d}_q p \\
&\quad + \frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta-\sigma_j)} \int_0^1 (1-qp)^{\eta-\sigma-1} g(p)\,\mathrm{d}_q p \\
&\quad + \frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta-1)} \int_0^1 (1-qp)^{\eta-2} g(p)\,\mathrm{d}_q p,
\end{aligned}$$

and

$$\begin{aligned}
t_1 &= \frac{2\mathbf{b}+2\mathbf{a}(\theta-1)}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta)} \int_0^\theta \int_0^p (p-qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&\quad + \frac{(1-\theta)}{\mathbf{b}\Gamma_q(\eta)} \int_0^\lambda \int_0^p (p-qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&\quad + \frac{\theta-1}{\mathbf{b}\Gamma_q(\eta)} \int_0^1 (1-qp)^{\eta-1} g(p)\,\mathrm{d}_q p \\
&\quad + \frac{\theta-1}{\mathbf{b}\Gamma_q(\eta-\sigma_j)} \int_0^1 (1-qp)^{\eta-\sigma-1} g(p)\,\mathrm{d}_q p \\
&\quad + \frac{\theta-1}{\mathbf{b}\Gamma_q(\eta-1)} \int_0^1 (1-qp)^{\eta-2} g(p)\,\mathrm{d}_q p,
\end{aligned}$$

Placing coefficients t_0 and t_1 in equation (3) provides the desired result. □

In order to obtain the result in our inclusion problem, it is necessary to apply the following hypotheses:

($\mathcal{A}_1$) The multifunction $\mathcal{T}_j : \mathcal{K} \times \mathbb{R}^4 \to P_{cp}(\mathbb{R})$ is integrable and bounded for all $j = 1, \dots, m$, therefore $\mathcal{T}_j(\cdot, x, y, u, v) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable.

($\mathcal{A}_2$) For $\Psi : [0,\infty) \to [0,\infty)$, which is nondecreasing and usc, i.e., $\forall p > 0$ we have $\liminf_{p\to\infty}(p - \Psi(p)) > 0$ and $\Psi(p) < p$.

($\mathcal{A}_3$) For all $\kappa \in \mathcal{K}, j = 1, \dots, m$ and $s_i, r_i \in \mathbb{R}$, with $i = 1, \dots, 4$, there exist $\mathbf{\Omega}_j \in \mathcal{C}(\mathcal{K}, [0,\infty))$, where

$$\begin{aligned}
&\mathcal{HM}(\mathcal{T}_j(\kappa, s_1, s_2, s_3, s_4), \mathcal{T}_j(\kappa, r_1, r_2, r_3, r_4)) \\
&\quad \le \frac{1}{\mathfrak{A}_{\mathbf{1}_j} + \mathfrak{A}_{\mathbf{2}_j} + \mathfrak{A}_{\mathbf{3}_j} + \mathfrak{A}_{\mathbf{4}_j}} \mathbf{\Omega}_j(\kappa) \Psi\left(\sum_{i=1}^4 |s_i - r_i|\right),
\end{aligned}$$

such that

$$\mathfrak{A}_{\mathbf{1}_j} = \|\mathbf{\Omega}_j\| \left[\frac{1}{\Gamma_q(\eta_j+1)} + \frac{|\mathbf{a}|\theta^{\eta_j+1}}{|\mathbf{b}|\Gamma_q(\eta_j+2)} + \left| \frac{(\theta^2-2)\lambda^{\eta_j+1}}{2\mathbf{b}\Gamma_q(\eta_j+2)} \right| \right.$$

$$
\begin{aligned}
&+\left|\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j+1)}\right|+\left|\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\right|+\left|\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-\sigma_j+1)}\right|\\
&+\left|\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j)}\right|+\left|\frac{2\mathbf{b}+2\mathbf{a}(\theta-1)\theta^{\eta_j+1}}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j+2)}\right|+\left|\frac{(1-\theta)\lambda^{\eta_j+1}}{\mathbf{b}\Gamma_q(\eta_j+2)}\right|\\
&+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j+1)}\right|+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j+1)}\right|+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j)}\right|\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{A}_{2_j}=\|\boldsymbol{\Omega}_j\|\Bigg[&\frac{1}{\Gamma_q(\eta_j)}+\left|\frac{2\mathbf{b}+2\mathbf{a}(\theta-1)\theta^{\eta_j+1}}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j+2)}\right|+\left|\frac{(1-\theta)\lambda^{\eta_j+1}}{\mathbf{b}\Gamma_q(\eta_j+2)}\right|\\
&+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j+1)}\right|+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j+1)}\right|+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j)}\right|\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{A}_{3_j}=\|\boldsymbol{\Omega}_j\|\Bigg[&\frac{1}{\Gamma_q(\eta_j-\sigma_j+1)}+\left|\frac{2\mathbf{b}+2\mathbf{a}(\theta-1)}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j+2)\Gamma_q(2-\sigma_j)}\right|\\
&+\left|\frac{(1-\theta)}{\mathbf{b}\Gamma_q(\eta_j+2)\Gamma_q(2-\sigma_j)}\right|+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j+1)\Gamma_q(2-\sigma_j)}\right|\\
&+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j+1)\Gamma_q(2-\sigma_j)}\right|+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j)\Gamma_q(2-\sigma_j)}\right|\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{A}_{4_j}=\|\boldsymbol{\Omega}_j\|\Bigg[&\frac{1}{\Gamma_q(\eta_j+\delta_j+1)}+\left|\frac{2\mathbf{b}+2\mathbf{a}(\theta-1)}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j+2)\Gamma_q(2+\delta_j)}\right|\\
&+\left|\frac{(1-\theta)}{\mathbf{b}\Gamma_q(\eta_j+2)\Gamma_q(2+\delta_j)}\right|+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j+1)\Gamma_q(2+\delta_j)}\right|\\
&+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j+1)\Gamma_q(2+\delta_j)}\right|+\left|\frac{\theta-1}{\mathbf{b}\Gamma_q(\eta_j)\Gamma_q(2+\delta_j)}\right|\Bigg]
\end{aligned}
$$

Theorem 3.2 *Let the hypotheses $\mathcal{A}_1$–$\mathcal{A}_3$ hold true. If the set-valued map $\mathcal{E}:\mathcal{G}\to 2^{\mathcal{G}}$ has the approximative endpoint property, then the quantum inclusion system mentioned in* (1)–(2) *has a solution.*

Proof To show that our problem (1)–(2) has a solution, we go to find the endpoint of $\mathcal{E}:\mathcal{G}\to 2^{\mathcal{G}}$. This endpoint is the solution of our inclusion. We do this in two steps.

