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On the intermixed method for mixed variational inequality problems: another look and some corrections

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Abstract

We explore the intermixed method for finding a common element of the intersection of the solution set of a mixed variational inequality and the fixed point set of a nonexpansive mapping. We point out that Khuangsatung and Kangtunyakarn's statement [J. Inequal. Appl. 2023:1, 2023] regarding the resolvent utilized in their paper is not correct. To resolve this gap, we employ the epigraphical projection and the product space approach. In particular, we obtain a strong convergence theorem with a weaker assumption.

Keywords: Nonexpansive mapping; Variational inequality; Fixed point; Epigraphical projection

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let $C \subset \mathcal{H}$, $S : C \to \mathcal{H}$, and $\alpha > 0$. We say that

- S is *α-Lipschitzian* if $\|Sx - Sy\| \le \alpha\|x - y\|$ for all $x, y \in C$;
- S is *α-inverse strongly monotone* if $\langle Sx - Sy, x - y\rangle \ge \alpha\|Sx - Sy\|^2$ for all $x, y \in C$.

An α-Lipschitzian mapping with $\alpha \in (0,1)$ ($\alpha = 1$, resp.) is called a *contraction* (a *nonexpansive* mapping, resp.). The following two classical nonlinear problems have been widely studied:

Fixed Point Problem: Find $x \in C$ such that $x = Sx$ (see [2]).

Variational Inequality Problem: Find $x \in C$ such that $\langle Sx, y - x\rangle \ge 0$ for all $y \in C$ (see [3]).

The solution sets of the preceding two problems are denoted by $\mathrm{Fix}(S)$ and $\mathrm{VI}(C, S)$, respectively. The following two observations are well known.

- If $S : C \to C$ is any mapping and $\mathrm{Id} : C \to C$ is the identity mapping, then $\mathrm{Fix}(S) = \mathrm{VI}(C, \mathrm{Id} - S)$. In fact, if $x = Sx$, then $\langle(\mathrm{Id} - S)x, y - x\rangle = 0$ for all $y \in C$. Hence $\mathrm{Fix}(S) \subset \mathrm{VI}(C, \mathrm{Id} - S)$. On the other hand, let $x \in C$ be such that $\langle(\mathrm{Id} - S)x, y - x\rangle \ge 0$ for all $y \in C$. Let $y := Sx \in C$. It follows that $-\|x - Sx\|^2 = \langle x - Sx, Sx - x\rangle \ge 0$, and hence $x = Sx$. This implies that reverse inclusion, and the statement is proved.
- If C is a closed convex subset of $\mathcal{H}$ and $S : C \to \mathcal{H}$ is any mapping, then $\mathrm{VI}(C, S) = \mathrm{Fix}(P_C \circ (\mathrm{Id} - S))$, where P_C is the metric projection onto C. Note that for

$x \in \mathcal{H}$ and $z \in C$, $z = P_C x$ if and only if $\langle z - x, y - z \rangle \geq 0$ for all $y \in C$ (for example, see [4]). To see this, let $x \in C$. It follows that

$$\langle Sx, y - x \rangle \geq 0 \text{ for all } y \in C \iff \langle x - (\mathrm{Id} - S)x, y - x \rangle \geq 0 \text{ for all } y \in C.$$

Hence $x \in \mathrm{VI}(C, S) \iff x = P_C(\mathrm{Id} - S)x \iff x \in \mathrm{Fix}(P_C \circ (\mathrm{Id} - S))$, and the statement is proved.

Recently, Khuangsatung and Kangtunyakarn [1] studied the following problem:

Let $f : \mathcal{H} \to (-\infty, \infty]$ be a proper, convex, and lower semicontinuous function. Let $C \subset \mathcal{H}$ be a closed convex set, and let $S : C \to C$. The *mixed variational inequality* problem is to find an element $x \in C$ such that

$$\langle Sx, y - x \rangle + f(y) - f(x) \geq 0 \quad \text{for all } y \in C.$$

The solution of this problem is denoted by $\mathrm{VI}(C, S, f)$. If $f \equiv 0$, then the mixed variational inequality problem becomes the (classical) variational inequality problem. They claimed in their Lemma 2.6 that

$$\mathrm{VI}(C, S, f) = \mathrm{Fix}\big((\mathrm{Id} + \gamma \partial f)^{-1} \circ (\mathrm{Id} - \gamma S)\big) \quad (\text{for all } \gamma > 0),$$

where ∂f is the *subdifferential* operator of f, that is,

$$\partial f(x) := \big\{ z \in \mathcal{H} : \langle z, y - x \rangle + f(x) \leq f(y) \text{ for all } y \in \mathcal{H} \big\}.$$

Unfortunately, their claim is *not* correct. To see this, let $C := [1, 2] \subset \mathbb{R}$, $Sx := 2x$ for all $x \in C$, and $f(x) := 0$ for all $x \in \mathbb{R}$. It follows that $\mathrm{VI}(C, S, f) = \{1\}$ and $\mathrm{Fix}((\mathrm{Id} + \gamma \partial f)^{-1} \circ (\mathrm{Id} - \gamma S)) = \mathrm{Fix}(\mathrm{Id} - \gamma S) = \varnothing$ for all $\gamma > 0$. In this paper, we propose an alternative way to address this gap. Moreover, we use the product space approach to deduce the *intermixed method* [5] and show that the convergence result can be established under a *weaker* assumption.

Let us recall their main result.