Step I. According to our assumptions, for all $(z_1,\dots,z_m)\in\mathcal{G}$, the map $\kappa\mapsto\mathcal{T}_j(\kappa,\mathbf{z}_j(\kappa),\mathbf{z}_j'(\kappa),{}^{C}\mathcal{D}_q^{\sigma_j}\mathbf{z}_j(\kappa),\mathcal{I}_q^{\delta_j}\mathbf{z}_j(\kappa))$ is measurable and closed-valued. Therefore, such a map has a nonempty measurable selection, i.e., $\mathcal{S}^*_{\mathcal{T}_j,\mathbf{z}}\neq\phi$.

Define the operator $\mathcal{E}:\mathcal{G}\to 2^{\mathcal{G}}$ by

$$
\mathcal{E}(z_1,\dots,z_m)=\begin{pmatrix}\mathcal{E}_1(z_1,\dots,z_m)\\ \mathcal{E}_2(z_1,\dots,z_m)\\ \vdots\\ \mathcal{E}_m(z_1,\dots,z_m)\end{pmatrix},
$$

which, for $\mathfrak{g}\in\mathcal{S}^*_{\mathcal{T}_j,(z_1,\dots,z_m)}$, reads as follows:

$$
\begin{aligned}
\mathcal{E}_j(z_1,z_2,\dots,z_m)=\Bigg\{\hbar\in\mathcal{G}_j:\hbar(\kappa)=&\frac{1}{\Gamma_q(\eta_j)}\int_0^{\kappa}(\kappa-qp)^{\eta_j-1}\mathfrak{g}(p)\,\mathrm{d}_qp\\
&+\frac{\mathbf{a}}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^{\theta}\int_0^{p}(p-qm)^{\eta_j-1}\mathfrak{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\theta^2-2)}{2\mathbf{b}\Gamma_q(\eta_j)}\int_0^\lambda\int_0^p(p-qm)^{\eta_j-1}\mathfrak{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p\\
&+\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j)}\int_0^1(1-qp)^{\eta_j-1}\mathfrak{g}(p)\,\mathrm{d}_qp\\
&+\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\int_0^1(1-qp)^{\eta_j-\sigma_j-1}\mathfrak{g}(p)\,\mathrm{d}_qp\\
&+\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-1)}\int_0^1(1-qp)^{\eta_j-2}\mathfrak{g}(p)\,\mathrm{d}_qp\\
&+\frac{(2\mathbf{b}+2\mathbf{a}(\theta-1))\kappa}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j)}\int_0^\theta\int_0^p(p-qm)^{\eta_j-1}\mathfrak{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p\\
&+\frac{(1-\theta)\kappa}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^\lambda\int_0^p(p-qm)^{\eta_j-1}\mathfrak{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p\\
&+\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^1(1-qp)^{\eta_j-1}\mathfrak{g}(p)\,\mathrm{d}_qp\\
&+\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\int_0^1(1-qp)^{\eta_j-\sigma_j-1}\mathfrak{g}(p)\,\mathrm{d}_qp\\
&+\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-1)}\int_0^1(1-qp)^{\eta_j-2}\mathfrak{g}(p)\,\mathrm{d}_qp\biggr\}.
\end{aligned}
$$

Now, assume that $\{(z_1^n,\dots,z_m^n)\}_{n\geq 1}$ is a sequence in $\mathcal{E}(z_1,\dots,z_m)$, which converges $(z_1^n,\dots,z_m^n)\to(z_1^0,\dots,z_m^0)$. Choose $(\mathfrak{g}_1^n,\dots,\mathfrak{g}_m^n)\in\mathcal{S}^*_{\mathcal{T}_1,(z_1^n,\dots,z_m^n)}\times\cdots\times\mathcal{S}^*_{\mathcal{T}_m,(z_1^n,\dots,z_m^n)}$, which, for all $\kappa\in\mathcal{K}$, $j=1\dots,m$, and $n\geq 1$, satisfies

$$
\begin{aligned}
z_j^n={}&\frac{1}{\Gamma_q(\eta_j)}\int_0^\kappa(\kappa-qp)^{\eta_j-1}\mathfrak{g}_j^n(p)\,\mathrm{d}_qp\\
&+\frac{\mathbf{a}}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^\theta\int_0^p(p-qm)^{\eta_j-1}\mathfrak{g}_j^n(m)\,\mathrm{d}_qm\,\mathrm{d}p\\
&+\frac{(\theta^2-2)}{2\mathbf{b}\Gamma_q(\eta_j)}\int_0^\lambda\int_0^p(p-qm)^{\eta_j-1}\mathfrak{g}_j^n(m)\,\mathrm{d}_qm\,\mathrm{d}p\\
&+\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j)}\int_0^1(1-qp)^{\eta_j-1}\mathfrak{g}_j^n(p)\,\mathrm{d}_qp\\
&+\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\int_0^1(1-qp)^{\eta_j-\sigma_j-1}\mathfrak{g}_j^n(p)\,\mathrm{d}_qp\\
&+\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-1)}\int_0^1(1-qp)^{\eta_j-2}\mathfrak{g}_j^n(p)\,\mathrm{d}_qp\\
&+\frac{(2\mathbf{b}+2\mathbf{a}(\theta-1))\kappa}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j)}\int_0^\theta\int_0^p(p-qm)^{\eta_j-1}\mathfrak{g}_j^n(m)\,\mathrm{d}_qm\,\mathrm{d}p\\
&+\frac{(1-\theta)\kappa}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^\lambda\int_0^p(p-qm)^{\eta_j-1}\mathfrak{g}_j^n(m)\,\mathrm{d}_qm\,\mathrm{d}p\\
&+\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^1(1-qp)^{\eta_j-1}\mathfrak{g}_j^n(p)\,\mathrm{d}_qp\\
&+\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\int_0^1(1-qp)^{\eta_j-\sigma_j-1}\mathfrak{g}_j^n(p)\,\mathrm{d}_qp
\end{aligned}
$$

$$+ \frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-1)}\int_0^1 (1-qp)^{\eta_j-2}\mathfrak{g}_j^n(p)\,\mathrm{d}_q p$$

The compactness of $\mathcal{T}_j$ for all $j=1,\dots,m$ implies that $\{\mathfrak{g}_j^n\}_{n\geq 1}$ has a subsequence (denoted again by $\{\mathfrak{g}_j^n\}_{n\geq 1}$), which converges to some $\mathfrak{g}_j^0 \in \mathcal{L}^1([0,1],\mathbb{R})$. It is easy to check that $\mathfrak{g}_j^0 \in \mathcal{S}^*_{\mathcal{T}_j,(z_1,\dots,z_m)}$, and for all $\kappa\in\mathcal{K}$, $j=1,\dots,m$,

$$\begin{aligned} z_j^n(\kappa)\to z_j^0(\kappa) ={}& \frac{1}{\Gamma_q(\eta_j)}\int_0^\kappa (\kappa-qp)^{\eta_j-1}\mathfrak{g}_j^0(p)\,\mathrm{d}_q p \\ &+ \frac{\mathbf{a}}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^\theta\int_0^p (p-qm)^{\eta_j-1}\mathfrak{g}_j^0(m)\,\mathrm{d}_q m\,\mathrm{d}p \\ &+ \frac{(\theta^2-2)}{2\mathbf{b}\Gamma_q(\eta_j)}\int_0^\lambda\int_0^p (p-qm)^{\eta_j-1}\mathfrak{g}_j^0(m)\,\mathrm{d}_q m\,\mathrm{d}p \\ &+ \frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j)}\int_0^1 (1-qp)^{\eta_j-1}\mathfrak{g}_j^0(p)\,\mathrm{d}_q p \\ &+ \frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\int_0^1 (1-qp)^{\eta_j-\sigma_j-1}\mathfrak{g}_j^0(p)\,\mathrm{d}_q p \\ &+ \frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-1)}\int_0^1 (1-qp)^{\eta_j-2}\mathfrak{g}_j^0(p)\,\mathrm{d}_q p \\ &+ \frac{(2\mathbf{b}+2\mathbf{a}(\theta-1))\kappa}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j)}\int_0^\theta\int_0^p (p-qm)^{\eta_j-1}\mathfrak{g}_j^0(m)\,\mathrm{d}_q m\,\mathrm{d}p \\ &+ \frac{(1-\theta)\kappa}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^\lambda\int_0^p (p-qm)^{\eta_j-1}\mathfrak{g}_j^0(m)\,\mathrm{d}_q m\,\mathrm{d}p \\ &+ \frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^1 (1-qp)^{\eta_j-1}\mathfrak{g}_j^0(p)\,\mathrm{d}_q p \\ &+ \frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\int_0^1 (1-qp)^{\eta_j-\sigma_j-1}\mathfrak{g}_j^0(p)\,\mathrm{d}_q p \\ &+ \frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-1)}\int_0^1 (1-qp)^{\eta_j-2}\mathfrak{g}_j^0(p)\,\mathrm{d}_q p. \end{aligned}$$

It can be concluded from this that, for all $j=1,\dots,m$, we have $z_j^0 \in \mathcal{E}_j(z_1,\dots,z_m)$, so $\mathcal{G}$ takes closed values. In addition, from the compactness of the value of $\mathcal{T}_j$, it follows that $(z_1^0,\dots,z_m^0)\in\mathcal{E}(z_1,\dots,z_m)$ is bounded.

Step II. Our goal at this step is to establish the inequality $\mathcal{HM}(\mathcal{E}(s),\mathcal{E}(r)) \leq \Psi(\|s-r\|)$. To do this, let $(s_1,\dots,s_m),(r_1,\dots,r_m)\in\mathcal{G}$, $(\hbar_1,\dots,\hbar_m)\in\mathcal{E}(r_1,\dots,r_m)$ be given. Then for all $\kappa\in\mathcal{K}$ and $j=1,\dots,m$, we can choose $(\mathfrak{g}_1,\dots,\mathfrak{g}_m)\in\mathcal{S}^*_{\mathcal{T}_1,(r_1,\dots,r_m)}\times\cdots\times\mathcal{S}^*_{\mathcal{T}_m,(r_1,\dots,r_m)}$ such that for all $\kappa\in\mathcal{K}$, we can write

$$\begin{aligned} \hbar_j(\kappa) ={}& \frac{1}{\Gamma_q(\eta_j)}\int_0^\kappa (\kappa-qp)^{\eta_j-1}\mathfrak{g}_j(p)\,\mathrm{d}_q p \\ &+ \frac{\mathbf{a}}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^\theta\int_0^p (p-qm)^{\eta_j-1}\mathfrak{g}_j(m)\,\mathrm{d}_q m\,\mathrm{d}p \\ &+ \frac{(\theta^2-2)}{2\mathbf{b}\Gamma_q(\eta_j)}\int_0^\lambda\int_0^p (p-qm)^{\eta_j-1}\mathfrak{g}_j(m)\,\mathrm{d}_q m\,\mathrm{d}p \end{aligned}$$

$$
\begin{aligned}
&+\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j)}\int_0^1 (1-qp)^{\eta_j-1}\mathfrak{g}_j(p)\,\mathrm{d}_q p\\
&+\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\int_0^1 (1-qp)^{\eta_j-\sigma_j-1}\mathfrak{g}_j(p)\,\mathrm{d}_q p\\
&+\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-1)}\int_0^1 (1-qp)^{\eta_j-2}\mathfrak{g}_j(p)\,\mathrm{d}_q p\\
&+\frac{(2\mathbf{b}+2\mathbf{a}(\theta-1))\kappa}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j)}\int_0^\theta\int_0^p (p-qm)^{\eta_j-1}\mathfrak{g}_j(m)\,\mathrm{d}_q m\,\mathrm{d}p\\
&+\frac{(1-\theta)\kappa}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^\lambda\int_0^p (p-qm)^{\eta_j-1}\mathfrak{g}_j(m)\,\mathrm{d}_q m\,\mathrm{d}p\\
&+\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^1 (1-qp)^{\eta_j-1}\mathfrak{g}_j(p)\,\mathrm{d}_q p\\
&+\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\int_0^1 (1-qp)^{\eta_j-\sigma_j-1}\mathfrak{g}_j(p)\,\mathrm{d}_q p\\
&+\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-1)}\int_0^1 (1-qp)^{\eta_j-2}\mathfrak{g}_j(p)\,\mathrm{d}_q p.
\end{aligned}
$$

But, in view of hypothesis $\mathcal{A}_3$, for all $j=1,\dots,m$, we have

$$
\begin{aligned}
&\mathcal{HM}\big(\mathcal{T}_j\big(\kappa,s_j(\kappa),s_j'(\kappa),{}^C\mathcal{D}_q^{\sigma_j}s_j(\kappa),\mathcal{I}_q^{\delta_j}s_j(\kappa)\big),\\
&\quad\mathcal{T}_j\big(\kappa,r_j(\kappa),r_j'(\kappa),{}^C\mathcal{D}_q^{\sigma_j}r_j(\kappa),\mathcal{I}_q^{\delta_j}r_j(\kappa)\big)\big)\\
&\quad\le\frac{1}{\mathfrak{A}_{1_j}+\mathfrak{A}_{2_j}+\mathfrak{A}_{3_j}+\mathfrak{A}_{4_j}}\mathbf{\Omega}_j(\kappa)\Psi\big(\big|s_j(\kappa)-r_j(\kappa)\big|+\big|s_j'(\kappa)-r_j'(\kappa)\big|\\
&\qquad+\big|{}^C\mathcal{D}_q^{\sigma_j}s_j(\kappa)-{}^C\mathcal{D}_q^{\sigma_j}r_j(\kappa)\big|+\big|\mathcal{I}_q^{\delta_j}s_j(\kappa)-\mathcal{I}_q^{\delta_j}r_j(\kappa)\big|\big).
\end{aligned}
$$