Theorem KK *Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$. Suppose that $A_1, A_2, B_1, B_2 : C \to \mathcal{H}$ are α-inverse strongly monotone operators and $T_1, T_2 : C \to C$ are nonexpansive mappings. Suppose that $f_1, f_2 : \mathcal{H} \to (-\infty, \infty]$ are proper, convex, and lower semicontinuous functions. Assume that for $i = 1, 2$,*

$$\Omega_i := \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A_i, f_i) \cap \mathrm{VI}(C, B_i, f_i) \neq \varnothing.$$

Suppose that $g_1, g_2 : \mathcal{H} \to \mathcal{H}$ are contractions and $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are iterative sequences generated by the following scheme:

$$\begin{aligned}
&x_1, y_1 \in C \text{ are arbitrarily chosen,} \\
&x_n' := b_1 x_n + (1 - b_1) T_1 x_n, \\
&y_n' := b_2 y_n + (1 - b_2) T_2 y_n,
\end{aligned}$$

$$
\begin{aligned}
x_n'' &:= J_{\gamma_1 f_1}\big(x_n - \gamma_1\big(a_1A_1 + (1-a_1)B_1\big)x_n\big),\\
y_n'' &:= J_{\gamma_2 f_2}\big(y_n - \gamma_2\big(a_2A_2 + (1-a_2)B_2\big)y_n\big),\\
x_{n+1} &:= (1-\beta_n)x_n' + \beta_n P_C\big(\alpha_n g_2(y_n) + (1-\alpha_n)x_n''\big),\\
y_{n+1} &:= (1-\beta_n)y_n' + \beta_n P_C\big(\alpha_n g_1(x_n) + (1-\alpha_n)y_n''\big),
\end{aligned}
$$

where $\gamma_1, \gamma_2 \in (0, 2\alpha)$, $a_1, a_2, b_1, b_2 \in (0,1)$, *and the sequences* $\{\alpha_n\}_{n=1}^\infty$ *and* $\{\beta_n\}_{n=1}^\infty \subset [0,1]$ *satisfy the following conditions*:

(C1) $\lim_n \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$,

(C2) $\beta_n \in [k, l] \subset (0,1)$ *for all* $n \geq 1$,

(C3) $\sum_n |\alpha_n - \alpha_{n+1}| < \infty$ *and* $\sum_n |\beta_n - \beta_{n+1}| < \infty$.

Then there are two elements x^* *and* y^* *such that* $x^* = P_{\Omega_1} g_2(y^*)$, $y^* = P_{\Omega_2} g_1(x^*)$, *and the iterative sequences* $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *converge strongly to* x^* *and* y^*, *respectively.*

We need the following lemma.

Lemma 1 ([6]) *Let* $\{s_n\}_{n=1}^\infty$ *be a sequence of nonnegative real numbers, let* $\{t_n\}_{n=1}^\infty$ *be a sequence of real numbers, and let* $\{\alpha_n\}_{n=1}^\infty$ *be a sequence in* $[0,1]$ *such that*

$$
s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n t_n \quad \textit{for all } n \geq 1.
$$

If $\sum_n \alpha_n = \infty$ *and* $\limsup_n t_n \leq 0$, *then* $\lim_n s_n = 0$.

Lemma 2 *Let* $C \subset \mathcal{H}$ *and* $S : C \to \mathcal{H}$. *Then*:

(a) *If* C *is closed and convex and* S *is nonexpansive, then* $\operatorname{Fix}(S)$ *is closed and convex.*

(b) *If* S *is* α*-inverse strongly monotone, then* $\operatorname{Id} - \lambda S$ *is nonexpansive for all* $\lambda \in [0, 2\alpha]$.

2 Main results

2.1 A Halpern-type method

Recall that a nonexpansive mapping $S : C \to C$ is *r-strongly quasi-nonexpansive* ($r > 0$) if $\operatorname{Fix}(S) \neq \varnothing$ and

$$
\|Sx - p\|^2 \leq \|x - p\|^2 - r\|x - Sx\|^2 \quad \text{for all } x \in C \text{ and } p \in \operatorname{Fix}(S).
$$

It is well known that every nonexpansive mapping $S : C \to C$ satisfies the *Browder demiclosedness principle*: $p \in \operatorname{Fix}(S)$ whenever $\{x_n\}_{n=1}^\infty$ is a sequence in C such that $\lim_n \|x_n - Sx_n\| = 0$ and $\{x_n\}_{n=1}^\infty$ converges weakly to $p \in C$ (see [7]). The technique we used in the following result is taken from Wang et al. [8].

Theorem 3 *Let* $C \subset \mathcal{H}$ *be closed and convex, and let* $S, U : C \to C$ *be nonexpansive mappings such that* $F := \operatorname{Fix}(S) \cap \operatorname{Fix}(U) \neq \varnothing$. *Suppose that* S *is r-strongly quasinonexpansive, where* $r > 0$. *Suppose that* $u \in \mathcal{H}$ *and* $\{x_n\}_{n=1}^\infty$ *is an iterative sequence generated by the following scheme*:

$$
\begin{aligned}
&x_1 \in C \textit{ is arbitrarily chosen},\\
&x_{n+1} := (1-\beta_n)Sx_n + \beta_n P_C\big(\alpha_n u + (1-\alpha_n)Ux_n\big) \quad (n \geq 1),
\end{aligned}
$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset [0,1]$ *satisfy the following conditions*:

$$\lim_n \frac{\alpha_n}{1-\beta_n} = 0 \quad \text{and} \quad \sum_n \alpha_n \beta_n = \infty.$$

Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $P_F u$.