Hence, $\exists\mathbf{x}_j\in\mathcal{T}_j(\kappa,\mathbf{z}_j(\kappa),\mathbf{z}_j'(\kappa),{}^C\mathcal{D}_q^{\sigma_j}\mathbf{z}_j(\kappa),\mathcal{I}_q^{\delta_j}\mathbf{z}_j(\kappa))$, where $\forall\kappa\in\mathcal{K}$ and $j=1,\dots,m$,

$$
\big|\mathfrak{g}_j(\kappa)-\mathbf{x}_j\big|\le\frac{1}{\mathfrak{A}_{1_j}+\mathfrak{A}_{2_j}+\mathfrak{A}_{3_j}+\mathfrak{A}_{4_j}}\mathbf{\Omega}_{\boldsymbol{j}}(\kappa)\Psi\left(\sum_{i=1}^4|s_i-r_i|\right).
$$

Now, for all $j=1,\dots,m$ define the map $\mathcal{F}_j:\mathcal{K}\to\mathcal{P}(\mathbb{R})$ such that

$$
\mathcal{F}_j(\kappa)=\left\{\mathbf{x}\in\mathbb{R}:\big|\mathfrak{g}_j(\kappa)-\mathbf{x}\big|\le\frac{1}{\mathfrak{A}_{1_j}+\mathfrak{A}_{2_j}+\mathfrak{A}_{3_j}+\mathfrak{A}_{4_j}}\mathbf{\Omega}_j(\kappa)\Psi\left(\sum_{i=1}^4|s_i-r_i|\right)\right\}.
$$

If $\frac{1}{\mathfrak{A}_{1_j}+\mathfrak{A}_{2_j}+\mathfrak{A}_{3_j}+\mathfrak{A}_{4_j}}\mathbf{\Omega}_{\boldsymbol{j}}(\kappa)\Psi(\sum_{i=1}^4|s_i-r_i|)$ and $\mathfrak{g}_j$ are measurable, then also the set-valued map $\mathcal{F}_j(\cdot)\cap\mathcal{T}_j(\cdot,\mathbf{z}_j(\cdot),\mathbf{z}_j'(\cdot),{}^C\mathcal{D}_q^{\sigma_j}\mathbf{z}_j(\cdot),\mathcal{I}_q^{\delta_j}\mathbf{z}_j(\cdot))$ is measurable.

Take $\mathfrak{g}_j^*(\kappa)\in\mathcal{T}_j(\kappa,\mathbf{z}_j(\kappa),\mathbf{z}_j'(\kappa),{}^C\mathcal{D}_q^{\sigma_j}\mathbf{z}_j(\kappa),\mathcal{I}_q^{\delta_j}\mathbf{z}_j(\kappa))$, which for all $\kappa\in\mathcal{K}$ and $j=1,\dots,m$, we have

$$
\begin{aligned}
\hbar_j^*(\kappa)&=\frac{1}{\Gamma_q(\eta_j)}\int_0^\kappa (\kappa-qp)^{\eta_j-1}\mathfrak{g}_j^*(p)\,\mathrm{d}_q p\\
&\quad+\frac{\mathbf{a}}{\mathbf{b}\Gamma_q(\eta_j)}\int_0^\theta\int_0^p (p-qm)^{\eta_j-1}\mathfrak{g}_j^*(m)\,\mathrm{d}_q m\,\mathrm{d}p
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(\theta^2-2)}{2\mathbf{b}\Gamma_q(\eta_j)} \int_0^{\lambda} \int_0^{p} (p-qm)^{\eta_j-1} \mathfrak{g}_j^*(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&+ \frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j)} \int_0^{1} (1-qp)^{\eta_j-1} \mathfrak{g}_j^*(p)\,\mathrm{d}_q p \\
&+ \frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-\sigma_j)} \int_0^{1} (1-qp)^{\eta_j-\sigma_j-1} \mathfrak{g}_j^*(p)\,\mathrm{d}_q p \\
&+ \frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-1)} \int_0^{1} (1-qp)^{\eta_j-2} \mathfrak{g}_j^*(p)\,\mathrm{d}_q p \\
&+ \frac{(2\mathbf{b}+2\mathbf{a}(\theta-1))\kappa}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j)} \int_0^{\theta} \int_0^{p} (p-qm)^{\eta_j-1} \mathfrak{g}_j^*(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&+ \frac{(1-\theta)\kappa}{\mathbf{b}\Gamma_q(\eta_j)} \int_0^{\lambda} \int_0^{p} (p-qm)^{\eta_j-1} \mathfrak{g}_j^*(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
&+ \frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j)} \int_0^{1} (1-qp)^{\eta_j-1} \mathfrak{g}_j^*(p)\,\mathrm{d}_q p \\
&+ \frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j)} \int_0^{1} (1-qp)^{\eta_j-\sigma_j-1} \mathfrak{g}_j^*(p)\,\mathrm{d}_q p \\
&+ \frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-1)} \int_0^{1} (1-qp)^{\eta_j-2} \mathfrak{g}_j^*(p)\,\mathrm{d}_q p.
\end{aligned}
$$

Subsequently, let $\sup_{\kappa\in\mathcal{K}} |\mathbf{\Omega}(\kappa)| = \|\mathbf{\Omega}\|$, therefore

$$
\begin{aligned}
&\left|\hbar_j(\kappa)-\hbar_j^*(\kappa)\right| \\
&\quad\le \frac{1}{\Gamma_q(\eta_j)} \int_0^{\kappa} (\kappa-qp)^{\eta_j-1} \left|\mathfrak{g}_j-\mathfrak{g}_j^*\right|\,\mathrm{d}_q p \\
&\qquad+ \left|\frac{\mathbf{a}}{\mathbf{b}\Gamma_q(\eta_j)}\right| \int_0^{\theta} \int_0^{p} (p-qm)^{\eta_j-1} \left|\mathfrak{g}_j(m)-\mathfrak{g}_j^*(m)\right|\,\mathrm{d}_q m\,\mathrm{d}p \\
&\qquad+ \left|\frac{(\theta^2-2)}{2\mathbf{b}\Gamma_q(\eta_j)}\right| \int_0^{\lambda} \int_0^{p} (p-qm)^{\eta_j-1} \left|\mathfrak{g}_j(m)-\mathfrak{g}_j^*(m)\right|\,\mathrm{d}_q m\,\mathrm{d}p \\
&\qquad+ \left|\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j)}\right| \int_0^{1} (1-qp)^{\eta_j-1} \left|\mathfrak{g}_j-\mathfrak{g}_j^*\right|\,\mathrm{d}_q p \\
&\qquad+ \left|\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\right| \int_0^{1} (1-qp)^{\eta_j-\sigma_j-1} \left|\mathfrak{g}_j-\mathfrak{g}_j^*\right|\,\mathrm{d}_q p \\
&\qquad+ \left|\frac{2-\theta^2}{2\mathbf{b}\Gamma_q(\eta_j-1)}\right| \int_0^{1} (1-qp)^{\eta_j-2} \left|\mathfrak{g}_j-\mathfrak{g}_j^*\right|\,\mathrm{d}_q p \\
&\qquad+ \left|\frac{(2\mathbf{b}+2\mathbf{a}(\theta-1))\kappa}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j)}\right| \int_0^{\theta} \int_0^{p} (p-qm)^{\eta_j-1} \left|\mathfrak{g}_j(m)-\mathfrak{g}_j^*(m)\right|\,\mathrm{d}_q m\,\mathrm{d}p \\
&\qquad+ \left|\frac{(1-\theta)\kappa}{\mathbf{b}\Gamma_q(\eta_j)}\right| \int_0^{\lambda} \int_0^{p} (p-qm)^{\eta_j-1} \left|\mathfrak{g}_j(m)-\mathfrak{g}_j^*(m)\right|\,\mathrm{d}_q m\,\mathrm{d}p \\
&\qquad+ \left|\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j)}\right| \int_0^{1} (1-qp)^{\eta_j-1} \left|\mathfrak{g}_j-\mathfrak{g}_j^*\right|\,\mathrm{d}_q p \\
&\qquad+ \left|\frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j)}\right| \int_0^{1} (1-qp)^{\eta_j-\sigma_j-1} \left|\mathfrak{g}_j-\mathfrak{g}_j^*\right|\,\mathrm{d}_q p
\end{aligned}
$$

$$
\begin{aligned}
&+ \left| \frac{(\theta-1)\kappa}{\mathbf{b}\Gamma_q(\eta_j-1)} \right| \int_0^1 (1-qp)^{\eta_j-2} \left| \mathfrak{g}_j - \mathfrak{g}_j^* \right| \mathrm{d}_q p \\
&\leq \frac{\mathfrak{A}_{\mathbf{1}_j}}{\mathfrak{A}_{\mathbf{1}_j} + \mathfrak{A}_{\mathbf{2}_j} + \mathfrak{A}_{\mathbf{3}_j} + \mathfrak{A}_{\mathbf{4}_j}} \Psi\big(\big\|(s_1 - r_1, \ldots, s_m - r_m)\big\|\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| \hbar_j'(\kappa) - \hbar_j^{*'}(\kappa) \right| \\
&\leq \frac{1}{\Gamma_q(\eta_j-1)} \int_0^\kappa (\kappa-qp)^{\eta_j-2} \left| \mathfrak{g}_j - \mathfrak{g}_j^* \right| \mathrm{d}_q p \\
&\quad + \left| \frac{(2\mathbf{b} + 2\mathbf{a}(\theta-1))}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j)} \right| \int_0^\theta \int_0^p (p-qm)^{\eta_j-1} \left| \mathfrak{g}_j(m) - \mathfrak{g}_j^*(m) \right| \mathrm{d}_q m \, \mathrm{d}p \\
&\quad + \left| \frac{(1-\theta)}{\mathbf{b}\Gamma_q(\eta_j)} \right| \int_0^\lambda \int_0^p (p-qm)^{\eta_j-1} \left| \mathfrak{g}_j(m) - \mathfrak{g}_j^*(m) \right| \mathrm{d}_q m \, \mathrm{d}p \\
&\quad + \left| \frac{(\theta-1)}{\mathbf{b}\Gamma_q(\eta_j)} \right| \int_0^1 (1-qp)^{\eta_j-1} \left| \mathfrak{g}_j - \mathfrak{g}_j^* \right| \mathrm{d}_q p \\
&\quad + \left| \frac{(\theta-1)}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j)} \right| \int_0^1 (1-qp)^{\eta_j-\sigma_j-1} \left| \mathfrak{g}_j - \mathfrak{g}_j^* \right| \mathrm{d}_q p \\
&\quad + \left| \frac{(\theta-1)}{\mathbf{b}\Gamma_q(\eta_j-1)} \right| \int_0^1 (1-qp)^{\eta_j-2} \left| \mathfrak{g}_j - \mathfrak{g}_j^* \right| \mathrm{d}_q p \\
&\leq \frac{\mathfrak{A}_{\mathbf{2}_j}}{\mathfrak{A}_{\mathbf{1}_j} + \mathfrak{A}_{\mathbf{2}_j} + \mathfrak{A}_{\mathbf{3}_j} + \mathfrak{A}_{\mathbf{4}_j}} \Psi\big(\big\|(s_1 - r_1, \ldots, s_m - r_m)\big\|\big).
\end{aligned}
$$