Proof Note that F is closed and convex. Let $z := P_F u$. It follows that $z = P_F z = Sz = Uz$ and

$$\begin{aligned}
\|x_{n+1} - z\| &\leq (1-\beta_n)\|Sx_n - z\| + \beta_n \big\| P_C\big(\alpha_n u + (1-\alpha_n)Ux_n\big) - P_F z\big\| \\
&\leq (1-\beta_n)\|x_n - z\| + \beta_n\alpha_n\|u - z\| + \beta_n(1-\alpha_n)\|Ux_n - z\| \\
&\leq (1-\beta_n\alpha_n)\|x_n - z\| + \beta_n\alpha_n\|u - z\| \\
&\leq \max\big\{\|x_n - z\|, \|u - z\|\big\}.
\end{aligned}$$

It follows by induction that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. In particular, the sequences $\{Sx_n\}_{n=1}^{\infty}$, $\{Ux_n\}_{n=1}^{\infty}$, and $\{P_C(\alpha_n u + (1-\alpha_n)Ux_n)\}_{n=1}^{\infty}$ are all bounded. For convenience, we denote

$$u_n := \alpha_n u + (1-\alpha_n)Ux_n.$$

We refine the preceding estimates by considering $\|\cdot\|^2$ as follows:

$$\|Sx_n - z\|^2 \leq \|x_n - z\|^2 - r\|x_n - Sx_n\|^2,$$

and

$$\begin{aligned}
\|u_n - z\|^2 &= \big\|\alpha_n(u - z) + (1-\alpha_n)(Ux_n - z)\big\|^2 \\
&\leq \big\|(1-\alpha_n)(Ux_n - z)\big\|^2 + 2\big\langle\alpha_n(u - z), u_n - z\big\rangle \\
&\leq (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, u_n - z\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad = \big\|(1-\beta_n)(Sx_n - z) + \beta_n(P_C u_n - z)\big\|^2 \\
&\quad = (1-\beta_n)\|Sx_n - z\|^2 + \beta_n\|P_C u_n - z\|^2 - \beta_n(1-\beta_n)\|Sx_n - P_C u_n\|^2 \\
&\quad \leq (1-\beta_n)\big(\|x_n - z\|^2 - r\|x_n - Sx_n\|^2\big) + \beta_n\big((1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, u_n - z\rangle\big) \\
&\qquad - \beta_n(1-\beta_n)\|Sx_n - P_C u_n\|^2 \\
&\quad = (1-\alpha_n\beta_n)\|x_n - z\|^2 + 2\alpha_n\beta_n\langle u - z, u_n - z\rangle \\
&\qquad - r(1-\beta_n)\|x_n - Sx_n\|^2 - \beta_n(1-\beta_n)\|Sx_n - P_C u_n\|^2.
\end{aligned}$$

Since $\sum_n \alpha_n\beta_n = \infty$, we have

$$\limsup_n \|x_n - z\|^2 \leq L,$$

where

$$L := \limsup_n \left(2\langle u - z, u_n - z\rangle - \frac{r(1-\beta_n)}{\alpha_n\beta_n}\|x_n - Sx_n\|^2 - \frac{1-\beta_n}{\alpha_n}\|Sx_n - P_C u_n\|^2 \right).$$

Note that $L \le 2\limsup_n \langle u - z, u_n - z\rangle < \infty$ because $\{u_n\}_{n=1}^{\infty}$ is bounded. If $L = -\infty$, then it follows that $\limsup_n \|x_n - z\|^2 \le 0$, and we are done. We now assume that L is finite. Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence such that $\{u_{n_k}\}_{k=1}^{\infty}$ converges weakly to some element $q \in C$ and

$$\lim_k \left(2\langle u - z, u_{n_k} - z\rangle - \frac{r(1-\beta_{n_k})}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k} - Sx_{n_k}\|^2 - \frac{1-\beta_{n_k}}{\alpha_{n_k}}\|Sx_{n_k} - P_C u_{n_k}\|^2 \right) = L.$$

In particular, the sequences

$$\left\{ \frac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k} - Sx_{n_k}\|^2 \right\}_{k=1}^{\infty} \quad \text{and} \quad \left\{ \frac{1-\beta_{n_k}}{\alpha_{n_k}}\|Sx_{n_k} - P_C u_{n_k}\|^2 \right\}_{k=1}^{\infty}$$

are both bounded. Note that $\lim_n \frac{\alpha_n\beta_n}{1-\beta_n} = \lim_n \frac{\alpha_n}{1-\beta_n} = 0$. It follows that

$$\lim_k \|x_{n_k} - Sx_{n_k}\|^2 = \lim_k \|Sx_{n_k} - P_C u_{n_k}\|^2 = 0.$$

Moreover, we have $\lim_k \|u_{n_k} - Ux_{n_k}\| = 0$. In particular, $\lim_k \|P_C u_{n_k} - Ux_{n_k}\| = 0$ and $x_{n_k} \rightharpoonup q$. Then it follows that

$$\lim_k \|x_{n_k} - Ux_{n_k}\| \le \lim_k \big(\|x_{n_k} - Sx_{n_k}\| + \|Sx_{n_k} - P_C u_{n_k}\| + \|P_C u_{n_k} - Ux_{n_k}\| \big) = 0.$$

In particular, it follows from the Browder demiclosedness principle that $q \in F$, and hence $\langle z - u, q - z\rangle \ge 0$. This implies that $\limsup_n \|x_n - z\|^2 \le L \le 2\lim_k \langle u - z, u_{n_k} - z\rangle = 2\langle u - z, q - z\rangle \le 0$. $\square$

Corollary 4 *Suppose that $C, S, U, F, r, \{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are as in the preceding theorem. Suppose that $h : \mathcal{H} \to \mathcal{H}$ is a contraction and $\{x_n\}_{n=1}^{\infty}$ is an iterative sequence generated by the following scheme*:

$$\begin{aligned} &x_1 \in C \textit{ is arbitrarily chosen},\\ &x_{n+1} := (1-\beta_n)Sx_n + \beta_n P_C\big(\alpha_n h(x_n) + (1-\alpha_n)Ux_n\big) \quad (n \ge 1). \end{aligned}$$

Proof Note that $P_F \circ h : \mathcal{H} \to \mathcal{H}$ is a contraction, and thus it follows that there exists a unique element $z \in \mathcal{H}$ such that $z = (P_F \circ h)(z)$. It is clear that $z \in F$. We let $u := h(z)$ and define

$$\begin{aligned} &y_1 := x_1,\\ &y_{n+1} := (1-\beta_n)Sy_n + \beta_n P_C\big(\alpha_n u + (1-\alpha_n)Uy_n\big) \quad (n \ge 1). \end{aligned}$$

It follows from the preceding theorem that $\lim_n \|y_n - z\| = 0$. Suppose that h is γ-Lipschitzian with $\gamma \in (0,1)$. We have the following estimate:

$$\|x_{n+1} - y_{n+1}\| \le (1-\beta_n)\|Sx_n - Sy_n\| + \beta_n \big\| P_C\big(\alpha_n h(x_n) + (1-\alpha_n)Ux_n\big)$$

$$
\begin{aligned}
&\quad - P_C\big(\alpha_n u + (1-\alpha_n)Uy_n\big)\big\| \\
&\le (1-\beta_n)\|x_n - y_n\| + \beta_n\big\|\alpha_n h(x_n) + (1-\alpha_n)Ux_n \\
&\quad - \big(\alpha_n h(z) + (1-\alpha_n)Uy_n\big)\big\| \\
&\le (1-\beta_n)\|x_n - y_n\| + \alpha_n\beta_n\big\|h(x_n) - h(z)\big\| + (1-\alpha_n)\beta_n\|Ux_n - Uy_n\| \\
&\le (1-\alpha_n\beta_n)\|x_n - y_n\| + \alpha_n\beta_n\big\|h(x_n) - h(z)\big\|.
\end{aligned}
$$

It follows from $\sum_n \alpha_n\beta_n = \infty$ that

$$
\begin{aligned}
\limsup_n \|x_n - y_n\| &\le \limsup_n \big\|h(x_n) - h(z)\big\| \\
&\le \limsup_n \gamma\|x_n - z\| \\
&\le \gamma \limsup_n \big(\|x_n - y_n\| + \|y_n - z\|\big) \\
&= \gamma \limsup_n \|x_n - y_n\|.
\end{aligned}
$$

In particular, since $\gamma < 1$, we have $\lim_n \|x_n - y_n\| = 0$, and hence $\lim_n \|x_n - z\| = 0$. The proof is complete. □

Let $S :=$ Id and $u \in C$. We immediately obtain the following result.

Corollary 5 *Let $C \subset \mathcal{H}$ be closed and convex, and let $U : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(U) \neq \emptyset$. *Suppose that $u \in C$ and $\{x_n\}_{n=1}^\infty$ is an iterative sequence generated by the following scheme*:

$$
\begin{aligned}
&x_1 \in C \textit{ is arbitrarily chosen}, \\
&x_{n+1} := (1-\beta_n)x_n + \beta_n\big(\alpha_n u + (1-\alpha_n)Ux_n\big) \quad (n \ge 1),
\end{aligned}
$$

where the sequences $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty \subset [0,1]$ satisfy the following conditions:

$$
\lim_n \frac{\alpha_n}{1-\beta_n} = 0 \quad \textit{and} \quad \sum_n \alpha_n\beta_n = \infty.
$$

Then the iterative sequence $\{x_n\}_{n=1}^\infty$ converges strongly to $P_{\mathrm{Fix}(U)}u$.

2.2 Comments and remarks on the mixed variational inequality problem

Let $C \subset \mathcal{H}$ be closed and convex, let $A : C \to \mathcal{H}$, and let $f : \mathcal{H} \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. The *mixed variational inequality problem* is to find $x \in C$ such that

$$
\langle Ax, y - x\rangle + f(y) - f(x) \ge 0 \quad \text{for all } y \in C. \tag{$\star$}
$$

As pointed out in the introduction of the paper, the resolvent proposed by Khuangsatung and Kangtunyakarn [1] is not correct. Moreover, without any further assumption on C and $\mathrm{dom} f$, it is possible to encounter the experession $\infty - \infty$ in ($\star$). For example, let $Ax := 0$

for all $x \in C := [1,2] \subset \mathbb{R}$. and let $f(x) := 0$ if $x \in [3,4]$ and $f(x) := \infty$ if $x \notin [3,4]$. To be on the right track, we discuss the problem with an additional assumption.

This mixed type problem was also considered by Mosco [9] in 1969. From now on, we also assume that $\mathrm{dom} f \subset C$ is as in Mosco's setting. In particular, we also have $\mathrm{VI}(C,A,f) \subset \mathrm{dom} f$.