Also, one can obtain

$$
\begin{aligned}
&\left| {}^C\mathcal{D}_q^{\sigma_j} \hbar_j(\kappa) - {}^C\mathcal{D}_q^{\sigma_j} \hbar_j^*(\kappa) \right| \\
&\leq \frac{1}{\Gamma_q(\eta_j-\sigma_j)} \int_0^\kappa (\kappa-qp)^{\eta_j-\sigma_j-1} \left| \mathfrak{g}_j - \mathfrak{g}_j^* \right| \mathrm{d}_q p \\
&\quad + \left| \frac{(2\mathbf{b} + 2\mathbf{a}(\theta-1))\kappa^{1-\sigma_j}}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j)\Gamma_q(2-\sigma_j)} \right| \\
&\quad \times \int_0^\theta \int_0^p (p-qm)^{\eta_j-1} \left| \mathfrak{g}_j(m) - \mathfrak{g}_j^*(m) \right| \mathrm{d}_q m \, \mathrm{d}p \\
&\quad + \left| \frac{(1-\theta)\kappa^{1-\sigma_j}}{\mathbf{b}\Gamma_q(\eta_j)\Gamma_q(2-\sigma_j)} \right| \int_0^\lambda \int_0^p (p-qm)^{\eta_j-1} \left| \mathfrak{g}_j(m) - \mathfrak{g}_j^*(m) \right| \mathrm{d}_q m \, \mathrm{d}p \\
&\quad + \left| \frac{(\theta-1)\kappa^{1-\sigma_j}}{\mathbf{b}\Gamma_q(\eta_j)\Gamma_q(2-\sigma_j)} \right| \int_0^1 (1-qp)^{\eta_j-1} \left| \mathfrak{g}_j - \mathfrak{g}_j^* \right| \mathrm{d}_q p \\
&\quad + \left| \frac{(\theta-1)\kappa^{1-\sigma_j}}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j)\Gamma_q(2-\sigma_j)} \right| \int_0^1 (1-qp)^{\eta_j-\sigma_j-1} \left| \mathfrak{g}_j - \mathfrak{g}_j^* \right| \mathrm{d}_q p \\
&\quad + \left| \frac{(\theta-1)\kappa^{1-\sigma_j}}{\mathbf{b}\Gamma_q(\eta_j-1)\Gamma_q(2-\sigma_j)} \right| \int_0^1 (1-qp)^{\eta_j-2} \left| \mathfrak{g}_j - \mathfrak{g}_j^* \right| \mathrm{d}_q p \\
&\leq \frac{\mathfrak{A}_{\mathbf{3}_j}}{\mathfrak{A}_{\mathbf{1}_j} + \mathfrak{A}_{\mathbf{2}_j} + \mathfrak{A}_{\mathbf{3}_j} + \mathfrak{A}_{\mathbf{4}_j}} \Psi\big(\big\|(s_1 - r_1, \ldots, s_m - r_m)\big\|\big).
\end{aligned}
$$

and

$$\begin{aligned}
&\left|\mathcal{I}_q^{\delta_j}\hbar_j(\kappa)-\mathcal{I}_q^{\delta_j}\hbar_j^*(\kappa)\right|\\
&\quad\le\frac{1}{\Gamma_q(\eta_j+\sigma_j)}\int_0^\kappa(\kappa-qp)^{\eta_j+\sigma_j-1}\left|\mathfrak{g}_j-\mathfrak{g}_j^*\right|\mathrm{d}_qp\\
&\qquad+\left|\frac{(2\mathbf{b}+2\mathbf{a}(\theta-1))\kappa^{1+\delta_j}}{\mathbf{b}(2-\theta^2)\Gamma_q(\eta_j)\Gamma_q(2+\delta_j)}\right|\\
&\qquad\times\int_0^\theta\int_0^p(p-qm)^{\eta_j-1}\left|\mathfrak{g}_j(m)-\mathfrak{g}_j^*(m)\right|\mathrm{d}_qm\,\mathrm{d}p\\
&\qquad+\left|\frac{(1-\theta)\kappa^{1+\delta_j}}{\mathbf{b}\Gamma_q(\eta_j)\Gamma_q(2+\delta_j)}\right|\int_0^\lambda\int_0^p(p-qm)^{\eta_j-1}\left|\mathfrak{g}_j(m)-\mathfrak{g}_j^*(m)\right|\mathrm{d}_qm\,\mathrm{d}p\\
&\qquad+\left|\frac{(\theta-1)\kappa^{1+\delta_j}}{\mathbf{b}\Gamma_q(\eta_j)\Gamma_q(2+\delta_j)}\right|\int_0^1(1-qp)^{\eta_j-1}\left|\mathfrak{g}_j-\mathfrak{g}_j^*\right|\mathrm{d}_qp\\
&\qquad+\left|\frac{(\theta-1)\kappa^{1+\delta_j}}{\mathbf{b}\Gamma_q(\eta_j-\sigma_j)\Gamma_q(2+\delta_j)}\right|\int_0^1(1-qp)^{\eta_j-\sigma_j-1}\left|\mathfrak{g}_j-\mathfrak{g}_j^*\right|\mathrm{d}_qp\\
&\qquad+\left|\frac{(\theta-1)\kappa^{1+\delta_j}}{\mathbf{b}\Gamma_q(\eta_j-1)\Gamma_q(2+\delta_j)}\right|\int_0^1(1-qp)^{\eta_j-2}\left|\mathfrak{g}_j-\mathfrak{g}_j^*\right|\mathrm{d}_qp\\
&\quad\le\frac{\mathfrak{A}_{4_j}}{\mathfrak{A}_{1_j}+\mathfrak{A}_{2_j}+\mathfrak{A}_{3_j}+\mathfrak{A}_{4_j}}\Psi\left(\left\|(s_1-r_1,\dots,s_m-r_m)\right\|\right).
\end{aligned}$$

It can be inferred from the above relationships that

$$\begin{aligned}
\left\|\hbar_j-\hbar_j^*\right\|&=\sup_{\kappa\in\mathcal{K}}\left|\hbar_j(\kappa)-\hbar_j^*(\kappa)\right|+\sup_{\kappa\in\mathcal{K}}\left|\hbar_j'(\kappa)-\hbar_j^{*'}(\kappa)\right|\\
&\quad+\sup_{\kappa\in\mathcal{K}}\left|{}^C\mathcal{D}_q^\sigma\hbar_j(\kappa)-{}^C\mathcal{D}_q^\sigma\hbar_j^*(\kappa)\right|+\sup_{\kappa\in\mathcal{K}}\left|\mathcal{I}_q^{\delta_j}\hbar_j(\kappa)-\mathcal{I}_q^{\delta_j}\hbar_j^*(\kappa)\right|\\
&\le\frac{1}{\mathfrak{A}_{1_j}+\mathfrak{A}_{2_j}+\mathfrak{A}_{3_j}+\mathfrak{A}_{4_j}}\Psi\left(\left\|(s_1-r_1,\dots,s_m-r_m)\right\|\right)\\
&\quad\times(\mathfrak{A}_{1_j}+\mathfrak{A}_{2_j}+\mathfrak{A}_{3_j}+\mathfrak{A}_{4_j})\\
&=\Psi\left(\left\|(s_1-r_1,\dots,s_m-r_m)\right\|\right).
\end{aligned}$$

Thus, for all $(s_1,\dots,s_m),(r_1,\dots,r_m)\in\mathcal{G}$, we have

$$\mathcal{HM}\left(\mathcal{E}(s_1,\dots,s_m),\mathcal{E}(r_1,\dots,r_m)\right)\le\Psi\left(\left\|(s_1-r_1,\dots,s_m-r_m)\right\|\right).$$