Mosco proved that the mixed and the classical variational inequality problems are equivalent. To see this, let $\widehat{\mathcal{H}} := \mathcal{H} \times \mathbb{R}$ with $\langle\!\langle \widehat{x}, \widehat{y} \rangle\!\rangle := \langle x, y\rangle + rs$ for all $\widehat{x} := (x,r)$ and $\widehat{y} := (y,s) \in \widehat{\mathcal{H}}$, and let $\widehat{C} := C \times \mathbb{R}$. Note that $|\!|\!|\widehat{x}|\!|\!|^2 = \langle\!\langle \widehat{x}, \widehat{x} \rangle\!\rangle = \|x\|^2 + r^2$. Define $\widehat{A} : \widehat{C} \to \widehat{\mathcal{H}}$ by

$$\widehat{A}(x,r) := (Ax, 1) \quad \text{for all } (x,r) \in \widehat{C}.$$

Here $\mathrm{epi} f := \{(x,r) \in \widehat{\mathcal{H}} : f(x) \le r\}$ is the *epigraph* of f, which is closed and convex because of the lower semicontinuity and convexity of f.

Theorem 6 *Suppose that* $\mathrm{dom} f \subset C$. *The following statements are true*:

(1) $\mathrm{VI}(C,A,f) = \{x \in C : \langle Ax, y - x\rangle + f(y) - f(x) \ge 0$ *for all* $y \in \mathrm{dom} f\}$;

(2) $(x,r) \in \mathrm{VI}(\mathrm{epi} f, \widehat{A}) \iff x \in \mathrm{VI}(C,A,f)$ *and* $r = f(x)$;

(3) *If A is α-inverse strongly monotone, then so is $\widehat{A}$, and hence* $\mathrm{Id} - \lambda\widehat{A}$ *is nonexpansive for all* $\lambda \in (0, 2\alpha]$.

Proof (1) is straight forward. (2) was proved by Mosco. For completeness, we give a proof of (2).

($\Rightarrow$) Let $(x,r) \in \mathrm{VI}(\mathrm{epi} f, \widehat{A})$, and let $y \in \mathrm{dom} f$. This implies that $(y, f(y)) \in \mathrm{epi} f$ and

$$\langle Ax, y - x\rangle + f(y) - r = \langle\!\langle \widehat{A}(x,r), \big(y, f(y)\big) - (x,r) \rangle\!\rangle \ge 0.$$

Note that $f(x) \le r$. This implies that $\langle Ax, y - x\rangle + f(y) - f(x) \ge 0$. Moreover, we have

$$f(x) - r = \langle\!\langle \widehat{A}(x,r), \big(x, f(x)\big) - (x,r) \rangle\!\rangle \ge 0.$$

This implies that $f(x) \ge r$, and hence $r = f(x)$. In particular, we have

$$\langle Ax, y - x\rangle + f(y) - f(x) \ge 0.$$

($\Leftarrow$) Suppose that $x \in \mathrm{VI}(C,A,f)$. We prove that $(x, f(x)) \in \mathrm{VI}(\mathrm{epi} f, \widehat{A})$. To see this, let $(y,s) \in \mathrm{epi} f$. It follows that $f(y) \le s$ and

$$\langle\!\langle \widehat{A}\big(x, f(x)\big), (y,s) - \big(x, f(x)\big) \rangle\!\rangle = \langle Ax, y - x\rangle + s - f(x) \ge \langle Ax, y - x\rangle + f(y) - f(x) \ge 0.$$

(3) Suppose that A is α-inverse strongly monotone. We show that $\widehat{A} : \widehat{C} \to \widehat{\mathcal{H}}$ is also α-inverse strongly monotone. To see this, let $\widehat{x} := (x,r), \widehat{y} := (y,s) \in \widehat{C}$. It follows that

$$\langle\!\langle \widehat{A}\widehat{x} - \widehat{A}\widehat{y}, \widehat{x} - \widehat{y} \rangle\!\rangle = \langle Ax - Ay, x - y\rangle \ge \alpha \|Ax - Ay\|^2 = \alpha \left|\!\left|\!\left| \widehat{A}\widehat{x} - \widehat{A}\widehat{y} \right|\!\right|\!\right|^2.$$

In particular, $\mathrm{Id} - \lambda\widehat{A}$ is nonexpansive for $\lambda \in (0, 2\alpha]$. □

Because of the error of the resolvent proposed by the authors of [1], we cannot infer the closedness and the convexity of $\mathrm{VI}(C,A,f)$. However, the conclusion remains true as follows.

Corollary 7 *Let $A : C \to \mathcal{H}$ be α-inverse strongly monotone, and let $f : \mathcal{H} \to (-\infty,\infty]$ be a proper convex and lower semicontinuous function. Suppose that* $\operatorname{dom} f \subset C$. *Then* $\mathrm{VI}(C,A,f)$ *is closed and convex.*

Proof We assume that $\mathrm{VI}(C,A,f)$ is nonempty. Note that $\mathrm{VI}(\operatorname{epi} f,\widehat{A}) = \mathrm{Fix}(P_{\operatorname{epi} f}\circ(\mathrm{Id} - \alpha\widehat{A}))$ is closed and convex. To prove the closedness of $\mathrm{VI}(C,A,f)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathrm{VI}(C,A,f)$ and assume that $\{x_n\}_{n=1}^{\infty}$ is strongly convergent to a point $x \in C$. It suffices to show that $(x,f(x)) \in \mathrm{VI}(\operatorname{epi} f,\widehat{A})$. Put $r := f(x)$ and $r_n := f(x_n)$. From the lower semicontinuity of f it follows that $r \le \liminf_n r_n$. Note that for $(y,s) \in \widehat{C} := C \times \mathbb{R}$, we have

$$\langle Ax_n, y - x_n\rangle + s - r_n = \langle\langle\widehat{A}(x_n,r_n),(y,s) - (x_n - r_n)\rangle\rangle \ge 0.$$

Since A is $(1/\alpha)$-Lipschitzian and hence continuous, we obtain that $\lim_n\langle Ax_n, y - x_n\rangle = \langle Ax, y - x\rangle$. In particular, $\langle Ax, y - x\rangle + s \ge \limsup_n r_n \ge r$. Hence $\langle\langle\widehat{A}(x,r),(y,s) - (x,r)\rangle\rangle = \langle Ax, y - x\rangle + s - r \ge 0$, that is, $(x,f(x)) = (x,r) \in \mathrm{VI}(\operatorname{epi} f,\widehat{A})$.