Now, according to Lemma 2.14, and the endpoint property of $\mathcal{E}$, there exists $w^*\in\mathcal{G}$ such that $\mathcal{E}(w^*)=\{w^*\}$. Hence, w^* is a solution for the fractional q-inclusion system mentioned (1)–(2). □

Table 3 Numerical values of $\mathfrak{A}_{1_j}, \mathfrak{A}_{2_j}, \mathfrak{A}_{3_j}, \mathfrak{A}_{4_j}$, where $j = 1, 2$, for different values of q in Example 4.1

	$q_1 = 0.13$	$q_2 = 0.25$	$q_3 = 0.39$	$q_4 = 0.58$	$q_5 = 0.72$	$q_6 = 0.89$
$\mathfrak{A}_{1_1}$	1.1721	1.1289	1.0752	0.9981	0.9381	0.8615
$\mathfrak{A}_{2_1}$	1.0400	1.0833	1.1377	1.2178	1.2811	1.3629
$\mathfrak{A}_{3_1}$	2.1729	2.3780	2.6406	3.0314	3.3423	3.7441
$\mathfrak{A}_{4_1}$	2.3068	2.6489	3.1006	3.8042	4.3897	5.1778
Ξ_1	0.0261	0.0241	0.0220	0.0193	0.0175	0.0157
$\mathfrak{A}_{1_2}$	0.5743	0.5497	0.5201	0.4784	0.4463	0.4054
$\mathfrak{A}_{2_2}$	0.5208	0.5430	0.5709	0.6121	0.6447	0.6869
$\mathfrak{A}_{3_2}$	1.0726	1.1650	1.2853	1.4658	1.6100	1.7965
$\mathfrak{A}_{4_2}$	1.1324	1.2790	1.4701	1.7632	2.0037	2.3233
Ξ_2	0.0265	0.0247	0.0227	0.0203	0.0186	0.0168

4 Examples

Example 4.1 Consider the following fractional quantum integro-differential inclusion problem:

$$\begin{cases} {}^{C}\mathcal{D}_q^{\frac{11}{8}} \mathbf{z}_1(\kappa) \in \mathcal{T}_1(\kappa, \mathbf{z}_1(\kappa), \mathbf{z}_1'(\kappa), {}^{C}\mathcal{D}_q^{\frac{7}{8}} \mathbf{z}_1(\kappa), \mathcal{I}_q^{\frac{3}{5}} \mathbf{z}_1(\kappa)), \\ {}^{C}\mathcal{D}_q^{\frac{7}{5}} \mathbf{z}_2(\kappa) \in \mathcal{T}_2(\kappa, \mathbf{z}_1(\kappa), \mathbf{z}_2'(\kappa), {}^{C}\mathcal{D}_q^{\frac{5}{8}} \mathbf{z}_2(\kappa), \mathcal{I}_q^{\frac{2}{5}} \mathbf{z}_2(\kappa)), \end{cases} \tag{4}$$

with boundary conditions

$$\begin{cases} \mathbf{z}_1(0) + \mathbf{z}_1'(0) + \mathcal{I}_q^{\frac{3}{5}} \mathbf{z}_1(0) = \int_0^{\frac{1}{2}} \mathbf{z}(p)\, \mathrm{d}p, \\ \mathbf{z}_1(1) + \mathbf{z}_1'(1) + {}^{C}\mathcal{D}_q^{\frac{7}{8}} \mathbf{z}_1(1) = \int_0^{\frac{1}{4}} \mathbf{z}(p)\, \mathrm{d}p, \end{cases} \tag{5}$$

and

$$\begin{cases} \mathbf{z}_2(0) + \mathbf{z}_2'(0) + \mathcal{I}_q^{\frac{2}{5}} \mathbf{z}_2(0) = \int_0^{\frac{1}{2}} \mathbf{z}(p)\, \mathrm{d}p, \\ \mathbf{z}_2(1) + \mathbf{z}_2'(1) + {}^{C}\mathcal{D}_q^{\frac{5}{8}} \mathbf{z}_2(1) = \int_0^{\frac{1}{4}} \mathbf{z}(p)\, \mathrm{d}p, \end{cases} \tag{6}$$

where $\kappa \in \mathcal{K} = [0,1]$. Here, we put $m = 2$, $\eta_1 = \frac{11}{8}, \eta_2 = \frac{7}{5}$, $\sigma_1 = \frac{7}{8}, \sigma_2 = \frac{5}{8}$, $\delta_1 = \frac{3}{5}, \delta_2 = \frac{2}{5}$, and multifunctions $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \times \mathbb{R}^4 \to 2^{\mathbb{R}}$ are defined as follows:

$$\mathcal{T}_1(\kappa, s_1, s_2, s_3, s_4) = \left[0, \frac{7}{40}\kappa \sin(\kappa) + \frac{7\kappa^3 |s_1|}{40(\kappa^2 + |s_1|)} + \frac{7}{40}\kappa \cos(s_2) + \frac{7\kappa(e^{s_3} + \cos(s_4))}{40(1 + e^{s_3})}\right]$$

and

$$\mathcal{T}_2(\kappa, s_1, s_2, s_3, s_4) = \left[0, \frac{7\kappa(\kappa^2 + 1)}{80\cos(\kappa)} + \frac{7\kappa}{80}\sin(s_1) + \frac{7\kappa}{80}\left(\frac{e^{s_2} + e^{s_3} + e^{s_4}}{1 + e^{s_2} + e^{s_3} + e^{s_4}}\right)\right].$$

We choose $\mathbf{\Omega}_j : [0,1] \to [0,\infty)$ to be $\mathbf{\Omega}_1(\kappa) = \frac{7}{40}\kappa$, $\mathbf{\Omega}_2(\kappa) = \frac{7}{80}\kappa$ with $\|\mathbf{\Omega}_1\| = \frac{7}{40}$, $\|\mathbf{\Omega}_2\| = \frac{7}{80}$, and $\Psi(\kappa) = \frac{\kappa}{31}$. Obviously, Ψ is nondecreasing and usc on $\mathcal{K}$. Furthermore, $\mathfrak{A}_{1_j}, \mathfrak{A}_{2_j}, \mathfrak{A}_{3_j}$, $\mathfrak{A}_{4_j}$, for $j = 1, 2$, are calculated for $q = 0.13$, 0.25, 0.39, 0.58, 0.72, and 0.89 in Table 3. Also, the heatmap of data in Table 3 is presented in Fig. 1. Notice that, for convenience, we set $\Xi_j = (\mathfrak{A}_{1_j} + \mathfrak{A}_{2_j} + \mathfrak{A}_{3_j} + \mathfrak{A}_{4_j})^{-1} \|\Omega_j\|$, for $j = 1, 2$.