Finally, we prove that $\mathrm{VI}(C,A,f)$ is convex. To this end, let $x, x' \in \mathrm{VI}(C,A,f)$ and $t \in (0,1)$. It follows that $(x,r),(x',r') \in \mathrm{VI}(\operatorname{epi} f,\widehat{A})$, where $r := f(x)$ and $r' := f(x')$. Put $x'' := (1-t)x + tx'$. Since $\mathrm{VI}(C,A,f)$ is convex, it follows that $(x'',(1-t)r + tr') \in \mathrm{VI}(\operatorname{epi} f,\widehat{A})$. In particular, for $(y,s) \in \widehat{C} := C \times \mathbb{R}$ and $r'' := f(x'')$, we have $r'' \le (1-t)r + tr'$ and

$$\begin{aligned}\langle\langle\widehat{A}(x'',r''),(y,s) - (x'',r'')\rangle\rangle &= \langle Ax'', y - x''\rangle + s - r'' \\ &\ge \langle Ax'', y - x''\rangle + s - ((1-t)r + tr') \\ &= \langle\langle\widehat{A}(x'',(1-t)r + tr'),(y,s) - (x'',(1-t)r + tr')\rangle\rangle \ge 0.\end{aligned}$$

It follows that $(x'',r'') \in \mathrm{VI}(\operatorname{epi} f,\widehat{A})$, and hence $x'' \in \mathrm{VI}(C,A,f)$. □

2.3 Another look at the intermixed method via a product space approach

Suppose that C, $\mathcal{H}$, A_i, B_i, T_i, f_i, g_i $(i = 1,2)$ are as in Theorem KK. Note that we can show that $\mathrm{VI}(C,A_1,f_1) \cap \mathrm{VI}(C,B_1,f_1) = \mathrm{VI}(C,a_1A_1 + (1-a_1)B_1,f_1)$ for $0 < a_1 < 1$ if $\mathrm{VI}(C,A_1,f_1) \cap \mathrm{VI}(C,B_1,f_1) \ne \emptyset$ and if A_1 and B_1 are α-inverse strongly monotone. Corresponding to this note, we assume for simplicity that $A_1 = B_1$ and $A_2 = B_2$. We also assume that

$$\Omega_i := \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,A_i,f_i) \ne \emptyset \quad \text{for } i = 1,2.$$

To deduce and correct the conclusion in Theorem KK, let us fix the following notation.

Let

$$\boldsymbol{\mathcal{H}} := \widehat{\mathcal{H}} \times \widehat{\mathcal{H}} \quad \text{and} \quad \boldsymbol{C} := \widehat{C} \times \widehat{C},$$

where $\widehat{\mathcal{H}} := \mathcal{H} \times \mathbb{R}$ and $\widehat{C} := C \times \mathbb{R}$. Note that $\boldsymbol{\mathcal{H}}$ is a Hilbert space endowed with the inner product $[\cdot,\cdot]$ defined by

$$[\boldsymbol{x},\boldsymbol{x}'] := \langle x,x'\rangle + rr' + \langle y,y'\rangle + ss'$$

for all $\boldsymbol{x} := ((x,r),(y,s))$ and $\boldsymbol{x'} := ((x',r'),(y',s')) \in \boldsymbol{\mathcal{H}}$. Moreover, the induced norm of each element $\boldsymbol{x} := ((x,r),(y,s)) \in \boldsymbol{\mathcal{H}}$ is given by

$$\|\boldsymbol{x}\| := \left(\|x\|^2 + r^2 + \|y\|^2 + s^2\right)^{1/2}.$$

Define $\boldsymbol{A} : \boldsymbol{C} \to \boldsymbol{\mathcal{H}}$ and $\boldsymbol{S} : \boldsymbol{C} \to \boldsymbol{C}$ by

$$\boldsymbol{Ax} := \left((A_1 x, 1), (A_2 y, 1)\right)$$

and

$$\boldsymbol{Sx} := \left(\left(b_1 x + (1 - b_1) T_1 x, r\right), \left(b_2 y + (1 - b_2) T_2 y, s\right)\right)$$

for $\boldsymbol{x} := ((x,r),(y,s)) \in \boldsymbol{C}$.

Using the preceding setting, we obtain the following results.

Proposition 8 (Properties of $\boldsymbol{A}$) *Let* $\boldsymbol{x} := ((x,r),(y,s)) \in \boldsymbol{\mathcal{H}}$ *and* $\boldsymbol{E} := \operatorname{epi} f_1 \times \operatorname{epi} f_2$. *Then the following two statements are equivalent*:

(a) $\boldsymbol{x} \in \mathrm{VI}(\boldsymbol{E}, \boldsymbol{A})$;

(b) $x \in \mathrm{VI}(C, A_1, f_1)$, $y \in \mathrm{VI}(C, A_2, f_2)$, $r = f_1(x)$, *and* $s = f_2(x)$.