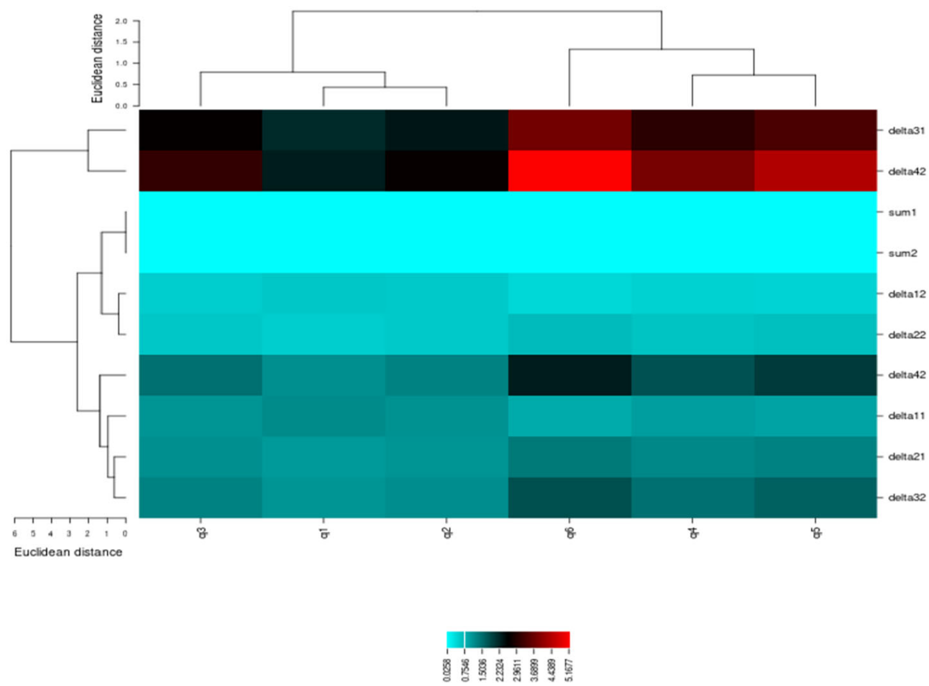

Figure 1 The heatmap of Table 3 in Example 4.1

Now, it is easy to check that for $j = 1, 2$,

$$\begin{aligned}\mathcal{HM}(\mathcal{T}_j(\kappa, s_1, s_2, s_3, s_4), \mathcal{T}_j(\kappa, r_1, r_2, r_3, r_4) &\leq \frac{1}{\mathfrak{A}_{1_j} + \mathfrak{A}_{2_j} + \mathfrak{A}_{3_j} + \mathfrak{A}_{4_j}} \boldsymbol{\Omega}_j(\kappa) \\ &\quad \times \Psi\left(\sum_{i=1}^{4} |s_i - r_i|\right),\end{aligned}$$

and $\inf_{w \in \mathcal{G}}(\sup_{z \in \mathcal{E}(z)} \|w - z\|) = 0$. Now all the conditions of Theorem 3.2 are satisfied. Thanks to the endpoint property and Theorem 3.2, our problem formulated in (4.1) has a solution. To help illustrate this example, graphs of several functions are presented in Figs. 2 and 3.

5 Conclusion

Trying to provide models, which are more accurate and have less errors, using the capabilities of fractional calculus is one of the broad new research topics in mathematics, physics, biology, engineering, and economics. But now, many classical methods are no longer able to interpret complex phenomena. For example, multifunctions have recently been considered for modeling phenomena with frequent shocks. In this work, we also quantum analyzed the m-dimensional system of integro-differential inclusions with the help of a new technique in the fixed point theory, namely, we applied the endpoint property. The existing derivative operators in our problem are of q-Caputo type. Here, unlike previous research in the literature, for the first time, the endpoint technique was used to solve an m-dimensional system. Quantum calculus has provided the necessary prerequisites for using computers in our computations, and the effectiveness of our proposed method has been shown in the final example.

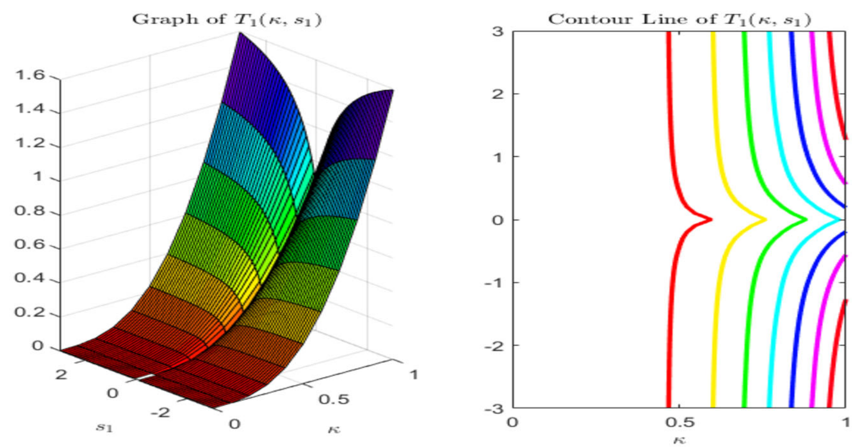

Figure 2 The graph of $\mathcal{T}_1(\kappa, s_1)$ in Example 4.1

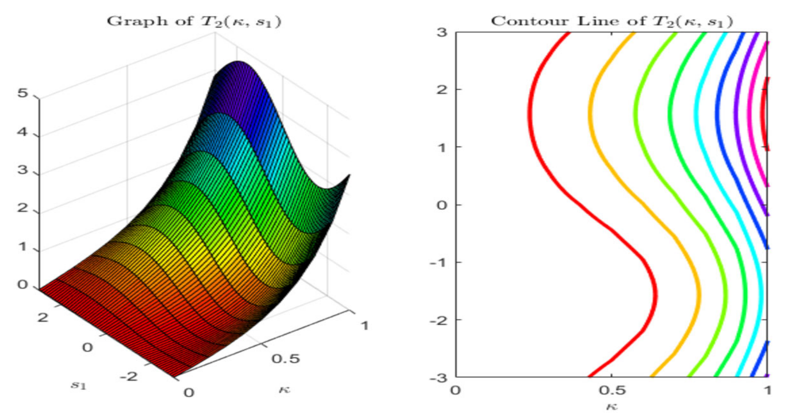

Figure 3 The graph of $\mathcal{T}_2(\kappa, s_1)$ in Example 4.1

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Author contributions
M.G. and S.R. dealt with conceptualization, supervision, methodology, investigation, writing-original draft, formal analysis, editing and preparing the figures and the table. The authors read and approved the final manuscript.

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The authors declare no competing interests.

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