If, in addition, $A_1, A_2 : C \to \mathcal{H}$ *are* α*-inverse strongly monotone, then* $\boldsymbol{A} : \boldsymbol{C} \to \boldsymbol{\mathcal{H}}$ *is* α*-inverse strongly monotone.*

Proof (a) $\implies$ (b) Let $\boldsymbol{x} := ((x,r),(y,s)) \in \mathrm{VI}(\boldsymbol{E}, \boldsymbol{A})$. Let $\boldsymbol{x'} := ((x',r'),(y,s))$, where $(x',r') \in \operatorname{epi} f_1$. It follows that $\boldsymbol{x'} \in \boldsymbol{E}$, and hence $\langle\!\langle \widehat{A_1}(x,r), (x',r') - (x,r) \rangle\!\rangle = [\boldsymbol{Ax}, \boldsymbol{x'} - \boldsymbol{x}] \geq 0$. This means that $(x,r) \in \mathrm{VI}(\operatorname{epi} f_1, \widehat{A_1})$. It follows from Theorem 6 that $x \in \mathrm{VI}(C, A_1, f_1)$ and $r = f_1(x)$. Using a similar technique, we obtain the remaining conclusion.

(b) $\implies$ (a) is trivial.

Suppose that $A_1, A_2 : C \to \mathcal{H}$ are α-inverse strongly monotone. To see that $\boldsymbol{A} : \boldsymbol{C} \to \boldsymbol{\mathcal{H}}$ is α-inverse strongly monotone, let $\boldsymbol{x} := ((x,r),(y,s))$ and $\boldsymbol{x'} := ((x',r'),(y',s')) \in \boldsymbol{C}$. It follows that

$$\begin{aligned}
&\left[\boldsymbol{Ax} - \boldsymbol{Ax'}, \boldsymbol{x} - \boldsymbol{x'}\right] \\
&\quad = \left[\left((A_1 x - A_1 x', 0), (A_2 y - A_2 y', 0)\right), \left((x - x', r - r'), (y - y', s - s')\right)\right] \\
&\quad = \left\langle A_1 x - A_1 x', x - x' \right\rangle + \left\langle A_2 y - A_2 y', y - y' \right\rangle \\
&\quad \geq \alpha\left(\left\|A_1 x - A_1 x'\right\|^2 + \left\|A_2 y - A_2 y'\right\|^2\right) = \alpha \left\|\boldsymbol{Ax} - \boldsymbol{Ax'}\right\|^2.
\end{aligned}$$

This completes the proof. □

Proposition 9 (Properties of $\boldsymbol{S}$) *Let* $\boldsymbol{x} := ((x,r),(y,s)) \in \boldsymbol{C}$. *Then the following two statements are equivalent*:

(a) $\boldsymbol{x} \in \mathrm{Fix}(\boldsymbol{S})$;

(b) $x \in \mathrm{Fix}(T_1)$ *and* $y \in \mathrm{Fix}(T_2)$.

If, in addition, T_1, T_2 *are nonexpansive and* $\mathrm{Fix}(T_1) \times \mathrm{Fix}(T_2) \neq \emptyset$, *then* $\boldsymbol{S}$ *is nonexpansive and* r*-strongly quasinonexpansive where* $r := \min\{b_1(1 - b_1), b_2(1 - b_2)\}$.

Proof (a) $\iff$ (b) is trivial. Now we suppose that T_1 and T_2 are nonexpansive and $\mathrm{Fix}(T_1) \times \mathrm{Fix}(T_2) \neq \emptyset$. It is clear that $\boldsymbol{S}$ is nonexpansive. Let $r := \min\{b_1(1-b_1), b_2(1-b_2)\}$. We show that $\boldsymbol{S}$ is r-strongly quasinonexpansive. To see this, let $\boldsymbol{x} := ((x,r),(y,s)) \in \boldsymbol{C}$ and $\boldsymbol{p} := ((p,r'),(q,s')) \in \mathrm{Fix}(\boldsymbol{S})$. It follows that

$$\begin{aligned}\left\|\left(b_1x + (1-b)T_1x\right) - p\right\|^2 &\leq b_1\|x-p\|^2 + (1-b_1)\|T_1x-p\|^2 - b_1(1-b_1)\|x-T_1x\|^2 \\ &\leq \|x-p\|^2 - r\|x-T_1x\|^2.\end{aligned}$$

Similarly, $\|(b_2y + (1-b_2)T_2y) - q\|^2 \leq \|y-q\|^2 - r\|y-T_2y\|^2$. This implies that

$$\begin{aligned}\|\boldsymbol{Sx} - \boldsymbol{p}\|^2 &= \left\|\left(b_1x + (1-b)T_1x\right) - p\right\|^2 + \left(r-r'\right)^2 \\ &\quad + \left\|\left(b_2y + (1-b_2)T_2y\right) - q\right\|^2 + \left(s-s'\right)^2 \\ &\leq \|x-p\|^2 + \left(r-r'\right)^2 + \|y-q\|^2 + \left(s-s'\right)^2 - r\left(\|x-T_1x\|^2 + \|y-T_2y\|^2\right) \\ &= \|\boldsymbol{x}-\boldsymbol{p}\|^2 - r\|\boldsymbol{x}-\boldsymbol{Sx}\|^2.\end{aligned}$$

The proof is complete. □

Proposition 10 *If $g_1, g_2 : \mathcal{H} \to \mathcal{H}$ are α-Lipschitzian, then $\boldsymbol{h} : \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}$ defined by*

$$\boldsymbol{h}(\boldsymbol{x}) := \left(\left(g_2(y), \alpha s\right), \left(g_1(x), \alpha r\right)\right) \quad \textit{for all } \boldsymbol{x} := \left((x,r),(y,s)\right) \in \boldsymbol{\mathcal{H}}$$

is also α-Lipschitzian.

Proof To see this, let $\boldsymbol{x} := ((x,r),(y,s))$, $\boldsymbol{x'} := ((x',r'),(y',s')) \in \boldsymbol{\mathcal{H}}$. It follows that

$$\begin{aligned}\left\|\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{h}\left(\boldsymbol{x'}\right)\right\|^2 &= \left\|\left(\left(g_2(y) - g_2\left(y'\right), \alpha\left(s-s'\right)\right), \left(g_1(x) - g_1\left(x'\right), \alpha\left(r-r'\right)\right)\right)\right\|^2 \\ &= \left\|g_2(y) - g_2\left(y'\right)\right\|^2 + \alpha^2\left(s-s'\right)^2) + \left\|g_1(x) - g_1\left(x'\right)\right\|^2 + \alpha^2\left(r-r'\right)^2 \\ &\leq \alpha^2\left(\left\|y-y'\right\|^2 + \left(s-s'\right)^2 + \left\|x-x'\right\|^2 + \left(r-r'\right)^2\right) \\ &= \alpha^2\left\|\boldsymbol{x} - \boldsymbol{x'}\right\|^2.\end{aligned}$$

This completes the proof. □

The intermixed algorithm can be regarded as a classical algorithm of Theorem 3, and we obtain the following convergence theorem.

Theorem 11 *Let $\boldsymbol{U} := \boldsymbol{P_E}(\mathbf{Id} - \lambda\boldsymbol{A})$ and $\boldsymbol{F} := \mathrm{VI}(\boldsymbol{E}, \boldsymbol{A}) \cap \mathrm{Fix}(\boldsymbol{S})$. Suppose that $\boldsymbol{x}_1 \in \boldsymbol{C}$ is arbitrarily chosen and*

$$\boldsymbol{x_{n+1}} := (1-\beta_n)\boldsymbol{Sx_n} + \beta_n\boldsymbol{P_C}\left(\alpha_n\boldsymbol{h}(\boldsymbol{x_n}) + (1-\alpha_n)\boldsymbol{Ux_n}\right),$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset [0,1]$ satisfy the following conditions:

$$\lim_n \frac{\alpha_n}{1-\beta_n} = 0 \quad \textit{and} \quad \sum_n \alpha_n\beta_n = \infty.$$

Then the iterative sequence $\{\boldsymbol{x_n}\}_{n=1}^{\infty}$ converges to $\boldsymbol{z} = \boldsymbol{P_F} \circ \boldsymbol{h}(\boldsymbol{z})$.

Remark 12 Our result is simultaneously a correction and an improvement of Theorem KK in the following ways.

(1) We use a product space approach to consider the mixed variational inequality problem and the intermixed algorithm.
(2) The resolvent proposed for the mixed variational inequality problem in the original work is not correct, and we propose a correction.
(3) The assumptions on the parameters $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are more general than those in Theorem KK. Moreover, Condition (C3) is superfluous. The choice $\alpha_n = \beta_n := 1/\sqrt{n}$ is applicable in our result, but it is not in Theorem KK.

Finally, we express the iterative sequence in our Theorem 11 as follows:

$$
\begin{aligned}
&(x_1, r_1), (y_1, s_1) \in C \times \mathbb{R} \text{are arbitrarily chosen,} \\
&\left(x_n', r_n'\right) := \left(b_1 x_n + (1 - b_1) T_1 x_n, r_n\right), \\
&\left(y_n', s_n'\right) := \left(b_2 y_n + (1 - b_2) T_2 y_n, s_n\right), \\
&\left(x_n'', r_n''\right) := P_{\operatorname{epi} f_1}(x_n - \lambda_n A_1 x_n, r_n - \lambda_n), \\
&\left(y_n'', s_n''\right) := P_{\operatorname{epi} f_2}(x_n - \lambda_n A_2 x_n, s_n - \lambda_n), \\
&(x_{n+1}, r_{n+1}) := \Bigl((1 - \beta_n) x_n' + \beta_n P_C\bigl(\alpha_n h(x_n) + (1 - \alpha_n) x_n''\bigr), \\
&\qquad\qquad (1 - \beta_n) r_n' + \beta_n\bigl(\alpha_n \alpha r_n + (1 - \alpha_n) r_n''\bigr)\Bigr), \\
&(y_{n+1}, s_{n+1}) := \Bigl((1 - \beta_n) y_n' + \beta_n P_C\bigl(\alpha_n h(y_n) + (1 - \alpha_n) y_n''\bigr), \\
&\qquad\qquad (1 - \beta_n) s_n' + \beta_n\bigl(\alpha_n \alpha s_n + (1 - \alpha_n) s_n''\bigr)\Bigr).
\end{aligned}
$$

For more detail on epigraphical projection, we refer to the book of Bauschke and Combettes [4]. It follows from our Theorem 11 that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge strongly to x^* and y^*, respectively, where $x^* = P_{\mathrm{VI}(C,A_1,f_1)\cap \mathrm{Fix}(T_1)} g_2(y^*)$ and $y^* = P_{\mathrm{VI}(C,A_2,f_2)\cap \mathrm{Fix}(T_2)} g_1(x^*)$.

Author contributions
I am the sole author of the manuscript.

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Data Availability
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Declarations

Competing interests
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